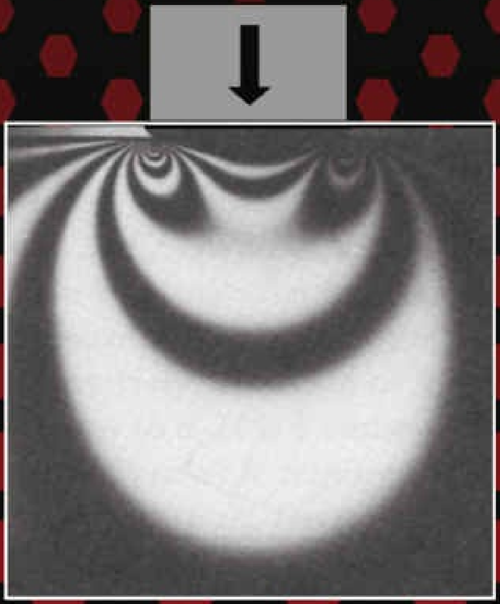
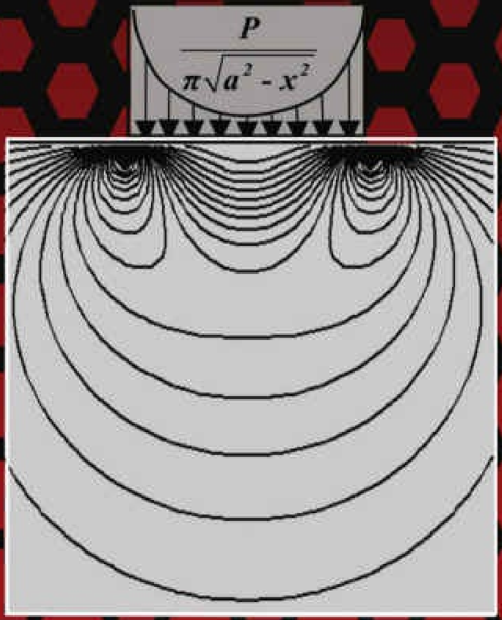
MATLAB examples

Fourth Edition

Elasticity

Theory, Applications, and Numerics

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Martin H. Sadd

Elasticity

Theory, Applications, and Numerics

FOURTH EDITION

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Preface

As with the previous works, this fourth edition continues the author's efforts to present linear elasticity with complete and concise theoretical development, numerous and contemporary applications, and enriching numerics to aid in problem solution and understanding. Over the years the author has given much thought on what should be taught to students in this field and what educational outcomes would be expected. Theoretical topics that are related to the foundations of elasticity should be presented in sufficient detail that will allow students to read and generally understand contemporary research papers. Related to this idea, students should acquire necessary vector and tensor notational skills and understand fundamental development of the basic field equations. Students should also have a solid understanding of the formulation and solution of various elasticity boundary-value problems that include a variety of domain and loading geometries. Finally, students should be able to apply modern engineering software (MATLAB, Maple or Mathematica) to aid in the solution, evaluation and graphical display of various elasticity problem solutions. These points are all emphasized in this text.

In addition to making numerous small corrections and clarifications, several new items have been added. A new section in Chapter 5 on singular elasticity solutions has been introduced to generally acquaint students with this type of behavior. Cubic anisotropy has now been presented in Chapter 11 as another particular form of elastic anisotropy. Inequality elastic moduli restrictions for various anisotropic material models have been better organized in a new table in Chapter 11. The general Naghdi-Hsu solution has now been introduced in Chapter 13. An additional micromechanical model of gradient elasticity has been added in Chapter 15. A couple of new MATLAB codes in Appendix C have been added and all codes are now referenced in the text where they are used. With the addition of 31 new exercises, the fourth edition now has 441 total exercises. These problems should provide instructors with many new and previous options for homework, exams, or material for in-class presentations or discussions. The online Solutions Manual has been updated and corrected and includes solutions to all exercises in the new edition. All text editions follow the original lineage as an outgrowth of lecture notes that I have used in teaching a two-course sequence in the theory of elasticity. Part I of the text is designed primarily for the first course, normally taken by beginning graduate students from a variety of engineering disciplines. The purpose of the first course is to introduce students to theory and formulation, and to present solutions to some basic problems. In this fashion students see how and why the more fundamental elasticity model of deformation should replace elementary strength of materials analysis. The first course also provides foundation for more advanced study in related areas of solid mechanics. Although the more advanced material included in Part II has normally been used for a second course, I often borrow selected topics for use in the first course. The elasticity presentation in this book reflects the words used in the title - theory, applications, and numerics. Because theory provides the fundamental cornerstone of this field, it is important to first provide a sound theoretical development of elasticity with sufficient rigor to give students a good foundation for the development of solutions to a broad class of problems. The theoretical development is carried out in an organized and concise manner in order to not lose the attention of the less mathematically inclined students or the focus of applications. With a primary goal of solving problems of engineering interest, the text offers numerous applications in contemporary areas, including anisotropic composite and functionally graded materials, fracture mechanics, micromechanics modeling, thermoelastic problems, and computational finite and boundary element methods. Numerous solved example problems and exercises are included in all chapters.

The new edition continues the special use of integrated numerics. By taking the approach that applications of theory need to be observed through calculation and graphical display, numerics is accomplished through the use of MATLAB, one of the most popular engineering software packages. This software is used throughout the text for applications such as stress and strain transformation, evaluation and plotting of stress and displacement distributions, finite element calculations, and comparisons between strength of materials and analytical and numerical elasticity solutions. With numerical and graphical evaluations, application problems become more interesting and useful for student learning. Other software such as Maple or Mathematica could also be used.

Contents summary

Part I of the book emphasizes formulation details and elementary applications. Chapter 1 provides a mathematical background for the formulation of elasticity through a review of scalar, vector, and tensor field theory. Cartesian tensor notation is introduced and is used throughout the book's formulation sections. Chapter 2 covers the analysis of strain and displacement within the context of small deformation theory. The concept of strain compatibility is also presented in this chapter. Forces, stresses, the equilibrium concept and various stress contour lines are developed in Chapter 3. Linear elastic material behavior leading to the generalized Hooke's law is discussed in Chapter 4, which also briefly presents nonhomogeneous, anisotropic, and thermoelastic constitutive forms. Later chapters more fully investigate these types of applications. Chapter 5 collects the previously derived equations and formulates the basic boundary value problems of elasticity theory. Displacement and stress formulations are constructed and general solution strategies are identified. This is an important chapter for students to put the theory together. Chapter 6 presents strain energy and related principles, including the reciprocal theorem, virtual work, and minimum potential and complementary energy. Two-dimensional formulations of plane strain, plane stress, and antiplane strain are given in Chapter 7. An extensive set of solutions for specific two dimensional problems is then presented in Chapter 8, and many applications employing MATLAB are used to demonstrate the results. Analytical solutions are continued in Chapter 9 for the Saint-Venant extension, torsion, and flexure problems. The material in Part I provides a logical and orderly basis for a sound one-semester beginning course in elasticity. Selected portions of the text's second part could also be incorporated into such a course. Part II delves into more advanced topics normally covered in a second course. The powerful method of complex variables for the plane problem is presented in Chapter 10, and several applications to fracture mechanics are given. Chapter 11 extends the previous isotropic theory into the behavior of anisotropic solids with emphasis on composite materials. This is an important application and, again, examples related to fracture mechanics are provided. Curvilinear anisotropy including both cylindrical and spherical orthotropy is included in this chapter to explore some basic problem solutions with this type of material structure. An introduction to thermoelasticity is developed in Chapter 12, and several specific application problems are discussed, including stress concentration and crack problems. Potential methods, including both displacement potentials and stress functions, are presented in Chapter 13. These methods are used to develop several three-dimensional elasticity solutions.

Chapter 14 covers nonhomogeneous elasticity, and this material is unique among current standard elasticity texts. After briefly covering theoretical formulations, several two-dimensional solutions are generated along with comparison field plots with the corresponding homogeneous cases. Chapter 15 presents a collection of elasticity applications to problems involving micromechanics modeling. Included are applications for dislocation modeling, singular stress states, solids with distributed cracks, micropolar, distributed voids, doublet mechanics and higher gradient theories. Chapter 16 provides a brief introduction to the powerful numerical methods of finite and boundary element techniques. Although only two-dimensional theory is developed, the numerical results in the example problems provide interesting comparisons with previously generated analytical solutions from earlier chapters. This fourth edition of Elasticity concludes with four appendices that contain a concise summary listing of basic field equations; transformation relations between Cartesian, cylindrical, and spherical coordinate systems; a MATLAB primer; and a self-contained review of mechanics of materials.

The subject

Elasticity is an elegant and fascinating subject that deals with determination of the stress, strain, and displacement distribution in an elastic solid under the influence of external forces. Following the usual assumptions of linear, small-deformation theory, the formulation establishes a mathematical model that allows solutions to problems that have applications in many engineering and scientific fields such as:

- Civil engineering applications include important contributions to stress and deflection analysis of structures, such as rods, beams, plates, and shells. Additional applications lie in geomechanics involving the stresses in materials such as soil, rock, concrete, and asphalt.
- Mechanical engineering uses elasticity in numerous problems in analysis and design of machine elements. Such applications include general stress analysis, contact stresses, thermal stress analysis, fracture mechanics, and fatigue.
- Materials engineering uses elasticity to determine the stress fields in crystalline solids, around dislocations, and in materials with microstructure.
- Applications in aeronautical and aerospace engineering typically include stress, fracture, and fatigue analysis in aerostructures.
- Biomechanical engineering uses elasticity to study the mechanics of bone and various types of soft tissue.

The subject also provides the basis for more advanced work in inelastic material behavior, including plasticity and viscoelasticity, and the study of computational stress analysis employing finite and boundary element methods. Since elasticity establishes a mathematical model of the deformation problem, it requires mathematical knowledge to understand formulation and solution procedures. Governing partial differential field equations are developed using basic principles of continuum mechanics commonly formulated in vector and tensor language. Techniques used to solve these field equations can encompass Fourier methods, variational calculus, integral transforms, complex variables, potential theory, finite differences, finite elements, and so forth. To prepare students for this subject, the text provides reviews of many mathematical topics, and additional references are given for further study. It is important for students to be adequately prepared for the theoretical developments, or else they will not be able to understand necessary formulation details. Of course, with emphasis on applications, the text concentrates on theory that is most useful for problem solution.

The concept of the elastic force–deformation relation was first proposed by Robert Hooke in 1678. However, the major formulation of the mathematical theory of elasticity was not developed until the nineteenth century. In 1821 Navier presented his investigations on the general equations of equilibrium; this was quickly followed by Cauchy, who studied the basic elasticity equations and developed the concept of stress at a point. A long list of prominent scientists and mathematicians continued development of the theory, including the Bernoullis, Lord Kelvin, Poisson, Lame', Green, Saint-Venant, Betti, Airy, Kirchhoff, Rayleigh, Love, Timoshenko, Kolosoff, Muskhelishvilli, and others.

During the two decades after World War II, elasticity research produced a large number of analytical solutions to specific problems of engineering interest. The 1970s and 1980s included considerable work on numerical methods using finite and boundary element theory. Also during this period, elasticity applications were directed at anisotropic materials for applications to composites. More recently, elasticity has been used in modeling materials with internal microstructures or heterogeneity and in inhomogeneous, graded materials. The rebirth of modern continuum mechanics in the 1960s led to a review of the foundations of elasticity and established a rational place for the theory within the general framework. Historical details can be found in the texts by Todhunter and Pearson, *History of* *the Theory of Elasticity*; Love, A *Treatise on the Mathematical Theory of Elasticity*; and Timoshenko, *A History of Strength of Materials*.

Exercises and web support

Of special note in regard to this text is the use of exercises and the publisher's website, www.textbooks.elsevier.com. Numerous exercises are provided at the end of each chapter for homework assignments to engage students with the subject matter. The exercises also provide an ideal tool for the instructor to present additional application examples during class lectures. Many places in the text make reference to specific exercises that work out details to a particular topic. Exercises marked with an asterisk (*) indicate problems that require numerical and plotting methods using the suggested MATLAB software. Solutions to all exercises are provided to registered instructors online at the publisher's website, thereby providing instructors with considerable help in using this material. In addition, downloadable MATLAB software is available to aid both students and instructors in developing codes for their own particular use to allow easy integration of the numerics. As with the previous edition, an on-line collection of PowerPoint slides is available for Chapters 1-9. This material includes graphical figures and summaries of basic equations that have proven to be useful during class presentations.

Feedback

The author is strongly interested in continual improvement of engineering education and welcomes feedback from users of the book. Please feel free to send comments concerning suggested improvements or corrections via surface mail or email (saddm@uri.edu). It is likely that such feedback will be shared with the text's user community via the publisher's website.

Acknowledgments

Many individuals deserve acknowledgment for aiding in the development of this textbook. First, I would like to recognize the many graduate students who sat in my elasticity classes. They have been a continual source of challenge and inspiration, and certainly influenced my efforts to find more effective ways to present this material. A special recognition goes to one particular student, Qingli Dai, who developed most of the original exercise solutions and did considerable proofreading. . I would also like to acknowledge the support of my institution, the University of Rhode Island, for providing me time, resources and the intellectual climate to complete this and previous editions. Several photoelastic pictures have been graciously provided by the URI Dynamic Photomechanics Laboratory (Professor Arun Shukla, director). Development and production support from several Elsevier staff was greatly appreciated.

As with the previous editions, this book is dedicated to the late Professor Marvin Stippes of the University of Illinois; he was the first to show me the elegance and beauty of the subject. His neatness, clarity, and apparently infinite understanding of elasticity will never be forgotten by his students.

Martin H. Sadd

About the Author

Martin H. Sadd is Professor Emeritus of Mechanical Engineering at the University of Rhode Island. He received his Ph.D. in mechanics from the Illinois Institute of Technology in 1971 and began his academic career at Mississippi State University. In 1979 he joined the faculty at Rhode Island and served as department chair from 1991 to 2000. He is a member of Phi Kappa Phi, Pi Tau Sigma, Tau Beta Pi, Sigma Xi, and is a Fellow of ASME. Professor Sadd's teaching background has been in the area of solid mechanics with emphasis in elasticity, continuum mechanics, wave propagation, and computational methods. His research has included analytical and computational modeling of materials under static and dynamic loading conditions. He has authored over 75 publications and is the author of *Continuum Mechanics Modeling of Material Behavior* (Elsevier, 2019).

PART 1

Foundations and elementary applications

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CHAPTER 1

Mathematical preliminaries

Abstract

Elasticity theory is formulated in terms of a variety of variables including scalar, vector, and tensor fields, and this calls for the use of tensor notation along with tensor algebra and calculus. Through the use of particular principles from continuum mechanics, the theory is formulated as a system of partial differential field equations that are to be solved in a region of space coinciding with the body under study. Solution techniques used on these field equations commonly employ Fourier methods, variational techniques, integral transforms, complex variables, potential theory, finite differences, and finite and boundary elements. Therefore, in order to develop proper formulation methods and solution techniques for elasticity problems, it is necessary to have an appropriate mathematical background. The purpose of this initial chapter is to provide this background primarily for the formulation part of our study. The chapter includes material on scalars, vectors, matrices, tensors, index notation, coordinate transformation, matrix principal value problem, calculus of Cartesian tensors, and curvilinear coordinates.

Keywords

Scalars; Vectors; Matrices; Tensors; Index notation; Coordinate transformation; Curvilinear coordinates

Similar to other field theories such as fluid mechanics, heat conduction, and electromagnetics, the study and application of elasticity theory requires knowledge of several areas of applied mathematics. The theory is formulated in terms of a variety of variables including scalar, vector, and tensor fields, and this calls for the use of tensor notation along with tensor algebra and calculus. Through the use of particular principles from continuum mechanics, the theory is developed as a system of partial differential field equations that are to be solved in a region of space coinciding with the body under study. Solution techniques used on these field equations commonly employ Fourier methods, variational techniques, integral transforms, complex variables, potential theory, finite differences, and finite and boundary elements. Therefore, to develop proper formulation methods and solution techniques for elasticity problems, it is necessary to have an appropriate mathematical background. The purpose of this initial chapter is to provide a background primarily for the formulation part of our study. Additional review of other mathematical topics related to problem solution technique is provided in later chapters where they are to be applied.

1.1. Scalar, vector, matrix, and tensor definitions

Elasticity theory is formulated in terms of many different types of variables that are either specified or sought at spatial points in the body under study. Some of these variables are *scalar quantities*, representing a single magnitude at each point in space. Common examples include the material density ρ and temperature T. Other variables of interest are *vector quantities* that are expressible in terms of components in a two- or three-dimensional coordinate system. Examples of vector variables are the displacement and rotation of material points in the elastic continuum. Formulations within the theory also require the need for *matrix variables*, which commonly require more than three components to quantify. Examples of such variables include stress and strain. As shown in subsequent chapters, a three-dimensional formulation requires nine components (only six are independent) to quantify the stress or strain at a point. For this case, the variable is normally expressed in a matrix format with three rows and three columns. To summarize this discussion, in a three-dimensional Cartesian coordinate system, scalar, vector, and matrix variables can thus be written as follows

$$\text{mass density scalar} = \rho$$

$$\text{displacement vector} = \boldsymbol{u} = u\boldsymbol{e}_1 + v\boldsymbol{e}_2 + w\boldsymbol{e}_3$$

$$\text{stress matrix} = [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are the usual unit basis vectors in the coordinate directions. Thus, scalars, vectors, and matrices are specified by one, three, and nine components respectively.

The formulation of elasticity problems not only involves these types of variables, but also incorporates additional quantities that require even more components to characterize. Because of this, most field theories such as elasticity make use of a *tensor formalism* using index notation. This enables efficient representation of all variables and governing equations using a single standardized scheme. The tensor concept is defined more precisely in a later section, but for now we can simply say that scalars, vectors, matrices, and other higher-order variables can all be represented by tensors of various orders. We now proceed to a discussion on the notational rules of order for the tensor formalism. Additional information on tensors and index notation can be found in many texts such as Goodbody (1982), Simmons (1994), Itskov (2015) and Sadd (2019).

1.2. Index notation

Index notation is a shorthand scheme whereby a whole set of numbers (elements or components) is represented by a single symbol with subscripts. For example, the three numbers a_1, a_2, a_3 are denoted by the symbol a_i, where index i has range 1, 2, 3. In a similar fashion, a_{ij} represents the nine numbers $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$. Although these representations can be written in any manner, it is common to use a scheme related to vector and matrix formats such that

$$a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{1.2.1}$$

In the matrix format, a_{1j} represents the first row, while a_{i1} indicates the first column. Other columns and rows are indicated in similar fashion, and thus the first index represents the row, while the second index denotes the column.

In general a symbol $a_{ij\ldots k}$ with N distinct indices represents 3^N distinct numbers. It should be apparent that a_i and a_j represent the same three numbers, and likewise a_{ij} and a_{mn} signify the same matrix. Addition, subtraction, multiplication, and equality of index symbols are defined in the normal fashion. For example, addition and subtraction are given by

$$a_i \pm b_i = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}, a_{ij} \pm b_{ij} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix} \tag{1.2.2}$$

and scalar multiplication is specified as

$$\lambda a_i = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{bmatrix}, \lambda a_{ij} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} \tag{1.2.3}$$

The multiplication of two symbols with different indices is called *outer multiplication*, and a simple example is given by

$$a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \tag{1.2.4}$$

The previous operations obey usual commutative, associative, and distributive laws, for example

$$
\begin{aligned}
&a_i + b_i = b_i + a_i \\
&a_{ij} b_k = b_k a_{ij} \\
&a_i + (b_i + c_i) = (a_i + b_i) + c_i \\
&a_i (b_{jk} c_l) = (a_i b_{jk}) c_l \\
&a_{ij}(b_k + c_k) = a_{ij} b_k + a_{ij} c_k
\end{aligned} \tag{1.2.5}
$$

Note that the simple relations $a_i = b_i$ and $a_{ij} = b_{ij}$ imply that $a_1 = b_1$, $a_2 = b_2$, … and $a_{11} = b_{11}$, $a_{12} = b_{12}$, … However, relations of the form $a_i = b_j$ or $a_{ij} = b_{kl}$ have ambiguous meaning because the distinct indices on each term are not the same, and these types of expressions are to be avoided in this notational scheme. In general, the distinct subscripts on all individual terms in an equation should match.

It is convenient to adopt the convention that if a subscript appears twice in the same term, then *summation* over that subscript from one to three is implied, for example

$$
\begin{aligned}
&a_{ii} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} \\
&a_{ij} b_j = \sum_{j=1}^{3} a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3
\end{aligned} \tag{1.2.6}
$$

It should be apparent that $a_{ii} = a_{jj} = a_{kk} = \dots$, and therefore the *repeated* subscripts or indices are sometimes called *dummy* subscripts. Unspecified indices that are not repeated are called *free* or *distinct* subscripts. The summation convention may be suspended by underlining one of the repeated indices or by writing *no sum*. The use of three or more repeated indices in the same term (e.g., a_{iii} or $a_{iij} b_{ij}$) has ambiguous meaning and is to be avoided. On a given symbol, the process of setting two free indices equal is called *contraction*. For example, a_{ii} is obtained from a_{ij} by contraction on i and j. The operation of outer multiplication of two indexed symbols followed by contraction with respect to one index from each symbol generates an *inner multiplication*; for example, $a_{ij} b_{jk}$ is an inner product obtained from the outer product $a_{ij} b_{mk}$ by contraction on indices j and m.

A symbol $a_{ij\ldots m\ldots n\ldots k}$ is said to be *symmetric* with respect to index pair mn if

$$a_{ij\ldots m\ldots n\ldots k} = a_{ij\ldots n\ldots m\ldots k} \tag{1.2.7}$$

while it is *antisymmetric* or *skewsymmetric* if

$$a_{ij\ldots m\ldots n\ldots k} = -a_{ij\ldots n\ldots m\ldots k} \tag{1.2.8}$$

Note that if $a_{ij\ldots m\ldots n\ldots k}$ is symmetric in mn while $b_{pq\ldots m\ldots n\ldots r}$ is antisymmetric in mn, then the product is zero

$$a_{ij\ldots m\ldots n\ldots k} b_{pq\ldots m\ldots n\ldots r} = 0 \tag{1.2.9}$$

A useful identity may be written as

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}) = a_{(ij)} + a_{[ij]} \quad \textbf{(1.2.10)}$$

The first term $a_{(ij)} = 1/2(a_{ij} + a_{ji})$ is symmetric, while the second term $a_{[ij]} = 1/2(a_{ij} - a_{ji})$ is antisymmetric, and thus an arbitrary symbol a_{ij} can be expressed as the sum of symmetric and antisymmetric pieces. Note that if a_{ij} is symmetric, it has only six independent components. On the other hand, if a_{ij} is antisymmetric, its diagonal terms a_{ii} (no sum on i) must be zero, and it has only three independent components. Since $a_{[ij]}$ has only three independent components, it can be related to a quantity with a single index, for example a_i (see Exercise 1.15).

Example 1.1 Index notation examples

The matrix a_{ij} and vector b_i are specified by

$$a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix}, \quad b_i = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

Determine the following quantities: a_{ii}, $a_{ij}a_{ij}$, $a_{ij}a_{jk}$, $a_{ij}b_j$, $a_{ij}b_ib_j$, b_ib_i, b_ib_j, $a_{(ij)}$, $a_{[ij]}$, and indicate whether they are a scalar, vector, or matrix.

Following the standard definitions given in Section 1.2

$$a_{ii} = a_{11} + a_{22} + a_{33} = 7 \text{ (scalar)}$$

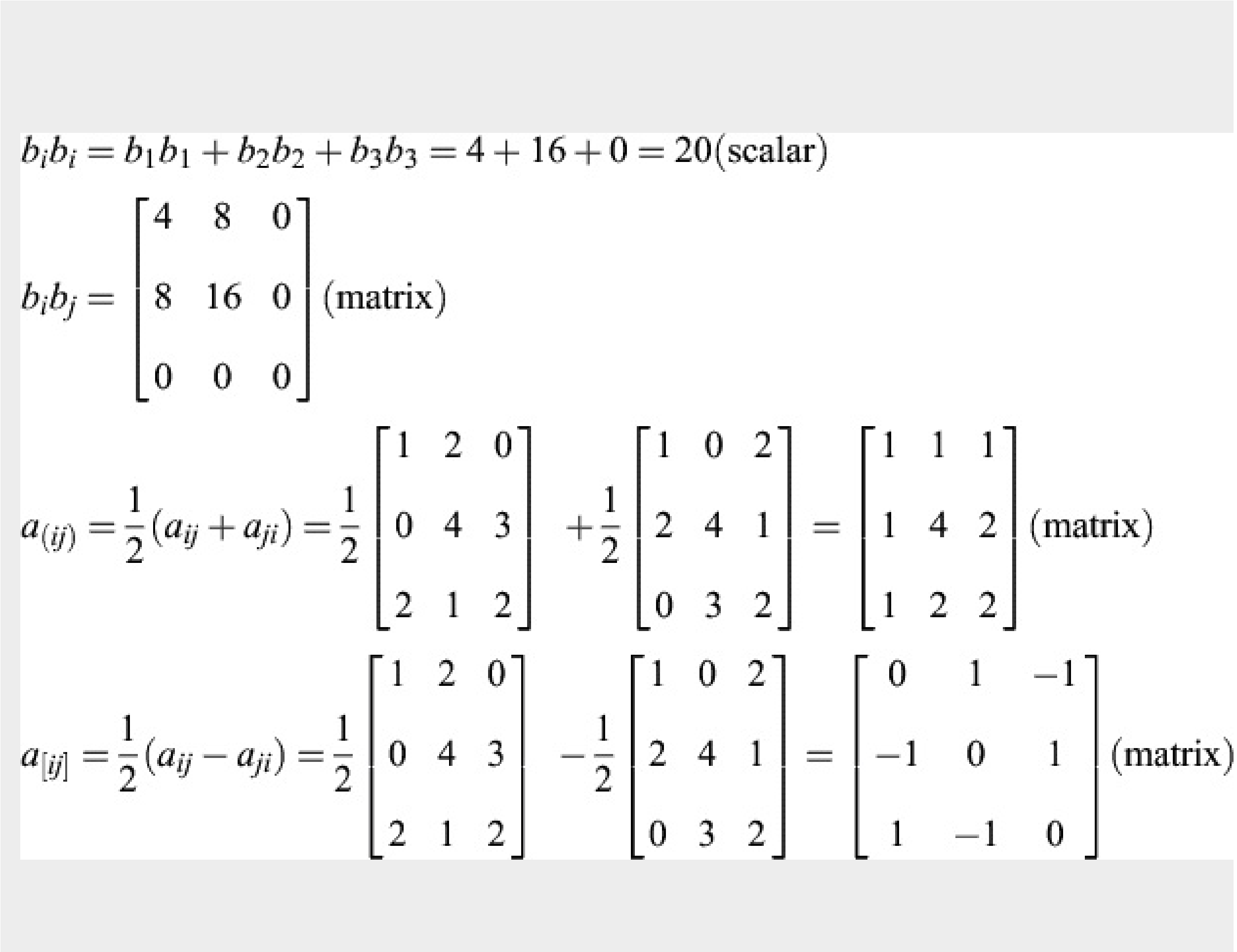
$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$

$$= 1 + 4 + 0 + 0 + 16 + 9 + 4 + 1 + 4 = 39 \text{ (scalar)}$$

$$a_{ij}a_{jk} = a_{i1}a_{1k} + a_{i2}a_{2k} + a_{i3}a_{3k} = \begin{bmatrix} 1 & 10 & 6 \\ 6 & 19 & 18 \\ 6 & 10 & 7 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 10 \\ 16 \\ 8 \end{bmatrix} \text{ (vector)}$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + \ldots = 84 \text{(scalar)}$$



$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 4 + 16 + 0 = 20 (\text{scalar})$$

$$b_ib_j = \begin{bmatrix} 4 & 8 & 0 \\ 8 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\text{matrix})$$

$$a_{(ij)} = \frac{1}{2}(a_{ij} + a_{ji}) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix} (\text{matrix})$$

$$a_{[ij]} = \frac{1}{2}(a_{ij} - a_{ji}) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} (\text{matrix})$$

1.3. Kronecker delta and alternating symbol

A useful special symbol commonly used in index notational schemes is the *Kronecker delta* defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i=j \text{ (no sum)} \\ 0, & \text{if } i \neq j \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.3.1}$$

Within usual matrix theory, it is observed that this symbol is simply the unit matrix. Note that the Kronecker delta is a symmetric symbol. Particularly useful properties of the Kronecker delta include the following

$$\delta_{ij} = \delta_{ji}$$
$$\delta_{ii} = 3,\ \delta_{\underline{ii}} = 1\delta_{ij}a_j = a_i,\ \delta_{ij}a_i = a_j\delta_{ij}a_{jk} = a_{ik},\ \delta_{jk}a_{ik} = a_{ij}\delta_{ij}a_{ij} = a_{ii},\ \delta_{ij}\delta_{ij} = 3 \tag{1.3.2}$$

Another useful special symbol is the *alternating* or *permutation symbol* defined by

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ is an even permutation of } 1,\ 2,\ 3 \\ -1, & \text{if } ijk \text{ is an odd permutation of } 1,\ 2, 3 \\ 0, & \text{otherwise} \end{cases} \quad \textbf{(1.3.3)}$$

Consequently, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$, $\varepsilon_{112} = \varepsilon_{131} = \varepsilon_{222} = \ldots = 0$. Therefore, of the 27 possible terms for the alternating symbol, three are equal to +1, three are equal to −1, and all others are 0. The alternating symbol is antisymmetric with respect to any pair of its indices.

This particular symbol is useful in evaluating determinants and vector cross products, and the determinant of an array a_{ij} can be written in two equivalent forms

$$\det[a_{ij}] = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = \varepsilon_{ijk}a_{i1}a_{j2}a_{k3} \quad \textbf{(1.3.4)}$$

where the first index expression represents the row expansion, while the second form is the column expansion. Using the property

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \tag{1.3.5}$$

another form of the determinant of a matrix can be written as

$$\det[a_{ij}] = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}a_{ip}a_{jq}a_{kr} \tag{1.3.6}$$

1.4. Coordinate transformations

It is convenient and in fact necessary to express elasticity variables and field equations in several different coordinate systems (see Appendix A). This situation requires the development of particular transformation rules for scalar, vector, matrix, and higher-order variables. This concept is fundamentally connected with the basic definitions of tensor variables and their related tensor transformation laws. We restrict our discussion to transformations only between Cartesian coordinate systems, and thus consider the two systems shown in Fig. 1.1. The two Cartesian frames (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) differ only by orientation, and the unit basis vectors for each frame are $\{\boldsymbol{e}_i\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\{\boldsymbol{e}'_i\} = \{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$.

Let Q_{ij} denote the cosine of the angle between the x'_i-axis and the x_j-axis

$$Q_{ij} = \cos(x'_i, x_j) \tag{1.4.1}$$

Using this definition, the basis vectors in the primed coordinate frame can be easily expressed in terms of those in the unprimed frame by the relations

$$\begin{aligned} \boldsymbol{e}_1' &= Q_{11}\boldsymbol{e}_1 + Q_{12}\boldsymbol{e}_2 + Q_{13}\boldsymbol{e}_3 \\ \boldsymbol{e}_2' &= Q_{21}\boldsymbol{e}_1 + Q_{22}\boldsymbol{e}_2 + Q_{23}\boldsymbol{e}_3 \\ \boldsymbol{e}_3' &= Q_{31}\boldsymbol{e}_1 + Q_{32}\boldsymbol{e}_2 + Q_{33}\boldsymbol{e}_3 \end{aligned} \qquad \textbf{(1.4.2)}$$

or in index notation

$$\boldsymbol{e}_i' = Q_{ij}e_j \qquad \textbf{(1.4.3)}$$