Chapter 1

Overview and Background

Delta Functions

1.1. (a) The area of $g(x)$ on $[-1, 2.5]$ is

$$\begin{aligned}\int_{-1}^{2.5} g(x)dx &= 5\int_{-1}^{2.5}\delta(x)dx + 3\int_{-1}^{2.5}\delta(x-2)dx - 2\int_{-1}^{2.5}\delta(x-3)dx \\ &= 5+3-0 = 8.\end{aligned} \tag{1.1}$$

The last delta function is outside the range of integration. (b) More generally for $x \in \mathcal{R}^+$:

$$\begin{aligned}h(x) &= \int_{-\infty}^{x} g(u)du = 5\int_{-\infty}^{x}\delta(u)du + 3\int_{-\infty}^{x}\delta(u-2)du - 2\int_{-\infty}^{x}\delta(u-3)du \\ &= 5u(x) + 3u(x-2) - 2u(x-3) \\ &= \begin{cases} 5, & 0 \le x < 2 \\ 8, & 2 \le x < 3 \\ 6, & x \ge 3, \end{cases}\end{aligned} \tag{1.2}$$

which verifies the result in part (a) when $x = 2.5$.

1.2. (a) From the sifting property of the Dirac delta function, the convolution is

$$\begin{aligned}g(x) * h(x) &= [2\delta(x-1) + 4\delta(x-2) + 8\delta(x-3)] * h(x) \\ &= 2h(x-1) + 4h(x-2) + 8h(x-3).\end{aligned} \tag{1.3}$$

Substituting $h(x) = \exp(-x)u(x)$ gives a sum of weighted and shifted exponentials:

$$g(x) * h(x) = 2\exp(-(x-1))u(x-1) + 4\exp(-(x-2))u(x-2) + 8\exp(-(x-3))u(x-3). \tag{1.4}$$

(b) From the scaling and shifting properties of the Dirac delta function, we have

$$\delta(ax+b) = \delta(a(x+b/a)) = (1/|a|)\delta(x+b/a). \tag{1.5}$$

Thus, from the sifting property:

$$\delta(ax+b) * h(x) = (1/|a|)h(x+b/a) = (1/|a|)\exp(-x-b/a)u(x+b/a). \tag{1.6}$$

Solutions Manual to Accompany Probability, Random Variables, and Random Processes: Theory and Signal Processing Applications.

1.3. (a) Composition with the differentiable function $g(x)$ gives

$$\delta(g(x)) = \sum_n \frac{\delta(x - x_n)}{|g'(x_n)|}, \tag{1.7}$$

where $\{x_n\}$ are the roots of $g(x)$ and $g'(x)$ is its ordinary derivative. Since $g(x) = x^2 - 1 = (x-1)(x+1)$ and $g'(x) = 2x$, we have

$$\delta(x^2 - 1) = (1/2)[\delta(x+1) + \delta(x-1)]. \tag{1.8}$$

(b) The function $u(1/4 - x^2) = 1$ is nonzero (equal to 1) when its argument is nonnegative: $1/4 - x^2 \geq 0$. This corresponds to $-1/2 \leq x \leq 1/2$, which is the support of the rectangle function. Thus

$$u(1/4 - x^2) = \text{rect}(x) = \begin{cases} 1, & |x| \leq 1/2 \\ 0, & \text{else.} \end{cases} \tag{1.9}$$

1.4. (a) The discrete convolution is

$$\begin{aligned} g[k] * h[k] &= (2\delta[k-1] + 4\delta[k-2] + 8\delta[k-3]) * h[k] \\ &= 2h[k-1] + 4h[k-2] + 8h[k-3] \\ &= 2(\delta[k-2] + \delta[k-3]) + 4(\delta[k-3] + \delta[k-4]) + 8(\delta[k-4] + \delta[k-5]) \\ &= 2\delta[k-2] + 6\delta[k-3] + 12\delta[k-4] + 8\delta[k-5]. \end{aligned} \tag{1.10}$$

(b) Similarly:

$$\begin{aligned} g[k] * h[k] &= (\delta[k] + 3\delta[k-1]) * h[k] = h[k] + 3h[k-1] \\ &= (1/3)^k u[k] + (1/3)^{k-2} u[k-1]. \end{aligned} \tag{1.11}$$

1.5. (a) The Dirac delta function with scaled argument can be written in terms of the rectangle function as follows:

$$\delta(\alpha x) = \lim_{a \to 0} (1/|a|)\text{rect}(\alpha x/a) = (1/|\alpha|) \lim_{a \to 0} (|\alpha/a|)\text{rect}(\alpha x/a). \tag{1.12}$$

Since the rectangle function is even, α/a in its argument can be negative. However, we must use the absolute value of α/a when multiplying the rectangle function so that it has positive area. Note that since

$$(|\alpha/a|)\text{rect}(\alpha x/a) = (|\alpha/a|) I_{[-1/2,1/2]}(\alpha x/a) = (|\alpha/a|) I_{[-|a/2\alpha|,|a/2\alpha|]}(x) \tag{1.13}$$

has unit area, we finally have in the limit as $a \to 0$:

$$\delta(\alpha x) = (1/|\alpha|)\delta(x). \tag{1.14}$$

(b) For the derivative of the Dirac delta function, consider a definition based on a smooth function such as the Gaussian distribution:

$$\delta(x) = \lim_{\sigma^2 \to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-x^2/2\sigma^2). \tag{1.15}$$

The derivative of the Gaussian pdf is an odd function:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \frac{d}{dx} \exp(-x^2/2\sigma^2) = -\frac{x}{\sqrt{2\pi}\sigma^3} \exp(-x^2/2\sigma^2), \tag{1.16}$$

such that

$$\int_{-\infty}^{\infty} \delta'(x) dx = 0. \tag{1.17}$$

Note also that

$$\begin{aligned}\int_{-\infty}^{\infty} x\delta'(x)dx &= -\lim_{\sigma^2\to 0}\int_{-\infty}^{\infty}\frac{x^2}{\sqrt{2\pi}\sigma^3}\exp(-x^2/2\sigma^2)dx \\ &= -\lim_{\sigma^2\to 0}(1/\sigma^2)\int_{-\infty}^{\infty}\frac{x^2}{\sqrt{2\pi\sigma^2}}\exp(-x^2/2\sigma^2)dx. \qquad (1.18)\end{aligned}$$

Since the integral on the right-hand side is σ^2 (the pdf has zero mean), we find that the left-hand side is -1.

1.6. (a) From the results of Problem 1.5, we conclude that

$$\delta(\alpha x - n) = \delta(\alpha(x - n/\alpha)) = 1/(|\alpha|)\delta(x - n/\alpha) \qquad (1.19)$$

and

$$s(\alpha x) = \sum_{n=-\infty}^{\infty}\delta(\alpha x - n) = (1/|\alpha|)\sum_{n=-\infty}^{\infty}\delta(x - n/\alpha). \qquad (1.20)$$

(b) The roots of $g(x) = \sin(\pi x)$ are $x_n = n \in \mathcal{Z}$. From Appendix B (or the solution of Problem 1.3):

$$\delta(\sin(\pi x)) = \sum_{n=-\infty}^{\infty}\frac{\delta(x-n)}{\pi|\cos(n\pi)|} = (1/\pi)\sum_{n=-\infty}^{\infty}\delta(x-n), \qquad (1.21)$$

where $|\cos(n\pi)| = 1$ for $n \in \mathcal{Z}$.

Continuous-Time Signals and Systems

1.7. The Laplace transform (with zero initial conditions) yields

$$s^2Y(s) + 2sY(s) + Y(s) = X(s), \qquad (1.22)$$

giving the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+2s+1}. \qquad (1.23)$$

It has two real poles at $s = -1$, and since the system is causal:

$$h(t) = t\exp(-t)u(t). \qquad (1.24)$$

1.8. Using the Laplace transform approach, we have

$$H(s) = \frac{2}{s+1} + \frac{\exp(-5s)}{s+1}. \qquad (1.25)$$

The unit-step response is obtained from the following PFE:

$$\begin{aligned}Y(s) &= \frac{2}{s(s+1)} + \exp(-5s)\frac{1}{s(s+1)} \\ &= \frac{2}{s} - \frac{2}{s+1} + \exp(-5s)\left[\frac{1}{s} - \frac{1}{s+1}\right]. \qquad (1.26)\end{aligned}$$

Thus

$$y(t) = 2u(t) - 2\exp(-t)u(t) + u(t-5) - \exp(-(t-5))u(t-5). \qquad (1.27)$$

Using instead convolution:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} [2\exp(-\tau)u(\tau)u(t-\tau) + \exp(-(\tau-5))u(\tau-5)u(t-\tau)]\,d\tau \\ &= 2\int_0^t \exp(-\tau)d\tau + \int_5^t \exp(-(\tau-5))d\tau, \end{aligned} \tag{1.28}$$

where unit-step functions have been used to give the limits of integration. Note that $t \geq 0$ for the first integral and $t \geq 5$ for the second integral. Thus

$$y(t) = -2\exp(-\tau)|_0^t\, u(t) - \exp(-(\tau-5))|_5^t\, u(t-5), \tag{1.29}$$

which is the same as (1.27).

1.9. For $x_1(t) = \exp(-5|t|)$:

$$\begin{aligned} X_1(s) &= \int_{-\infty}^{0} \exp((5-s)t)dt + \int_0^{\infty} \exp(-(5+s)t)dt \\ &= \int_0^{\infty} \exp(-(5-s)t)dt + \frac{1}{s+5}, \end{aligned} \tag{1.30}$$

where the ROC of the second term is $\text{Re}(s) > -5$. Thus

$$X_1(s) = \frac{1}{5-s} + \frac{1}{s+5} = \frac{10}{25-s^2}, \tag{1.31}$$

which has ROC $-5 < \text{Re}(s) < 5$. For $x_2(t) = \text{erf}(t)u(t)$, it is perhaps easier to first recognize that the error function can be written as the following convolution:

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-\tau^2)d\tau = \frac{2}{\sqrt{\pi}} \exp(-t^2)u(t) * u(t). \tag{1.32}$$

The Laplace transform of $u(t)$ is $1/s$ with ROC $\text{Re}(s) > 0$, and

$$\begin{aligned} \frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp(-t^2)\exp(-st)dt &= \frac{2}{\sqrt{\pi}} \exp(s^2/4) \int_0^{\infty} \exp(-(t+s/2)^2)dt \\ &= \frac{2}{\sqrt{\pi}} \exp(s^2/4) \int_{s/2}^{\infty} \exp(-v^2)dv, \end{aligned} \tag{1.33}$$

where we have completed the square and changed variables as $v \triangleq t + s/2$. The last integral can be written as the complementary error function:

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp(-t^2)\exp(-st)dt = \exp(s^2/4)\text{erfc}(s/2). \tag{1.34}$$

Multiplying this expression by $1/s$ gives the final result:

$$X_2(s) = (1/s)\exp(s^2/4)\text{erfc}(s/2), \tag{1.35}$$

which has ROC $\text{Re}(s) > 0$.

1.10. For $x_1(t) = \text{rect}(t)$:

$$\begin{aligned} X_1(s) &= \int_{-1/2}^{1/2} \exp(-st)dt = -(1/s)\exp(-st)|_{-1/2}^{1/2} \\ &= (1/s)[\exp(s/2) - \exp(-s/2)], \end{aligned} \tag{1.36}$$

with ROC anywhere on the s-plane (since the function has finite support). The Laplace transform for $x_2(t) = 1/t$ exists only for $\text{Re}(s) = 0$, corresponding to the Fourier transform, i.e., the "region" of convergence for $X_2(s)$ is the $j\omega$ axis. Consider first the Fourier transform of the signum function:

$$\int_{-\infty}^{\infty} \text{sgn}(t)\exp(-j\omega t)dt = -j\int_{-\infty}^{\infty} \text{sgn}(t)\sin(\omega t)dt, \tag{1.37}$$

where we have used Euler's formula and the fact that $\text{sgn}(t)$ is an odd function to eliminate the cosine term. Next, we demonstrate that technically the Fourier transform of the signum function does not exist because it is not absolutely integrable:

$$\int_{-\infty}^{\infty} |\text{sgn}(t)|dt \not< \infty. \tag{1.38}$$

Since the product of the signum and sine functions is even, we can write

$$\begin{aligned} -j\int_{-\infty}^{\infty} \text{sgn}(t)\sin(\omega t)dt &= -2j\int_{0}^{\infty} \sin(\omega t)dt \\ &= (2j/\omega)\cos(\omega t)|_0^{\infty} = (2j/\omega)\cos(\omega t)|_{t=\infty} + 2/j\omega, \end{aligned} \tag{1.39}$$

from which we see the first term in the last expression cannot be evaluated. The second term is actually the Fourier transform, which is shown by writing the signum function using exponential functions:

$$\text{sgn}(t) = \lim_{\alpha\to\infty} [\exp(-t/\alpha)u(t) - \exp(t/\alpha)u(-t)]. \tag{1.40}$$

Then

$$\int_0^{\infty} \exp(-t/\alpha)\exp(-j\omega t)dt = -\frac{\exp(-(j\omega + 1/\alpha)t)}{j\omega + 1/\alpha}\bigg|_0^{\infty} = \frac{1}{j\omega + 1/\alpha}, \tag{1.41}$$

and

$$\int_{-\infty}^{0} \exp(t/\alpha)\exp(-j\omega t)dt = \int_0^{\infty} \exp((j\omega - 1/\alpha)t)dt = -\frac{1}{j\omega - 1/\alpha}. \tag{1.42}$$

Subtracting these two expressions and letting $\alpha \to \infty$ gives $2/j\omega$. From the duality property of the Fourier transform, we can write

$$\int_{-\infty}^{\infty} (1/t)\exp(-j\omega t)dt = -j\pi\text{sgn}(\omega). \tag{1.43}$$

As an aside, we prove the frequency-integration property of the one-sided Laplace transform, which involves $1/t$ and is given by

$$x(t)/t \to \int_s^{\infty} X(s)ds. \tag{1.44}$$

Inserting the Laplace transform definition on the right-hand side and rearranging gives

$$\begin{aligned}\int_s^\infty X(s)ds &= \int_0^\infty x(t)\int_s^\infty \exp(-st)dsdt \\ &= \int_0^\infty x(t)(-1/t)\exp(-st)\Big|_s^\infty dt \\ &= \int_0^\infty [x(t)/t]\exp(-st)dt, \end{aligned} \qquad (1.45)$$

which is the Laplace transform of $x(t)/t$.

1.11. For $x_1(t) = t\exp(-3t)u(t-1)$, first note that

$$\frac{d}{ds}\int_{-\infty}^{\infty} \exp(-3t)u(t-1)\exp(-st)dt = -\int_{-\infty}^{\infty} t\exp(-3t)u(t-1)\exp(-st)dt, \qquad (1.46)$$

and from the shift property of the Laplace transform:

$$\begin{aligned}\int_{-\infty}^{\infty} \exp(-3t)u(t-1)\exp(-st)dt &= \exp(-3)\int_{-\infty}^{\infty} \exp(-3(t-1))u(t-1)\exp(-st)dt \\ &= \frac{\exp(-3)\exp(-s)}{s+3}. \end{aligned} \qquad (1.47)$$

Combining these results gives

$$\begin{aligned}X_1(s) &= -\exp(-3)\frac{d}{ds}\frac{\exp(-s)}{s+3} = -\exp(-3)\left[-\frac{\exp(-s)}{s+3} - \frac{\exp(-s)}{(s+3)^2}\right] \\ &= \exp(-(s+3))\frac{s+4}{(s+3)^2}, \end{aligned} \qquad (1.48)$$

with ROC $\text{Re}(s) > -3$. For $x_2(t) = \cos(2\pi t - 5)u(t-2)$, we have from Euler's formula:

$$\begin{aligned}\cos(2\pi t - 5)u(t-2) &= (1/2)[\exp(j(2\pi t - 5)) + \exp(-j(2\pi t - 5))]u(t-2) \\ &= (1/2)[\exp(-j(5-4\pi))\exp(2\pi j(t-2) \\ &\quad + \exp(j(5-4\pi))\exp(-2\pi j(t-2))]u(t-2). \end{aligned} \qquad (1.49)$$

Since

$$\int_{-\infty}^{\infty} \exp(\pm 2\pi j(t-2))u(t-2)\exp(-st)dt = \frac{\exp(-2s)}{s \mp 2\pi j}, \qquad (1.50)$$

the final result is

$$\begin{aligned}X_2(s) &= (1/2)\exp(-2s)\left[\frac{\exp(-j(5-4\pi))}{s-2\pi j} + \frac{\exp(j(5-4\pi))}{s+2\pi j}\right] \\ &= \exp(-2s)[s\cos(5-4\pi) - 2\pi\sin(5-4\pi)]/(s+4\pi^2), \end{aligned} \qquad (1.51)$$

with ROC $\text{Re}(s) > 0$.

1.12. The poles of the transfer function are $s = -1, -2$. A PFE yields:

$$H(s) = \frac{[2s/(s+1)]_{s=-2}}{s+2} + \frac{[2s/(s+2)]_{s=-1}}{s+1} = \frac{4}{s+2} - \frac{2}{s+1}. \qquad (1.52)$$

Since the ROC is $\text{Re}(s) > -1$ and contains the imaginary axis, $h(t)$ is a stable right-sided sequence:

$$h(t) = [4\exp(-2t) - 2\exp(-t)]u(t). \tag{1.53}$$

1.13. The poles of the transfer function are $s = 2, -1$. A PFE yields:

$$H(s) = \frac{[2s/(s+1)]_{s=2}}{s-2} + \frac{[2s/(s-2)]_{s=-1}}{s+1} = \frac{4/3}{s-2} + \frac{2/3}{s+1}. \tag{1.54}$$

Since the ROC is $-1 < \text{Re}(s) < 2$ and contains the imaginary axis, $h(t)$ is a stable two-sided sequence:

$$h(t) = (2/3)\exp(-t)u(t) - (4/3)\exp(2t)u(-t). \tag{1.55}$$

1.14. From the definition of convolution:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} [\exp(-2\tau)u(\tau) + \exp(-\tau+1)u(\tau-1)]u(t-\tau-2)d\tau \\ &= \int_{0}^{t-2} \exp(-2\tau)d\tau + \exp(1)\int_{1}^{t-2} \exp(-\tau)d\tau \\ &= (-1/2)\exp(-2\tau)|_0^{t-2} - \exp(1)\exp(-\tau)|_1^{t-2}. \end{aligned} \tag{1.56}$$

The functions in the first integral do not overlap until $t > 2$, and those in the second integral until $t > 3$. Thus

$$\begin{aligned} y(t) &= (1/2)[1-\exp(-2(t-2))]u(t-2) + \exp(1)[\exp(-1) - \exp(-(t-2))]u(t-3) \\ &= (1/2)[1-\exp(-2(t-2))]u(t-2) + [1-\exp(-(t-3))]u(t-3). \end{aligned} \tag{1.57}$$

1.15. Taking the Laplace transform of the DE yields the following transfer function:

$$sY(s) + 4Y(s) = X(s) \implies H(s) = Y(s)/X(s) = \frac{1}{s+4}. \tag{1.58}$$

The Laplace transform of the input is

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \exp(-t)u(t-1)\exp(-st)dt \\ &= \exp(-1)\int_{-\infty}^{\infty} \exp(-(t-1))u(t-1)\exp(-st)dt \\ &= \frac{\exp(-1)\exp(-s)}{s+1} = \frac{\exp(-(s+1))}{s+1}. \end{aligned} \tag{1.59}$$

Thus

$$\begin{aligned} Y(s) &= H(s)X(s) = \frac{\exp(-(s+1))}{(s+4)(s+1)} \\ &= (1/3)\exp(-(s+1))\left[\frac{1}{s+1} - \frac{1}{s+4}\right], \end{aligned} \tag{1.60}$$

and

$$\begin{aligned} y(t) &= (1/3)\exp(-1)[\exp(-(t-1)) + \exp(-4(t-1))]u(t-1) \\ &= (1/3)[\exp(-t) + \exp(-4t+3)]u(t-1). \end{aligned} \tag{1.61}$$

1.16. (a) The Fourier transform of the integral is

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t+\tau)d\tau \exp(-j\omega t)dt \\ &= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t+\tau)\exp(-j\omega t)dtd\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\exp(j\omega\tau)H(\omega)d\tau, \end{aligned} \tag{1.62}$$

where a change variables gives the last expression. Thus

$$Y(\omega) = H(\omega)\int_{-\infty}^{\infty} x(\tau)\exp(j\omega\tau)d\tau = H(\omega)X(-\omega). \tag{1.63}$$

(b) For the functions in Problem 1.14:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} [\exp(-2\tau)u(\tau) + \exp(-\tau+1)u(\tau-1)]\, u(t+\tau-2)d\tau \\ &= \int_{\max(0,2-t)}^{\infty} \exp(-2\tau)d\tau + \exp(1)\int_{\max(1,2-t)}^{\infty} \exp(-\tau)d\tau. \end{aligned} \tag{1.64}$$

For $t \geq 2$:

$$\begin{aligned} y(t) &= \int_{0}^{\infty} \exp(-2\tau)d\tau + \exp(1)\int_{1}^{\infty} \exp(-\tau)d\tau \\ &= 1/2 + 1 = 3/2. \end{aligned} \tag{1.65}$$

For $1 \leq t < 2$:

$$y(t) = \int_{2-t}^{\infty} \exp(-2\tau)d\tau + \exp(1)\int_{1}^{\infty} \exp(-\tau)d\tau = (1/2)\exp(2t-4) + 1. \tag{1.66}$$

For $t < 1$:

$$y(t) = \int_{2-t}^{\infty} \exp(-2\tau)d\tau + \exp(1)\int_{2-t}^{\infty} \exp(-\tau)d\tau = (1/2)\exp(2t-4) + \exp(t-1). \tag{1.67}$$

1.17. Substituting expressions for the inverse Fourier transform yields

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)\exp(2\pi ft)df \int_{-\infty}^{\infty} X^*(\alpha)\exp(-2\pi\alpha t)d\alpha dt, \tag{1.68}$$

where we have used $x(t)x^*(t)$ to obtain the right-hand side (since $x(t)$ is real-valued). Rearranging the integrals:

$$\begin{aligned} \int_{-\infty}^{\infty} x^2(t)dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)X^*(\alpha)\int_{-\infty}^{\infty} \exp(-2\pi(\alpha-f)t)dtd\alpha df \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)X^*(\alpha)\delta(\alpha-f)d\alpha df, \end{aligned} \tag{1.69}$$

we recognize that the third integral is the Fourier transform (with frequency $\alpha - f$) of a constant, yielding the Dirac delta function. Finally, from the sifting property of the Dirac delta function:

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} X(f)X^*(f)df = \int_{-\infty}^{\infty} |X(f)|^2 df. \tag{1.70}$$

Note that when $x(t)$ is complex-valued, the left-hand side of Parseval's theorem is $\int_{-\infty}^{\infty} |x(t)|^2 dt$.

1.18. For $x_1(t)$:

$$\begin{aligned} X_1(f) &= 5\int_{-1/10}^{1/10} \exp(-j2\pi ft)dt \\ &= \frac{5}{j2\pi f}[\exp(j2\pi f/10) - \exp(-j2\pi f/10)] \\ &= \frac{1}{\pi f/5}\sin(\pi f/5) = \text{sinc}(f/5). \end{aligned} \tag{1.71}$$

(b) For $x_2(t)$, consider each function separately:

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(-|t|)\exp(-j2\pi ft)dt &= \int_{-\infty}^{0} \exp(t - j2\pi ft)dt + \int_{0}^{\infty} \exp(-t - j2\pi ft)dt \\ &= \frac{1}{1-j2\pi f} + \frac{1}{1+j2\pi f} = \frac{2}{1+4\pi^2 f^2}, \end{aligned} \tag{1.72}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \cos(4\pi t) &= (1/2)\int_{-\infty}^{\infty} [\exp(j4\pi t) + \exp(-j4\pi t)]\exp(-j2\pi ft)dt \\ &= (1/2)[\delta(f-2) + \delta(f+2)]. \end{aligned} \tag{1.73}$$

From the product property of the Fourier transform:

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(-|t|)\cos(4\pi t)\exp(-j2\pi ft)dt &= \frac{2}{1+4\pi^2 f^2} * (1/2)[\delta(f-2) + \delta(f+2)] \\ &= \left[\frac{1}{1+4\pi^2(f-2)^2} + \frac{1}{1+4\pi^2(f+2)^2}\right]. \end{aligned} \tag{1.74}$$

1.19. Observe that $y(t)$ corresponds to the following convolution:

$$y(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau = x(t) * (1/\pi t). \tag{1.75}$$

From Problem 1.10, we found that although the Laplace transform of $1/t$ does not exist, its Fourier transform is $-j\pi\text{sgn}(\omega)$. Using this result, the convolution property of the Fourier transform yields

$$Y(\omega) = (1/\pi)X(\omega)[-j\pi\text{sgn}(\omega)] = -j\text{sgn}(\omega)X(\omega). \tag{1.76}$$

The output $y(t)$ is the Hilbert transform of $x(t)$, and $-j\text{sgn}(\omega)$ is the frequency response of the filter $1/\pi t$.

Discrete-Time Signals and Systems

1.20. A z-transform of the expression gives

$$Y(z) + 2z^{-1}Y(z) + 4z^{-2}Y(z) = X(z) + 2z^{-1}X(z), \tag{1.77}$$

which has transfer function:

$$H(z) = Y(z)/X(z) = \frac{1+2z^{-1}}{1+2z^{-1}+4z^{-2}}. \tag{1.78}$$

Since $y[k]$ has been written as a difference equation, the impulse response of the filter is assumed to be right-sided. From the table of z-transform pairs in Appendix C, we have

$$a^k \cos(\omega_o k)u[k] \quad \rightarrow \quad \frac{1-a\cos(\omega_o)z^{-1}}{1-2a\cos(\omega_o)z^{-1}+a^2z^{-2}}, \tag{1.79}$$

$$a^k \sin(\omega_o k)u[k] \quad \rightarrow \quad \frac{a\sin(\omega_o)z^{-1}}{1-2a\cos(\omega_o)z^{-1}+a^2z^{-2}}. \tag{1.80}$$

In order to obtain these forms, the transfer function can rewritten as

$$H(z) = \frac{1+z^{-1}}{1+2z^{-1}+4z^{-2}} + \frac{z^{-1}}{1+2z^{-1}+4z^{-2}}. \tag{1.81}$$

Let $a=-2$ and $\cos(\omega_o) = 1/2$ such that $\omega_o = \pi/3$ and

$$\frac{1+z^{-1}}{1+2z^{-1}+4z^{-2}} \rightarrow (-2)^k\cos(k\pi/3)u[k]. \tag{1.82}$$

Since $\sin(\pi/3) = \sqrt{3}/2$, the second term of $H(z)$ with $a=-2$ needs to be rewritten as follows:

$$\frac{z^{-1}}{1+2z^{-1}+4z^{-2}} = -(1/\sqrt{3})\frac{-\sqrt{3}z^{-1}}{1+2z^{-1}+4z^{-2}} \rightarrow -(1/\sqrt{3})(-2)^k\sin(k\pi/3)u[k]. \tag{1.83}$$

Combining these results yields the following causal impulse response:

$$h[k] = (-2)^k\cos(k\pi/3)u[k] - (1/\sqrt{3})(-2)^k\sin(k\pi/3)u[k], \tag{1.84}$$

which has ROC $|z| > |2|$, meaning that it is an unstable system.

1.21. The unit-step response is obtained from the following convolution:

$$\begin{aligned} y[k] &= \sum_{n=-\infty}^{\infty} \left((1/3)^n u[n] - 2u[n-1]\right) u[k-n] \\ &= \sum_{n=0}^{k}(1/3)^n - \sum_{n=1}^{k} 2 = (3/2)[1-(1/3)^{k+1}]u[k] - 2ku[k-1], \end{aligned} \tag{1.85}$$

where a closed form for a finite sum in Appendix E has been used. Note that $u[k]$ can replace $u[k-1]$ in the second term because it is zero for $k=0$. In the z-domain with ROC $|z| > 1/3$:

$$h[k] = (1/3)^k u[k] - 2u[k-1] \rightarrow H(z) = \frac{1}{1-(1/3)z^{-1}} - \frac{2z^{-1}}{1-z^{-1}}, \tag{1.86}$$

so that

$$Y[z] = H(z)U(z) = \frac{1}{(1-(1/3)z^{-1})(1-z^{-1})} - \frac{2z^{-1}}{(1-z^{-1})^2}, \tag{1.87}$$

where the z-transform $U(z) = 1/(1-z^{-1})$ of the unit-step function has been inserted. A PFE of the first term gives:

$$\frac{1}{(1-(1/3)z^{-1})(1-z^{-1})} = \frac{-1/2}{1-(1/3)z^{-1}} + \frac{3/2}{1-z^{-1}}, \tag{1.88}$$

and from the table of z-transform pairs in Appendix C:

$$\frac{-1/2}{1-(1/3)z^{-1}} \rightarrow -(1/2)(1/3)^k u[k] \tag{1.89}$$

$$\frac{3/2}{1-z^{-1}} \rightarrow (3/2)u[k] \tag{1.90}$$

$$\frac{-2z^{-1}}{(1-z^{-1})^2} \rightarrow -2ku[k]. \tag{1.91}$$

These combine to give $y[k]$ above.

1.22. For $x_1[k]$, we change variables in order to use a closed-form result in Appendix E:

$$\begin{aligned} X_1(z) &= \sum_{k=-\infty}^{\infty} (k-1)^2 z^{-k} u[k-1] = z^{-1} \sum_{k=0}^{\infty} k^2 z^{-k} \\ &= z^{-1} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} = \frac{z^{-2}(1+z^{-1})}{(1-z^{-1})^3}, \end{aligned} \tag{1.92}$$

which has ROC $|z| > 1$. Likewise for $x_2[k]$:

$$\begin{aligned} X_2(z) &= \sum_{k=-\infty}^{\infty} ((1/2)^k u[k] + 4^k u[k+1]) z^{-k} \\ &= \sum_{k=0}^{\infty} (z^{-1}/2)^k + \sum_{k=-1}^{\infty} (4z^{-1})^k \\ &= \frac{1}{1-(1/2)z^{-1}} + \frac{z/4}{1-4z^{-1}}, \end{aligned} \tag{1.93}$$

which has ROC $|z| > 4$.

1.23. For $x_1[k]$:

$$\begin{aligned} X_1(z) &= \sum_{k=-\infty}^{\infty} 3^k u[-k+1] z^{-k} = \sum_{k=-\infty}^{1} (3z^{-1})^k = \sum_{k=1}^{\infty} (z/3)^k \\ &= \sum_{k=0}^{\infty} (z/3)^k - 1 = \frac{1}{1-z/3} - 1 = -\frac{z}{z-3}, \end{aligned} \tag{1.94}$$

which has ROC $|z| < 3$. Likewise for $x_2[k]$:

$$\begin{aligned} X_2(z) &= \sum_{k=-\infty}^{\infty} 2(u[k] - u[k-5]) z^{-k} \\ &= 2\sum_{k=0}^{4} z^{-k} = 2\frac{1-z^{-5}}{1-z^{-1}}. \end{aligned} \tag{1.95}$$

Note that although the ROC is the entire z-plane, the last expression holds only for $z \neq 1$.

1.24. The sinc and rectangle functions form a DTFT pair:

$$\begin{aligned} x[k] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \text{rect}(\omega/2\pi)\exp(j\omega k)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(j\omega k)d\omega = \frac{1}{2\pi}\left.\frac{\exp(j\omega k)}{jk}\right|_{-\pi}^{\pi} \\ &= (1/j2\pi k)[\exp(j\pi k) - \exp(-j\pi k)] = \text{sinc}[k]. \end{aligned} \tag{1.96}$$

Recall that π is implicit in the sinc function. Using the shift property of the DTFT for $x_1[k] = \text{sinc}(k-1)$ gives

$$X_1(j\omega) = \exp(-j\omega)\text{rect}(\omega/2\pi). \tag{1.97}$$

For $x_2[k]$:

$$\begin{aligned} X_2(j\omega) &= \sum_{k=-\infty}^{\infty} k(1/2)^{k-1}\exp(-j\omega k)u[k] \\ &= 2\sum_{k=0}^{\infty} k[\exp(-j\omega)/2]^k. \end{aligned} \tag{1.98}$$

A closed-form expression in Appendix E for the sum gives

$$X_2(j\omega) = \frac{\exp(-j\omega)}{[1-\exp(-j\omega)/2]^2}. \tag{1.99}$$

1.25. (a) In order to view the sample mean as a filter, we rewrite it as follows:

$$\bar{x}[k] = (1/k)\sum_{n=0}^{k} x[k-n]. \tag{1.100}$$

This corresponds to an FIR filter, and at time instant $k=2$, it has two complex zeros obtained as the roots of $1+z^{-1}+z^{-2}$. They are given by $z = -1/2 \pm j\sqrt{3}/2$, which lie on the unit circle. (b) Separating the $n=k$ term in the original sum (or the sum above) gives

$$\bar{x}[k] = \frac{k-1}{k}\frac{1}{k-1}\sum_{n=0}^{k-1} x[n] + x[k]/k = \frac{k-1}{k}\bar{x}[k-1] + x[k]/k, \tag{1.101}$$

which is a recursive (IIR) filter with time-varying coefficients. An implementation using the direct-form II filter is shown in Figure 1.1.

1.26. The impulse response $h[k]$ has a finite duration of N samples with support $k = \{0,\ldots,N-1\}$. The discrete-time convolution is

$$\begin{aligned} y[k] &= \sum_{n=-\infty}^{\infty} a^{n-1}u[n-1](u[k-n] - u[k-n-N]) \\ &= \sum_{n=1}^{k} a^{n-1} - \sum_{n=1}^{k-N} a^{n-1}, \end{aligned} \tag{1.102}$$

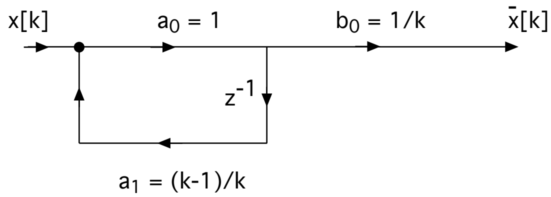

Figure 1.1: Recursive implementation of the sample mean using the direct-form II filter.

which is zero for $k \leq 0$. For $1 \leq k \leq N$, the second sum is ignored such that

$$y[k] = \sum_{n=1}^{k} a^{n-1} = \frac{1-a^k}{1-a}, \tag{1.103}$$

where $a \neq 1$ is assumed in the last expression. For $k > N$, both terms of (1.102) are included:

$$y[k] = \frac{1-a^k}{1-a} - \frac{1-a^{k-N}}{1-a} = \frac{a^{k-N}-a^k}{1-a}. \tag{1.104}$$

The second term is subtracted because the rectangular function has finite duration N. Using z-transforms:

$$x[k] \quad \rightarrow \quad X(z) = \frac{z^{-1}}{1-az^{-1}} \tag{1.105}$$

$$h[k] \quad \rightarrow \quad H(z) = \sum_{n=0}^{N-1} z^{-n}. \tag{1.106}$$

Thus

$$Y(z) = H(z)X(z) = \frac{z^{-1}}{1-az^{-1}} \sum_{n=0}^{N-1} z^{-n}, \tag{1.107}$$

corresponding to a right-sided sequence. It has inverse z-transform

$$y[k] = \sum_{n=0}^{N-1} a^{k-n-1} u[k-n-1]. \tag{1.108}$$

Note that with increasing k, the number of nonzero terms in the sum also increases, up to a maximum of N (the width of the rectangular function) when $k > N$. It is left as an exercise to show that the time- and z-domain results are the same.

1.27. From the definition of the cross-correlation function:

$$\begin{aligned} y[k] &= \sum_{n=-\infty}^{\infty} a^{n-1} u[n-1](u[k+n] - u[k+n-N]) \\ &= \sum_{n=\max(1,-k)}^{\infty} a^{n-1} - \sum_{n=\max(1,N-k)}^{\infty} a^{n-1}, \end{aligned} \tag{1.109}$$

which is zero for $k \geq N-1$. For $-1 < k < N-1$:

$$\begin{aligned} y[k] &= \sum_{n=1}^{\infty} a^{n-1} - \sum_{n=N-k}^{\infty} a^{n-1} = \sum_{n=1}^{N-k-1} a^{n-1} \\ &= \sum_{m=0}^{N-k-2} a^{m} = \frac{1-a^{N-k-1}}{1-a}, \end{aligned} \tag{1.110}$$

where $a \neq 1$ is assumed in the last expression. For $k \leq -1$:

$$\begin{aligned} y[k] &= \sum_{n=-k}^{\infty} a^{n-1} - \sum_{n=N-k}^{\infty} a^{n-1} = \sum_{n=-k}^{N-k-1} a^{n-1} \\ &= \sum_{n=1}^{N-k-1} a^{n-1} - \sum_{n=1}^{-k-1} a^{n-1} = \sum_{m=0}^{N-k-2} a^{m} - \sum_{m=0}^{-k-2} a^{m} \\ &= \frac{1-a^{N-k-1}}{1-a} - \frac{1-a^{-k-1}}{1-a} = \frac{a^{-k-1}-a^{N-k-1}}{1-a}. \end{aligned} \tag{1.111}$$

Using z-transforms:

$$x[k] \quad \rightarrow \quad X(z) = \frac{z^{-1}}{1-az^{-1}} \tag{1.112}$$

$$h[k] \quad \rightarrow \quad H(z) = \sum_{n=0}^{N-1} z^{-n}. \tag{1.113}$$

Thus

$$Y(z) = H(z)X(z^{-1}) = \frac{z}{1-az} \sum_{n=0}^{N-1} z^{-n} = \frac{1}{1-az} \sum_{m=-1}^{N-2} z^{-m}, \tag{1.114}$$

corresponding to a left-sided sequence. The inverse z-transform of $1/(1-az)$ is $a^{-k}u[-k]$, and so the delays in the sum yield the following result:

$$y[k] = \sum_{m=-1}^{N-2} a^{-k+m} u[-k+m]. \tag{1.115}$$

Note that with decreasing k, the number of nonzero terms in the sum increases, up to a maximum of N (the width of the rectangular function) when $k \leq -1$. It is left as an exercise to show that the time- and z-domain results are the same.

1.28. From the definition of the z-transform, differentiating gives

$$\begin{aligned} \frac{dX(z)}{dz} &= \sum_{k=-\infty}^{\infty} x[k] \frac{dz^{-k}}{dz} = -\sum_{k=-\infty}^{\infty} kx[k]z^{-k-1} \\ &= -z^{-1} \sum_{k=-\infty}^{\infty} kx[k]z^{-k}. \end{aligned} \tag{1.116}$$

Rearranging this expression completes the proof. This can also be shown as follows:

$$\begin{aligned}\sum_{k=-\infty}^{\infty} kx[k]z^{-k} &= -z\sum_{k=-\infty}^{\infty} x[k]\frac{dz^{-k}}{dz} \\ &= -z\frac{d}{dz}\sum_{k-\infty}^{\infty} x[k]z^{-k} = -z\frac{d}{dz}X(z). \qquad (1.117)\end{aligned}$$

1.29. Since the ROC is $|z| > 1/2$, both poles are inside the contour C, so that the sequence for each pole is right-sided. Using Cauchy's residue theorem, we evaluate the following expression for $k \geq 1$:

$$\begin{aligned}\sum_{n=1}^{2}\text{Res}[H(z)z^{k-1}, p_n] &= \text{Res}\left[\frac{2(z+1)z^{k-1}}{(z+1/2)(z-1/3)}, -1/2\right] \\ &\quad + \text{Res}\left[\frac{2(z+1)z^{k-1}}{(z+1/2)(z-1/3)}, 1/3\right] \\ &= \left.\frac{2(z+1)z^{k-1}}{z-1/3}\right|_{z=-1/2} + \left.\frac{2(z+1)z^{k-1}}{z+1/2}\right|_{z=1/3}. \qquad (1.118)\end{aligned}$$

For $k < 1$, Cauchy's residue theorem gives zero. Thus

$$h[k] = [(-6/5)(-1/2)^{k-1} + (16/5)(1/3)^{k-1}]u[k-1]. \qquad (1.119)$$

The result is verified using a PFE:

$$\begin{aligned}H(z) &= \frac{2(z+1)}{(z+1/2)(z-1/3)} = \frac{-6/5}{z+1/2} + \frac{16/5}{z-1/3} \\ &= (-6/5)\frac{z^{-1}}{1+1/2z^{-1}} + (16/5)\frac{z^{-1}}{1-1/3z^{-1}}, \qquad (1.120)\end{aligned}$$

which has been written in this form because the sequence is right-sided. The inverse z-transform of each part gives the result in (1.119)

1.30. Since the ROC is $1/3 < |z| < 1/2$, the sequence is two-sided and unstable because of the pole at $z = 1/2$. For $k \geq 1$, only the pole at $z = 1/3$ is inside the contour:

$$\text{Res}\left[\frac{2(z+1)z^{k-1}}{(z+1/2)(z-1/3)}, 1/3\right] = \left.\frac{2(z+1)z^{k-1}}{z+1/2}\right|_{z=1/3} = (16/5)(1/3)^{k-1}u[k-1]. \qquad (1.121)$$

For $k < 1$, it is easier to evaluate the residue at the pole outside the contour:

$$\begin{aligned}-\text{Res}\left[\frac{2(z+1)z^{k-1}}{(z+1/2)(z-1/3)}, -1/2\right] &= -\left.\frac{2(z+1)z^{k-1}}{z-1/3}\right|_{z=-1/2} \\ &= (6/5)(-1/2)^{k-1}u[-k]. \qquad (1.122)\end{aligned}$$

Note that for $k < 1$:

$$\text{Res}[H(z)z^{k-1}, \infty] = -\lim_{z\to\infty} zH(z)z^{k-1} = 0. \qquad (1.123)$$

Thus

$$h[k] = (16/5)(1/3)^{k-1}u[k-1] + (6/5)(-1/2)^{k-1}u[-k]. \qquad (1.124)$$

This result is verified using a PFE:

$$\begin{aligned} H(z) &= \frac{2(z+1)}{(z+1/2)(z-1/3)} = \frac{-6/5}{z+1/2} + \frac{16/5}{z-1/3} \\ &= (-6/5)\frac{1}{z+1/2} + (16/5)\frac{z^{-1}}{1-1/3z^{-1}}, \end{aligned} \tag{1.125}$$

which has been written in this form because the sequence is two-sided. The inverse z-transform of each part gives the same result as above.

1.31. For a left-side sequence, the ROC is $|z| < 1/2$. Using Cauchy's residue theorem, we evaluate the following expression for $k \leq 0$:

$$\begin{aligned} -\sum_{n=1}^{2} \text{Res}[H(z)z^{k-1}, q_n] &= \text{Res}\left[\frac{3z^{k-1}}{(z+1/2)(z-3/2)}, -1/2\right] \\ &\quad + \text{Res}\left[\frac{3z^{k-1}}{(z+1/2)(z-3/2)}, 3/2\right] \\ &= \left.\frac{3z^{k-1}}{z-3/2}\right|_{z=-1/2} + \left.\frac{3z^{k-1}}{z+1/2}\right|_{z=3/2}. \end{aligned} \tag{1.126}$$

Note that for $k \leq 0$:

$$\text{Res}[H(z)z^{k-1}, \infty] = -\lim_{z\to\infty} zH(z)z^{k-1} = 0. \tag{1.127}$$

Cauchy's residue theorem gives zero for $k > 0$, so that the final expression is

$$h[k] = (3/2)[(3/2)^{k-1} - (-1/2)^{k-1}]u[-k] = [(3/2)^k - 3(-1/2)^k]u[-k]. \tag{1.128}$$

A PFE gives

$$\begin{aligned} H(z) &= \frac{-3}{(z+1/2)(z-3/2)} = \frac{3/2}{z+1/2} + \frac{-3/2}{z-3/2} \\ &= -3\frac{1/2}{z+1/2} - \frac{3/2}{z-3/2}, \end{aligned} \tag{1.129}$$

which has the inverse z-transform given above.

1.32. The DTFT is obtained from the z-transform by substituting $z = \exp(j\omega)$:

$$H(j\omega) = \frac{5\exp(j\omega)}{(\exp(j\omega)-1/4)(\exp(j\omega)+1/5)}. \tag{1.130}$$

The magnitude and phase can be obtained by writing each component in the denominator in polar form:

$$\begin{aligned} \exp(j\omega) - 1/4 &= \sqrt{(\cos(\omega)-1/4)^2 + \sin^2(\omega)}\,\exp(j\tan^{-1}(\sin(\omega)/(\cos(\omega)-1/4))) \\ &= \sqrt{17/16 - (1/2)\cos(\omega)}\,\exp(j\tan^{-1}(\sin(\omega)/(\cos(\omega)-1/4))) \end{aligned} \tag{1.131}$$

$$\begin{aligned} \exp(j\omega) + 1/5 &= \sqrt{(\cos(\omega)+1/5)^2 + \sin^2(\omega)}\,\exp(j\tan^{-1}(\sin(\omega)/(\cos(\omega)+1/5))) \\ &= \sqrt{26/25 + (2/5)\cos(\omega)}\,\exp(j\tan^{-1}(\sin(\omega)/(\cos(\omega)+1/5))). \end{aligned} \tag{1.132}$$

Thus

$$|H(j\omega)| = \frac{5}{[17/16 - (1/2)\cos(\omega)][26/25 + (2/5)\cos(\omega)]}, \tag{1.133}$$

and

$$\begin{aligned}\arg(H(j\omega)) &= \omega - \tan^{-1}(\sin(\omega)/(\cos(\omega) - 1/4)) \\ &- \tan^{-1}(\sin(\omega)/(\cos(\omega) + 1/5)).\end{aligned} \tag{1.134}$$

1.33. The DTFT is

$$\begin{aligned}X(j\omega) &= \sum_{k=0}^{N-1} \exp(j\omega_o k)\exp(-j\omega k) = \sum_{k=0}^{N-1} \exp(-j(\omega - \omega_o)k) \\ &= \frac{1 - \exp(-j(\omega - \omega_o)N)}{1 - \exp(-j(\omega - \omega_o))},\end{aligned} \tag{1.135}$$

which we see is the DTFT of a delayed rectangle function, but with shifted frequency $\omega - \omega_o$. Thus, by factoring exponentials, we can write

$$X(j\omega) = \frac{\exp(-j(\omega - \omega_o)N/2)\sin((\omega - \omega_o)N/2)}{\exp(-j(\omega - \omega_o)/2)\sin((\omega - \omega_o)/2)}, \tag{1.136}$$

which has magnitude

$$|X(j\omega)| = \frac{\sin((\omega - \omega_o)N/2)}{\sin((\omega - \omega_o)/2)} \tag{1.137}$$

and linear phase

$$\arg[X(j\omega)] = -(\omega - \omega_o)(N-1)/2. \tag{1.138}$$

Sampling of Continuous-Time Signals

1.34. (a) The multiplication property of the Fourier transform yields the following convolution:

$$X_s(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(v)S(\omega - v)dv. \tag{1.139}$$

From Appendix C:

$$S(\omega) = \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n/T_s), \tag{1.140}$$

which has used the sifting property of the Dirac delta function. Thus

$$\begin{aligned}X_s(\omega) &= \frac{1}{T_s}\int_{-\infty}^{\infty} X(v)\sum_{n=-\infty}^{\infty} \delta(\omega - v - 2\pi n/T_s) \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - 2\pi n/T_s),\end{aligned} \tag{1.141}$$

which we see is the original spectrum repeated at integer multiples of $2\pi/T_s$ (and scaled by $1/T_s$). For completeness, we demonstrate that the Fourier transform of the impulse train is also an impulse train. Since $s(t)$ is periodic, it can be expressed as a Fourier series:

$$s(t) = \sum_{n=-\infty}^{\infty} s_n \exp(j2\pi nt/T_s), \tag{1.142}$$

where s_n is the following nth Fourier series coefficient:

$$s_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} s(t)\exp(-j2\pi nt/T_s)dt = 1/T_s, \tag{1.143}$$

which has used the sifting property of the Dirac delta function. Thus

$$s(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s}\exp(j2\pi nt/T_s), \tag{1.144}$$

and the shift property of the Fourier transform leads to the result in (1.140). (b) The DTFT of the sampled waveform is

$$X(j\omega) = \sum_{k=0}^{\infty} \exp(-\alpha kT_s)\exp(-j\omega kT_s) = \frac{1}{1-\exp(-(\alpha+j\omega)T_s)}, \tag{1.145}$$

where $T_s \neq 1$ must be included in the definition of the DTFT. The Fourier transform of $x(t)$ is

$$X(\omega) = \int_0^{\infty} \exp(-\alpha t)\exp(-j\omega t)dt = \frac{1}{j\omega+\alpha}, \tag{1.146}$$

which gives

$$X_s(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} \frac{1}{j(\omega - 2\pi n/T_s)+\alpha}. \tag{1.147}$$

This expression is equivalent to $X(j\omega)$. However, since $X(\omega)$ is not bandlimited, the overlapping components in the sum for $X_s(\omega)$ cause aliasing, and combine to give the closed-form result in (1.145).

1.35. For the rectangular pulse train:

$$p(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}(t/T_p), \tag{1.148}$$

the mth Fourier series coefficient is

$$\begin{aligned} p_m &= -\frac{1}{T_s}\frac{1}{j2\pi m/T_s}\exp(-j2\pi mt/T_s)\Big|_{-T_p}^{T_p} \\ &= \frac{1}{j2\pi m}[\exp(j\pi mT_p/T_s) - \exp(-j\pi mT_p/T_s)] \\ &= \frac{1}{\pi m}\sin(\pi mT_p/T_s) = (T_s/T_p)\mathrm{sinc}(mT_p/T_s). \end{aligned} \tag{1.149}$$

Note that the corresponding Fourier series coefficient for ideal impulse sampling is $1/T_s$. The sampled waveform is

$$x_p(t) = x(t)p(t) = \sum_{m=-\infty}^{\infty} (T_p/T_s)\mathrm{sinc}(mT_p/T_s)x(t)\exp(j2\pi mt/T_s), \tag{1.150}$$

which has Fourier transform

$$X_p(\omega) = \sum_{m=-\infty}^{\infty} (T_p/T_s)\mathrm{sinc}(mT_p/T_s)X(\omega - 2\pi m/T_s). \tag{1.151}$$

Although the sinc weighting changes across the spectrum replicas, it is not a function of ω and does not distort the spectrum. It is possible to reconstruct the original signal using a rectangular lowpass filter as is used for reconstruction after ideal sampling.

1.36. The Fourier transform of $x(t)$ in natural frequency is

$$\begin{aligned} X(f) &= \delta(f-100)+\delta(f+100)+2[\delta(f-200)+\delta(f+200)] \\ &\quad + (3/2)[\delta(f-300)+\delta(f+300)], \end{aligned} \tag{1.152}$$

and that of the ideal impulse train is

$$S(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-n/T_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-nf_s), \tag{1.153}$$

where $f_s = 1/T_s$ is the sampling frequency. Thus

$$\begin{aligned} X_s(f) &= \frac{1}{T_s}X(f) * \sum_{n=-\infty}^{\infty}\delta(f-nf_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}X(f-nf_s) \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}[\delta(f-100-nf_s)+\delta(f+100-nf_s)] \\ &\quad + 2[\delta(f-200-nf_s)+\delta(f+200-nf_s)] \\ &\quad + (3/2)[\delta(f-300-nf_s)+\delta(f+300-nf_s)]. \end{aligned} \tag{1.154}$$

(a) The spectrum for $f_s = 300$ Hz is shown in Figure 1.2(b) (only for $n = 0, \pm 1$). Since $f_s = 300 < 2\cdot 400$ Hz, we find that aliasing occurs, as seen by the overlapping spectral lines. (b) The spectrum for $f_s = 500$ Hz is shown in Figure 1.2(c), which also has aliasing because $f_s = 500 < 2\cdot 300$ Hz.

1.37. Since the spectrum of the reconstructed signal is obtained as $X(\omega) = H(\omega)X_s(\omega)$, the result in the time domain is a convolution $x(t) = h(t) * x_s(t)$ where $h(t)$ the inverse Fourier transform of the rectangle function:

$$h(t) = (T_s/2\pi)\int_{-W}^{W}\exp(j\omega t)d\omega = T_s W \operatorname{sinc}(Wt). \tag{1.155}$$

Thus

$$\begin{aligned} x(t) &= T_s W \operatorname{sinc}(Wt) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s) \\ &= T_s W \sum_{n=-\infty}^{\infty} x(nT_s)\operatorname{sinc}(W(t-nT_s)), \end{aligned} \tag{1.156}$$

where the sifting property of the Dirac delta function has been used. Observe that the samples $\{x(nT_s)\}$ are weighted by time-shifted sinc functions which perform an interpolation of the function in between the samples at integer multiples of T_s.

Discrete-Time Filters

1.38. (a) For a direct-form I implementation, we multiply terms and rewrite the expression as a function of z^{-1}:

$$H(z) = \frac{5z^{-2}-10z^{-3}}{1-0.2z^{-1}-0.63z^{-2}+1.8z^{-3}}, \tag{1.157}$$

which gives the structure in Figure 1.3(a). Rearranging the feed-forward and feedback sections, and combining the two delay lines into a single delay line gives the direct-form II implementation in Figure 1.3(b).

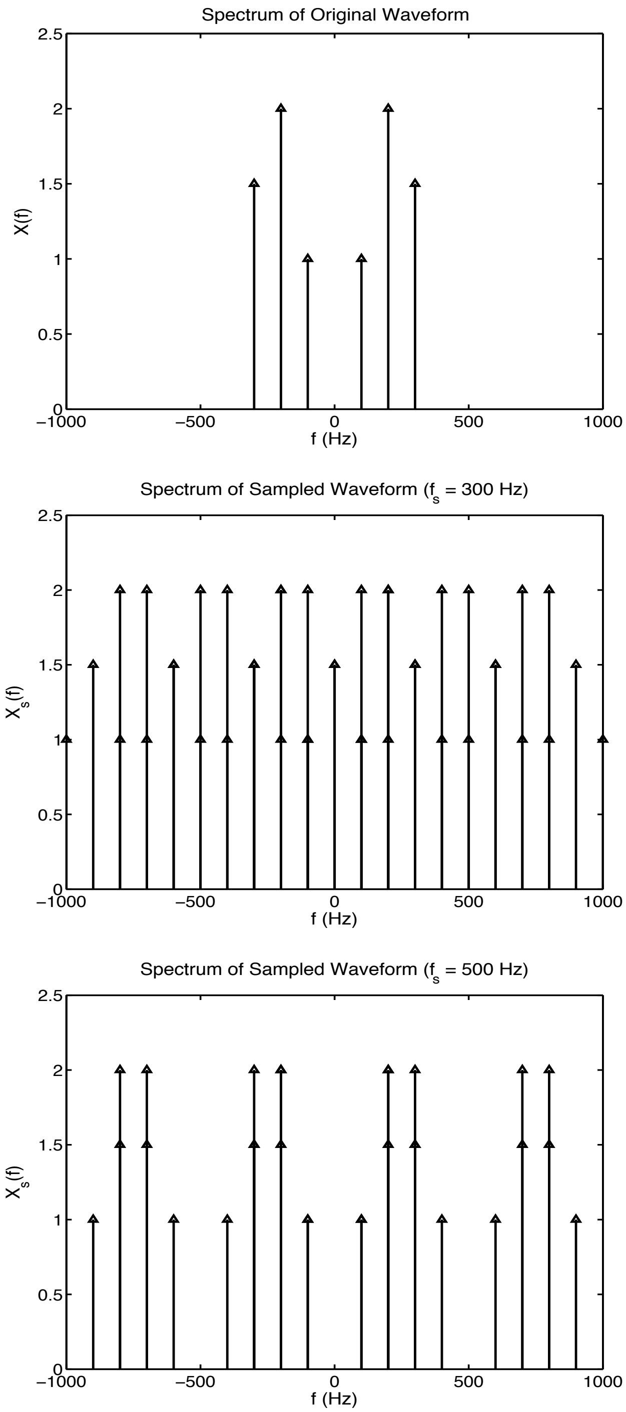

Figure 1.2: Spectra for Problem 1.36. (a) Original waveform. (b) $f_s = 300$ Hz. (c) $f_s = 500$ Hz. Note that the results only for $n = 0, \pm 1$ are shown in (b) and (c).