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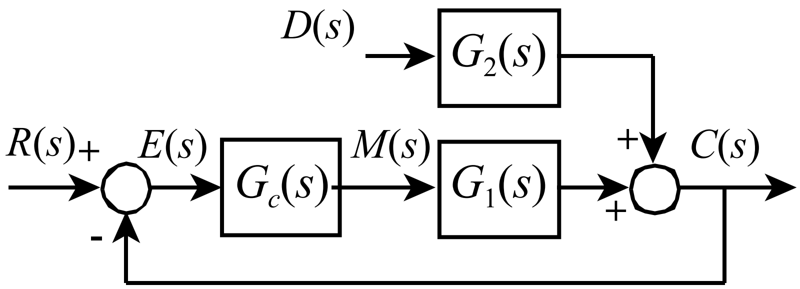
$\%TO := \%$ $\%CO := \%$

Problem 6-1. Second-order loop with proportional controller.



$$G_1(s) = \frac{K}{(\tau_1 \cdot s + 1)\cdot(\tau_2 \cdot s + 1)} \qquad G_c(s) = K_c$$

Problem parameters: $K := 0.10\frac{\%TO}{\%CO}$ $\tau_1 := 1min$ $\tau_2 := 0.8min$

(a) Closed loop transfer function and characteristic equation of the loop.

$$\frac{C(s)}{R(s)} = \frac{K_c \cdot \frac{K}{(\tau_1 \cdot s + 1)\cdot(\tau_2 \cdot s + 1)}}{1 + K_c \cdot \frac{K}{(\tau_1 \cdot s + 1)\cdot(\tau_2 \cdot s + 1)}} = \frac{K_c \cdot K}{(\tau_1 \cdot s + 1)\cdot(\tau_2 \cdot s + 1) + K_c \cdot K}$$

Characteristic equation: $\tau_1 \cdot \tau_2 \cdot s^2 + (\tau_1 + \tau_2)s + 1 + K_c \cdot K = 0$

Closed-loop transfer function: $\boxed{\frac{C(s)}{R(s)} = \frac{0.1K_c}{0.8s^2 + 1.8s + 1 + 0.1K_c}}$

Characteristic equation: $\boxed{0.8s^2 + 1.8s + 1 + 0.1K_c = 0}$

(b) Values of the controller gain for which the response is over-damped, critically damped, and under-damped

Roots of the characteristic equation:

$$r_1 = \frac{-1.8 + \sqrt{1.8^2 - 4 \cdot 0.8 \cdot (1 + 0.1K_c)}}{2 \cdot 0.8} = \frac{-1.8}{1.6} + \sqrt{\left(\frac{1.8}{1.6}\right)^2 - \frac{1 + 0.1K_c}{0.8}}$$

The response is critically damped when the term in the radical is zero: $\left(\frac{1.8}{1.6}\right)^2 - \frac{1 + 0.1K_c}{0.8} = 0$

Critically damped: $K_{ccd} := \frac{1}{0.1}\left[0.8\left(\frac{1.8}{1.6}\right)^2 - 1\right]$ $\quad K_{ccd} = 0.125\,\frac{\%CO}{\%TO}$

Over-damped (real roots): $K_c < 0.125\,\frac{\%CO}{\%TO}$ Under-damped: $K_c > 0.125\,\frac{\%CO}{\%TO}$

The loop cannot be unstable for positive gain because,

- for real roots the radical cannot be greater than the negative term, so both roots are negative
- for complex conjugate roots the real part is always negative, -1.8/1.6, or $-(\tau_1+\tau_2)/2\tau_1\tau_2$

This is true for all positive values of the time constants and the product K_cK.

(c) Equivalent time constants for different values of the gain:

$K_c := 0.1\,\frac{\%CO}{\%TO}$ (over-damped, two equivalent time constans)

$\tau_{e1} = \frac{-1}{r_1}$ $\quad \tau_{e1} := \frac{2\cdot\tau_1\cdot\tau_2}{(\tau_1+\tau_2) - \sqrt{(\tau_1+\tau_2)^2 - 4\cdot\tau_1\cdot\tau_2\cdot(1+K_c\cdot K)}}$ $\quad \tau_{e1} = 0.935\text{ min}$

$\tau_{e2} = \frac{-1}{r_2}$ $\quad \tau_{e2} := \frac{2\cdot\tau_1\cdot\tau_2}{(\tau_1+\tau_2) + \sqrt{(\tau_1+\tau_2)^2 - 4\cdot\tau_1\cdot\tau_2\cdot(1+K_c\cdot K)}}$ $\quad \tau_{e2} = 0.847\text{ min}$

$K_c := 0.125\,\frac{\%CO}{\%TO}$ (critically damped, two equal real time constants)

$\tau_{e1} = \frac{-1}{r_1}$ $\quad \tau_{e1} := \frac{2\cdot\tau_1\cdot\tau_2}{(\tau_1+\tau_2) - \sqrt{(\tau_1+\tau_2)^2 - 4\cdot\tau_1\cdot\tau_2\cdot(1+K_c\cdot K)}}$ $\quad \tau_{e1} = 0.889\text{ min}$

$\tau_{e2} = \frac{-1}{r_2}$ $\quad \tau_{e2} := \frac{2\cdot\tau_1\cdot\tau_2}{(\tau_1+\tau_2) + \sqrt{(\tau_1+\tau_2)^2 - 4\cdot\tau_1\cdot\tau_2\cdot(1+K_c\cdot K)}}$ $\quad \tau_{e2} = 0.889\text{ min}$

$K_c := 0.2\,\frac{\%CO}{\%TO}$ (under-damped, time constant and damping ratio)

$$\tau^2 s^2 + 2\zeta\cdot\tau\cdot s + 1 = \frac{\tau_1\cdot\tau_2}{1+K_c\cdot K}s^2 + \frac{\tau_1+\tau_2}{1+K_c\cdot K}s + 1$$

Match coefficients: $\tau := \sqrt{\frac{\tau_1\cdot\tau_2}{1+K_c\cdot K}}$ $\quad \zeta := \frac{\tau_1+\tau_2}{2\cdot\tau\cdot(1+K_c\cdot K)}$ $\quad \tau = 0.886\text{ min}$ $\quad \zeta = 0.996$

(d) Steady-state offset for a unit step change in set point.

Final value theorem: $\lim_{t\to\infty} Y(t) = \lim_{s\to 0} s\cdot Y(s)$ $\quad R(s) = \frac{1}{s}$ (Table 2-1.1)

$$K_c := 0.1\,\frac{\%CO}{\%TO} \qquad \lim_{s \to 0} s \cdot \frac{K_c \cdot K}{\tau_1 \cdot \tau_2 \cdot s^2 + (\tau_1 + \tau_2) \cdot s + 1 + K_c \cdot K} \frac{1}{s} \rightarrow 9.9009900990099009901 \cdot 10^{-3}$$

$$\text{offset} := (1 - 0.0099)\%TO \qquad \boxed{\text{offset} = 0.99\ \%TO}$$

$$K_c := 0.125\,\frac{\%CO}{\%TO} \qquad \lim_{s \to 0} s \cdot \frac{K_c \cdot K}{\tau_1 \cdot \tau_2 \cdot s^2 + (\tau_1 + \tau_2) \cdot s + 1 + K_c \cdot K} \frac{1}{s} \rightarrow 1.2345679012345679012 \cdot 10^{-2}$$

$$\text{offset} := (1 - 0.01235)\%TO \qquad \boxed{\text{offset} = 0.988\ \%TO}$$

$$K_c := 0.2\,\frac{\%CO}{\%TO} \qquad \lim_{s \to 0} s \cdot \frac{K_c \cdot K}{\tau_1 \cdot \tau_2 \cdot s^2 + (\tau_1 + \tau_2) \cdot s + 1 + K_c \cdot K} \frac{1}{s} \rightarrow 1.9607843137254901961 \cdot 10^{-2}$$

$$\text{offset} := (1 - 0.01961)\%TO \qquad \boxed{\text{offset} = 0.98\ \%TO}$$

These are very large offsets because the loop gains are so small.

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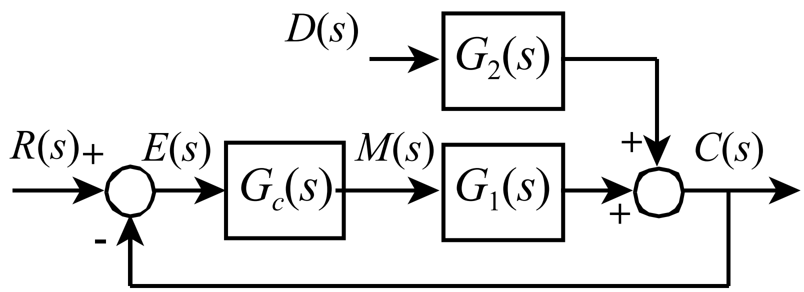
Problem 6-2. Inverse-response second-order system with proportional controller.



$$G_1(s) = \frac{6(1-s)}{(s+1)\cdot(0.5\cdot s+1)}\frac{\%TO}{\%CO} \qquad G_c(s) = K_c\cdot\frac{\%CO}{\%TO}$$

(a) Closed-loop transfer function and characteristic equation of the loop.

Closed-loop transfer function: $$\frac{C(s)}{R(s)} = \frac{K_c\cdot 6(1-s)}{(s+1)\cdot(0.5s+1)+K_c\cdot 6(1-s)}$$

Characteristic equation: $$0.5\cdot s^2 + \left(1.5-6K_c\right)s + 1 + 6K_c = 0$$

(b) Values of the gain for which the response is over-, critically, and under-damped

Roots:

$$r_1 = \frac{-\left(1.5-6K_c\right)+\sqrt{\left(1.5-6K_c\right)^2-4\cdot 0.5\cdot\left(1+6K_c\right)}}{2\cdot 0.5} = -1.5+6K_c+\sqrt{0.25-30K_c+36K_c^{\ 2}}$$

The response is critically damped when the term in the radical is zero:

$$0.25-30K_c+36K_c^{\ 2} = 0 \qquad K_c := \frac{30+\sqrt{30^2-4\cdot 0.25\cdot 36}}{2\cdot 36} \qquad K_c = 0.82491\,\frac{\%CO}{\%TO}$$

$$K_c := \frac{30-\sqrt{30^2-4\cdot 0.25\cdot 36}}{2\cdot 36} \qquad K_c = 0.00842\,\frac{\%CO}{\%TO}$$

Over-damped (two real roots): $K_c < 0.00842\,\frac{\%CO}{\%TO}$ and $K_c > 0.825\,\frac{\%CO}{\%TO}$

Under-damped (complex conjugate roots): $0.00842\,\frac{\%CO}{\%TO} < K_c < 0.825\,\frac{\%CO}{\%TO}$

The response is unstable when $K_c > 0.25\frac{\%CO}{\%TO}$ (one real root is positive or the real part of the complex roots i positive)

(c) Effective time constants or time constant and damping ratio for various values of the gain:

$$\frac{0.5}{1+6K_c}s^2 + \frac{1.5-6K_c}{1+6K_c}s + 1 = \tau^2 s^2 + 2\zeta\cdot\tau\cdot s + 1$$

$K_c := 0.1\frac{\%CO}{\%TO}$ $\quad \tau := \sqrt{\frac{0.5\text{min}^2}{1+6K_c}}$ $\quad \zeta := \frac{(1.5-6K_c)\text{min}}{2\cdot\tau\cdot(1+6K_c)}$ $\quad \tau = 0.559\ \text{min}$ $\quad \zeta = 0.503$

$K_c := 0.125\frac{\%CO}{\%TO}$ $\quad \tau := \sqrt{\frac{0.5\text{min}^2}{1+6K_c}}$ $\quad \zeta := \frac{(1.5-6K_c)\text{min}}{2\cdot\tau\cdot(1+6K_c)}$ $\quad \tau = 0.535\ \text{min}$ $\quad \zeta = 0.401$

$K_c := 0.2\frac{\%CO}{\%TO}$ $\quad \tau := \sqrt{\frac{0.5\text{min}^2}{1+6K_c}}$ $\quad \zeta := \frac{(1.5-6K_c)\text{min}}{2\cdot\tau\cdot(1+6K_c)}$ $\quad \tau = 0.477\ \text{min}$ $\quad \zeta = 0.143$

$K_c := 0.3\frac{\%CO}{\%TO}$ $\quad \tau := \sqrt{\frac{0.5\text{min}^2}{1+6K_c}}$ $\quad \zeta := \frac{(1.5-6K_c)\text{min}}{2\cdot\tau\cdot(1+6K_c)}$ $\quad \tau = 0.423\ \text{min}$ $\quad \zeta = -0.127$

(unstable)

Try values that result in equivalent time constants:

$K_c := 0.005\frac{\%CO}{\%TO}$ $\quad \tau_{e1} := \frac{1\text{min}}{1.5-6\cdot K_c-\sqrt{0.25-30K_c+36K_c^2}}$ $\quad \tau_{e1} = 0.868\ \text{min}$

$\tau_{e2} := \frac{1\text{min}}{1.5-6\cdot K_c+\sqrt{0.25-30K_c+36K_c^2}}$ $\quad \tau_{e2} = 0.559\ \text{min}$

$K_c := 1\frac{\%CO}{\%TO}$ $\quad \tau_{e1} := \frac{1\text{min}}{1.5-6\cdot K_c-\sqrt{0.25-30K_c+36K_c^2}}$ $\quad \tau_{e1} = -0.143\ \text{min}$

(unstable)

$\tau_{e2} := \frac{1\text{min}}{1.5-6\cdot K_c+\sqrt{0.25-30K_c+36K_c^2}}$ $\quad \tau_{e2} = -0.5\ \text{min}$

(d) Offset for various values of the gain and a unit step change in set point.

$K_c := 0.10\frac{\%CO}{\%TO}$ $\quad \lim_{s\to 0} s\cdot\frac{K_c\cdot 6\cdot(1-s)}{0.5s^2+(1.5-6\cdot K_c)s+1+6K_c}\frac{1}{s} \rightarrow .37500000000000000000$

$\text{offset} := 1-0.375$ $\quad \text{offset} = 0.625\frac{\%CO}{\%TO}$

$K_c := 0.125 \frac{\%CO}{\%TO}$ $\quad \lim_{s \to 0} s \cdot \frac{K_c \cdot 6 \cdot (1 - s)}{0.5s^2 + (1.5 - 6 \cdot K_c)s + 1 + 6K_c} \frac{1}{s} \rightarrow .42857142857142857143$

offset := 1 − 0.429 $\quad$ $\boxed{\text{offset} = 0.571 \frac{\%CO}{\%TO}}$

$K_c := 0.20 \frac{\%CO}{\%TO}$ $\quad \lim_{s \to 0} s \cdot \frac{K_c \cdot 6 \cdot (1 - s)}{0.5s^2 + (1.5 - 6 \cdot K_c)s + 1 + 6K_c} \frac{1}{s} \rightarrow .54545454545454545455$

offset := 1 − 0.545 $\quad$ $\boxed{\text{offset} = 0.455 \frac{\%CO}{\%TO}}$

The offsets are high because the gains are small. Of course, since for gains greater than 0.25%CO/%TO the loop is unstable, offsets can only be high with a proportional controller.

K := 1.8R

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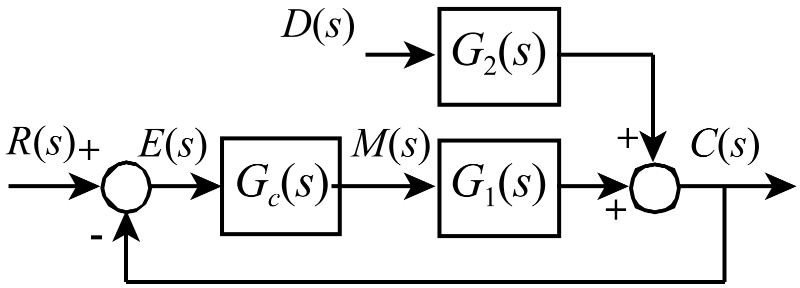
Problem 6-3. First-order process and proportional-integral controller.



$$G_1(s) = \frac{K}{\tau \cdot s + 1} \qquad G_c(s) = K_c \cdot \left(1 + \frac{1}{\tau_I \cdot s}\right)$$

To work in dimensionless units, t/τ, set: $\tau := 1$

(a) Closed-loop transfer function and characteristic equation of the loop. Offset.

Closed-loop transfer functon: $\frac{C(s)}{R(s)} = \frac{KK_c \cdot (\tau_I \cdot s + 1)}{\tau_I \cdot s \cdot (\tau \cdot s + 1) + KK_c \cdot (\tau_I \cdot s + 1)}$

Characteristic equation: $\tau_I \cdot \tau \cdot s^2 + (1 + KK_c)\tau_I \cdot s + KK_c = 0$

Offset: the steady state gain is: $\lim_{s \to 0} \frac{KK_c \cdot (\tau_I \cdot s + 1)}{\tau_I \cdot \tau \cdot s^2 + (1 + KK_c)\tau_I \cdot s + KK_c} = \frac{KK_c}{KK_c} = 1$

(no offset)

(b) Is there an ultimate gain for this loop?

Substitute s = iω: $-\tau_I \cdot \tau \cdot \omega_u^2 + KK_{cu} + i(1 + KK_{cu})\,\omega_u = 0 \qquad \omega_u := 0 \qquad KK_{cu} := 0$

No, there is no ultimate gain. This result just means that a negative loop gain will make the loop unstable. Another way to show it is to determine the roots of the characteristic equation:

$$r_1 = \frac{-(1 + KK_c) \cdot \tau_I + \sqrt{(1 + KK_c)^2 \cdot \tau_I^2 - 4 \cdot \tau_I \cdot \tau \cdot KK_c}}{2\tau_I \cdot \tau}$$

The real root cannot be negative for any positive value of the loop gain KK_c because the radical is always smaller than the negative term. Also, for complex conjugate roots, the real part is always negative:

$$\text{Real} = \frac{-(1 + KK_c)}{2\cdot\tau} < 0$$

(c) Response of the loop to a step change in set point for $\tau_I = \tau$ as the gain varies from 0 to infinity.

$$R(s) = \frac{1}{s} \quad \text{(Table 2-1.1)} \qquad Y(s) = \frac{KK_c\cdot(\tau\cdot s + 1)}{\tau\cdot s\cdot(\tau\cdot s + 1) + KK_c\cdot(\tau\cdot s + 1)}\frac{1}{s} = \frac{KK_c}{\tau\cdot s + KK_c}\frac{1}{s}$$

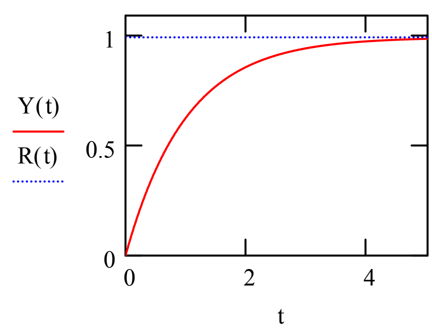
Let $\boxed{\tau_c = \frac{\tau}{KK_c}}$

$$Y(s) = \frac{1}{\tau_c\cdot s + 1}\frac{1}{s} = \frac{1}{s} - \frac{1}{s + \frac{1}{\tau_c}}$$

$$u(t) := \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Invert using Table 2-1.1: $\boxed{Y(t) := u(t) - e^{\frac{-t}{\tau_c}}}$ $\quad R(t) := u(t) \quad \tau_c := 1$

As KKc increases τc decreases and the response is faster.



$$R := \frac{K}{1.8}$$

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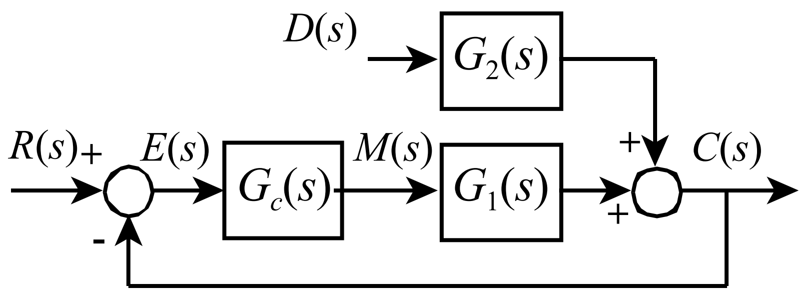
Problem 6-4. Second-order process with pure integral controller.



$$G_1(s) = \frac{K}{(\tau_1 \cdot s + 1)\cdot(\tau_2 \cdot s + 1)} \qquad G_c(s) = \frac{K_I}{s}$$

(a) Ultimate gain and period with the parameters of Problem 6-1:

$$K := 0.1\frac{\%CO}{\%TO} \qquad \tau_1 := 1\text{min} \qquad \tau_2 := 0.8\text{min}$$

Characteristic equation:

$$\tau_1 \cdot \tau_2 \cdot s^2 + (\tau_1 + \tau_2)s + 1 + \frac{K \cdot K_I}{s} = 0$$

$$\tau_1 \cdot \tau_2 s^3 + (\tau_1 + \tau_2)s^2 + s + KK_I = 0$$

Substitute s = $i\omega_u$:

$$-\tau_1 \cdot \tau_2 \cdot i \cdot \omega_u^3 - (\tau_1 + \tau_2)\omega_u^2 + i \cdot \omega_u + KK_{Iu} = 0 + 0i$$

$$-(\tau_1 + \tau_2)\,\omega_u^2 + KK_{Iu} = 0 \qquad -\tau_1 \cdot \tau_2 \omega_u^3 + \omega_u = 0$$

$$\omega_u := \sqrt{\frac{1}{\tau_1 \cdot \tau_2}} \qquad K_{Iu} := \frac{(\tau_1 + \tau_2)\cdot \omega_u^2}{K} \qquad \omega_u = 1.118\ \text{min}^{-1} \qquad \boxed{K_{Iu} = 22.5\frac{\%CO}{\%TO \cdot \text{min}}}$$

$$T_u := \frac{2\pi}{\omega_u} \qquad \boxed{T_u = 5.62\ \text{min}}$$

(b) Ultimate gain and period for other values of the smaller time constant:

$$\tau_2 := 0.1\text{min} \qquad \omega_u := \sqrt{\frac{1}{\tau_1 \cdot \tau_2}} \qquad K_{Iu} := \frac{(\tau_1 + \tau_2)\cdot \omega_u^2}{K} \qquad \omega_u = 3.162\ \text{min}^{-1} \qquad \boxed{K_{Iu} = 110\frac{\%CO}{\%TO \cdot \text{min}}}$$

$$T_u := 2\frac{\pi}{\omega_u} \qquad \boxed{T_u = 1.987\ \text{min}}$$

$\tau_2 := 2\text{min}$ $\quad \omega_u := \sqrt{\frac{1}{\tau_1 \cdot \tau_2}}$ $\quad K_{Iu} := \frac{(\tau_1 + \tau_2) \cdot \omega_u^2}{K}$ $\quad \omega_u = 0.707\,\text{min}^{-1}$ $\quad \boxed{K_{Iu} = 15 \frac{\%CO}{\%TO \cdot \text{min}}}$

$T_u := \frac{2\pi}{\omega_u}$ $\quad \boxed{T_u = 8.886\,\text{min}}$

Reducing the non-dominat time constant increases the ultimate gain and reduces the ultimate period, as expected. When τ_2 is increased to 2 min, it becomes the dominant time constant and the ultmate gain should be higher than for part (a). However, in this case K_I has units of rate and, since the loop is slower, it results in a smaller ultimate gain.