Chapter II: Selected Solutions

Exercise 2:

The likelihood ratio is given by

$$L(y) = \frac{3}{2(y+1)}, \quad 0 \le y \le 1.$$

a. With uniform costs and equal priors, the critical region for minimum Bayes error is given by $\{y \in [0,1] | L(y) \ge 1\} = \{y \in [0,1] | 3 \ge 2(y+1)\} = [0, 1/2]$. Thus the Bayes rule is given by

$$\delta_B(y) = \begin{cases} 1 & \text{if } 0 \le y \le 1/2 \\ 0 & \text{if } 1/2 < y \le 1 \end{cases}.$$

The corresponding minimum Bayes risk is

$$r(\delta_B) = \frac{1}{2}\int_0^{1/2} \frac{2}{3}(y+1)dy + \int_{1/2}^{1} dy = \frac{11}{24}.$$

b. With uniform costs, the least-favorable prior will be interior to $(0,1)$, so we examine the conditional risks of Bayes rules for an equalizer condition. The critical region for the Bayes rule δ_{π_0} is given by

$$\Gamma_1 = \left\{ y \in [0,1] \,\middle|\, L(y) \ge \frac{\pi_0}{1-\pi_0} \right\} = [0, \tau'],$$

where

$$\tau' = \begin{cases} 1 & \text{if } 0 \le \pi_0 \le \frac{3}{7} \\ \frac{1}{2}\left(\frac{3}{\pi_0} - 5\right) & \text{if } \frac{3}{7} < \pi_0 < \frac{3}{5} \\ 0 & \text{if } \frac{3}{7} \le \pi_0 \le 1 \end{cases}.$$

Thus, the conditional risks are:

$$R_0(\delta_{\pi_0}) = \int_0^{\tau'} \frac{2}{3}(y+1)dy = \begin{cases} 1 & \text{if } 0 \le \pi_0 \le \frac{3}{7} \\ \frac{2\tau'}{3}\left(\frac{\tau'}{2} + 1\right) & \text{if } \frac{3}{7} < \pi_0 < \frac{3}{5} \\ 0 & \text{if } \frac{3}{7} \le \pi_0 \le 1 \end{cases},$$

and

$$R_1(\delta_{\pi_0}) = \int_{\tau'}^{1} dy = \begin{cases} 0 & \text{if } 0 \le \pi_0 \le \frac{3}{7} \\ 1 - \tau' & \text{if } \frac{3}{7} < \pi_0 < \frac{3}{5} \\ 1 & \text{if } \frac{3}{7} \le \pi_0 \le 1 \end{cases}.$$

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