SOLUTIONS

Section 1

1.2. (a) Only β is onto. (b) All are one-to-one.
(c) $\alpha(\mathbb{N}) = \{2, 4, 6, \ldots\}$; $\beta(\mathbb{N}) = \{2, 3, 4, \ldots\}$; $\gamma(\mathbb{N}) = \{1, 8, 27, \ldots\}$.

1.4. 6

1.6. 6

1.8. ...there exist $x_1, x_2 \in S$ such that $x_1 \neq x_2$ and $\alpha(x_1) = \alpha(x_2)$.

1.10. ...there is exactly one $x \in S$ such that $\beta = y$.

1.12. Onto. One-to-one. $f(P) = \{x : x > -4\}$.

1.14. Not onto. Not one-to-one. $f(P) = P$.

1.16. Onto. Not one-to-one. $f(P) = \mathbb{R}$.

1.18. $\beta(A) = \mathbb{N}$. $\beta(B) = \mathbb{N}$. $\beta(C) = B$.

1.20. All non-zero n.

1.21. (a) n^m.

(b) If $m > n$, there are none. If $m \leq n$, there are

$$n(n-1)(n-2)\cdots(n-m+1)\ .$$

1.22. (a) 2. (b) 6. (c) $2^n - 2$.

1.23. (a) ...iff each horizontal line intersects the graph of f at most once.

(b) ...iff each horizontal line intersects the graph of f exactly once.

1.24. (a) $\{(a, b) : a = 1 \text{ or } a = -1\}$

(b) $\{(a, b) : a \neq 0\}$

1.25. (a)

$$\gamma(n) = \begin{cases} (n+2)/3 & \text{for } n = 1, 4, 7, \ldots \\ (n+1)/3 & \text{for } n = 2, 5, 8, \ldots \\ n/3 & \text{for } n = 3, 6, 9, \ldots \end{cases}$$

(b) $\gamma(1) = 1$, $\gamma(2) = \gamma(3) = 2$, $\gamma(4) = \gamma(5) = \gamma(6) = 3$, and so on. A formula is $\gamma(k) = n$ for $(n-1)n/2 < k \leq n(n+1)/2$, for each $n \in \mathbb{N}$.

(c) Let p_n denote the n^{th} prime ($p_1 = 2$, $p_2 = 3, \ldots$). Define γ by $\gamma(p_n^k) = n$ for $k = 1, 2, 3, \ldots$; $\gamma(m)$ can then be anything for m not a prime power. (There are many other solutions of course.)

1.26. Assume that $\alpha : S \to S$ is one-to-one but not onto. Choose $a \in S$, and define $\beta : S \to S$ by $\beta(x) = y$ iff $\alpha(y) = x$, for each $x \in \alpha(S)$, and $\beta(x) = a$ for $x \neq \alpha(S)$. Then β is onto but not one-to-one.

Assume that $\alpha : S \to S$ is onto but not one-to-one. For each $x \in S$, choose exactly one $y_x \in S$ such that $\alpha(y_x) = x$.

Define $\beta : S \to S$ by $\beta(x) = y_x$ for each $x \in S$. Then β is one-to-one but not onto.

1.27. For $y \in T$, $y \in \alpha(A \cup B)$ iff $y = \alpha(x)$ for some $x \in A \cup B$ iff $y = \alpha(x)$ for some $x \in A$ or some $x \in B$ iff $y \in \alpha(A) \cup \alpha(B)$.

1.28. (a) For $y \in T$, if $y \in \alpha(A \cap B)$ then $y = a(x)$ for some $x \in A \cap B$ and so $y \in \alpha(A)$ and $y \in \alpha(B)$, that is, $y \in \alpha(A) \cap \alpha(B)$.

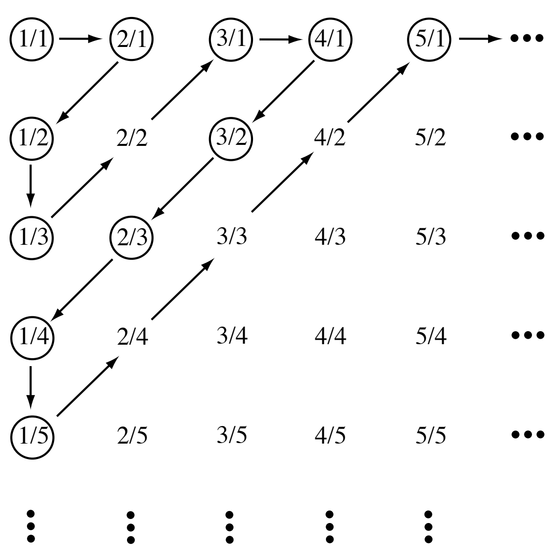
(b) Let $S = T = \{1, 2\}$, $A = \{1\}$, $B = \{2\}$, and $\alpha(1) = \alpha(2) = 1$.

1.29. Assume that α is one-to-one; by Problem 1.28(a) it suffices to prove that $\alpha(A \cap B) \supseteq \alpha(A) \cap \alpha(B)$ for every pair of subsets A and B of S. If $y \in \alpha(A) \cap \alpha(B)$, then $y = \alpha(x_1)$ for some $x_1 \in A$ and $y = \alpha(x_2)$ for some $x_2 \in B$. But α is one-to-one so $x_1 = x_2$. Therefore $y \in \alpha(A \cap B)$.

Assume, conversely, that $\alpha(A \cap B) = \alpha(A) \cap \alpha(B)$ for every pair of subsets A and B of S. If $\alpha(x_1) = \alpha(x_2)$ for $x_1, x_2 \in S$, then with $A = \{x_1\}$ and $B = \{x_2\}$ we have $\alpha(A) \cap \alpha(B) = \{\alpha(x_1)\} = \{\alpha(x_2)\}$, and so $\alpha(A \cap B) = \{a(x_1)\} = \{\alpha(x_2)\}$, whence $A = B$ and $x_1 = x_2$; therefore α is one-to-one.

1.30. Let T denote an infinite subset of S, and let $\alpha : T \to T$ be one-to-one but not onto. Define $\beta : S \to S$ by $\beta(t) = \alpha(t)$ for each $t \in T$, and $\beta(s) = s$ for all other $s \in S$.

1.31. List the positive fractions systematically, as shown.



Now follow the path indicated, circling an entry if it is not equal to some entry previously circled. Finally, write down the circled entries, in order, to obtain the one-to-one correspondence with the natural numbers.

Section 2

2.2. $(\gamma \circ \alpha)(n) = 4n^2$. Image $= \{4n^2 : n \in \mathbb{Z}\}$.

2.4. $(\beta \circ \beta)(n) = n + 2$. Image $= \mathbb{Z}$.

2.6. $(\gamma \circ \gamma)(n) = n^4$. Image $= \{n^4 : n \in \mathbb{Z}\}$.

2.8. Denote it by γ. Then $\gamma(a) = 3$, $\gamma(b) = 1$, $\gamma(c) = 2$.

2.10. (a) $g(x) = -2x$. (b) $g(x) = \sqrt[3]{x}$. (c) $g(x) = e^x$.

2.12. (a) True. (b) False. (c) True. (d) False.

2.14. None is invertible. (This can motivate a discussion of the inverse trigonometric functions.)

2.16. $\alpha \circ \beta = \iota_T$.

2.18. (a) Yes. Theorem 2.2 and Theorem 2.1(d).

(b) “If $\beta \circ \alpha$ is not invertible, then α is not one-to-one.” False. For an example, see the answer to Problem 2.17(a) in Appendix E.

(c) “If $\beta \circ \alpha$ is invertible, then α is one-to-one.” True. Theorem 2.2 and Theorem 2.1(d).

2.19. Assume $S < T$ and $T < U$. If $\alpha : T \to S$ is onto and $\beta : U \to T$ is onto, then $\alpha \circ \beta : U \to S$ is onto. However, if $S \not< U$, then there also exists $\gamma : S \to U$ that is onto; this yields an onto mapping $\beta \circ \gamma : S \to T$, contradicting $S < T$. (Remark: This makes use of ideas from Section 2. The problem is in Section 1 as a special challenge. The proof given can be conveyed without the notation for composition, of course. Or the problem can be assigned after Section 2 has been covered.)

2.20. The same equations that show β is an inverse of α show that α is an inverse of β.

2.21. If $S = T = \{1, 2\}$, $U = \{a\}$, $\alpha(1) = \alpha(2) = 1$, and $\beta(1) = \beta(2) = a$, then $\beta \circ \alpha$ is onto but α is not onto.

2.22. If $S = \{1\}$, $T = U = \{a, b\}$, $\alpha(1) = a$, and $\beta(a) = \beta(b) = a$, then $\beta \circ \alpha$ is one-to-one but β is not one-to-one.

2.23. If $y \in T$, then $y = \alpha(x)$ for some $x \in S$ because α is onto. Therefore $\beta(y) = (\beta \circ \alpha)(x) = (\gamma \circ \alpha)(x) = \gamma(y)$ because $\beta \circ \alpha = \gamma \circ \alpha$. Thus $\beta = \gamma$.

2.24. If $x \in S$, then $\alpha(\beta(x)) = \alpha(\gamma(x))$ because $\alpha \circ \beta = \alpha \circ \gamma$. Therefore $\beta(x) = \gamma(x)$ because α is one-to-one. Thus $\beta = \gamma$.

2.25. Let $S = \{1\}$, $T = U = \{a, b\}$, $\alpha(1) = a$, $\beta(a) = \gamma(a) = a$, $\beta(b) = b$, and $\gamma(b) = a$.

2.26. Let $S = U = \{1\}$, $T = \{a, b\}$, $\alpha(a) = \alpha(b) = 1$, $\beta(1) = a$, and $\gamma(1) = b$.

2.27. (a) If α and β are invertible, then both are one-to-one and onto by Theorem 2.2. Then $\beta \circ \alpha$ is one-to-one and onto by Theorem 2.1, parts (c) and (a). Therefore $\beta \circ \alpha$ is invertible by Theorem 2.2.

(b) If $\beta \circ \alpha$ is invertible, then it is one-to-one and onto by Theorem 2.2. Therefore β is onto by Theorem 2.1(b), and α is one-to-one by Theorem 2.1(d).

Section 3

3.2. Not an operation.

3.4. Operation. Neither associative nor commutative. No identity.

3.6. Not an operation.

3.8. Not an operation.

3.10. The identity mapping from S to S.

3.12. (b) Let $u * v = v$.

3.14. ...for some $a, b, c, \in S$.

3.16. ...$e * a \neq a$ or $a * e \neq a$ for some $a \in S$.

3.18. $a * c = c$, $b * a = b$, $b * c = d$, $b * d = a$, $d * a = d$, $d * c = b$.

3.20. The positive rational numbers. (Since $2 \in B$, $2/2 = 1 \in B$. Therefore each positive integer is in B, and so, by division again, each positive rational number is in B.)

3.21. In order, the answers are 1, 2^4, 3^9, and n^{n^2}.

3.22. In order, the answers are 1, 2^3, 3^6, and $n^{n(n+1)/2}$. (There is complete freedom of choice everywhere on and above the main diagonal of the Cayley table. This gives $1 + 2 + \cdots + n = n(n + 1)/2$ choices.)

3.23. (a) Let $u * u = u$ and $u * v = v * u = v$. Then $v * v$ can be either u or v, so it can be done in two ways.

(b) No, because otherwise $u = u * v = v$.

(c) If e and f were both identity elements, then $e = e * f = f$.

3.24. $w * x = z$, $w * y = w$, $x * y = x$, $x * z = y$, $y * w = w$, $y * x = x$,

$y * y = y$, $y * z = z$, $z * w = x$, $z * x = y$, $z * y = z$.

3.25. Straightforward, using commutativity of addition of real numbers.

3.26. One example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

3.27. Straightforward; verify that the product of the given matrix and the matrix on the right-hand side of equation is the identity matrix.

3.28. If $a = e$, then $e * (b * c) = b * c$ and $(e * b) * c = b * c$ for all b and c. The cases $b = e$ and $c = e$ are similar.

3.29. If $x \in C(a)$ and $y \in C(a)$, then $a * (x * y) = (a * x) * y = (x * a) * y = x * (a * y) = x * (y * a) = (x * y) * a$, so $x * y \in C(a)$.

3.30. For $a * (b * (c * d)) = (a * b) * (c * d)$, see the second paragraph of Section 14. The other cases are similar.

3.31. If $y, z \in S$, then $y * z = e * (y * z) = (e * z) * y = z * y$, where the middle equality is the given condition with $x = e$; thus $y * z = z * y$ and $*$ is commutative. Using commutativity, we can write $x * (y * z) = (x * z) * y$ as $x * (z * y) = (x * z) * y$, so $*$ is associative.

3.32. $b = b * e = b * (a * c) = (b * a) * c = e * c = c$.

Section 4

4.2. (a)

$\circ$	α	β	γ	δ
α	α	β	γ	δ
β	β	α	δ	γ
γ	γ	γ	γ	γ
δ	δ	δ	δ	δ

(b) Theorem 4.1(a).

(c) No. For example, $\beta \circ \gamma \neq \gamma \circ \beta$.

(d) α.

4.4. (a) Theorem 2.1(c) proves that composition is an operation on $N(S)$, and Theorem 4.1(a) proves that it is associative.

(b) Yes. (The identity mapping is one-to-one.)

(c) S finite.

4.5. (a) $\{\alpha, \alpha^2, \alpha^3, \alpha^4\}$, where $\alpha^2 = \alpha \circ \alpha$ and so on.

(b) $\{\alpha, \alpha^2, \ldots, \alpha^{12}\}$

(c) $\{\alpha, \alpha^2, \ldots, \alpha^{2k}\}$. These $2k$ elements are distinct.

4.6. (a) $\alpha_{1,0} \circ \alpha_{a,b} = \alpha_{a,b} = \alpha_{a,b} \circ \alpha_{1,0}$, because $\alpha_{a,b} \circ \alpha_{c,d} = \alpha_{ac,ad+b}$.

(b) One-to-one: $\alpha_{a,b}(x_1) = \alpha_{a,b}(x_2)$ iff $ax_1 + b = ax_2 + b$ iff $x_1 = x_2$.
Onto: If $y \in \mathbb{R}$, then $\alpha_{a,b}[(1/a)(y - b)] = y$.

(c) $(c, d) = (1/a, -b/a)$.

4.7. (a) $\alpha_{a,b}$ is a dilation; the line shrinks toward the origin by a factor of a.

(b) $\alpha_{a,b}$ reflects points through the origin and also magnifies or dilates if $|a| > 1$ or $|a| < 1$, respectively.

(c) $\alpha_{a,b}$ translates each point $|b|$ units to the left.

4.8. (a) The operation is associative and commutative and $\alpha_{1,0}$ is an identity element.

(b) Same as (a).

(c) $\alpha_{a,b} = \alpha_{1,b} \circ \alpha_{a,0}$.

4.9. $D = \{\alpha_{1,n} : n \in \mathbb{N}\}$.

4.10. Let $S = \{a, b, \dots\}$, and define π and τ in $M(S)$ by $\pi(x) = a$ for all $x \in S$ and $\tau(x) = b$ for all $x \in S$. Then $\pi \circ \tau = \pi$ but $\tau \circ \pi = \tau$.

4.11. (a)

$\circ$	α_1	α_2	α_3	α_4	α_5	α_6
α_1	α_1	α_2	α_3	α_4	α_5	α_6
α_2	α_2	α_1	α_5	α_6	α_3	α_4
α_3	α_3	α_4	α_1	α_2	α_6	α_5
α_4	α_4	α_3	α_6	α_5	α_1	α_2
α_5	α_5	α_6	α_2	α_1	α_4	α_3
α_6	α_6	α_5	α_4	α_3	α_2	α_1

(b) α_1

(c) The inverse of α_4 is α_5, the inverse of α_5 is α_4, and each of the other elements is its own inverse.

(d) No.

(e) Theorem 4.1.

4.12. Use the calculation in the proof of Theorem 4.1.

4.13. $\beta = \beta \circ \iota_T = \beta \circ (\alpha \circ \gamma) = (\beta \circ \alpha) \circ \gamma = \iota_S \circ \gamma = \gamma$.

4.14. Yes.

4.15. If α and β are linear, $x, y \in V$, and a and b are scalars, then $(\alpha \circ \beta)(ax+by) = \alpha[\beta(ax+by)] = \alpha[a\beta(x)+b\beta(y)] = a\alpha[\beta(x)]+b\alpha[\beta(y)] = a(\alpha\circ\beta)(x)+b(\alpha\circ\beta)(y)$, and so $\alpha \circ \beta$ is linear.

4.16. (a) If $\beta \circ \alpha = \iota_S$, then α is one-to-one by Theorem 2.1(c). If α is one-to-one, define β by $\beta(x) = y$ iff $\alpha(y) = x$ for each x in the image of α, and define β arbitrarily for the other elements of S (if there are any); then $\beta \circ \alpha = \iota_S$.

(b) If $\alpha \circ \beta = \iota_S$, then α is onto by Theorem 2.1(b). If α is onto, then for each $y \in S$ choose $x \in S$ such that $\alpha(x) = y$ and let $\beta(y) = x$; then $\alpha \circ \beta = \iota_S$.

Section 5

5.2. Group; 1 is the identity, and s/r is the inverse of r/s.

5.4. Not a group, because 0 has no inverse.

5.6. Not a group, because of lack of closure.

5.8. Group, 0 is the identity, and $-n$ is the inverse of n.

5.10. Not a group, because subtraction is not associative (also, there is no identity).

5.12. $1 = 2^0 3^0$ is the identity and $2^{-m}3^{-n}$ is the inverse of $2^m 3^n$.

5.14. Associativity is similar to that in Problem 5.13. The identity here is e defined by $e(x) = 1$ for each $x \in \mathbb{R}$. The inverse of f is g, defined by $g(x) = 1/f(x)$ for each $x \in \mathbb{R}$. The operation in Example 5.6 is composition.

5.15. If $|S| > 1$, then $M(S)$ contains elements without inverses. (See Problem 4.1, for example.)

5.16. The elements of the identity matrix are obviously rational. And it follows from equation (3.13) that if a matrix has rational entries, then so does its inverse. The group is non-Abelian.

5.17. Associativity is proved in linear algebra. The identity is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

5.18. Here is a typical calculation involving a (there are 19 such products): $b * (a * c) = b * c = (b * a) * c$. Here is a typical calculation without a (there are 8 such products): $b * (c * b) = b * a = b$ and $(b * c) * b = a * b = b$.

5.19. (a) $\alpha_{a,b} \circ \alpha_{a^{-1},-a^{-1}b} = \alpha_{aa^{-1},a(-a^{-1}b)+b} = \alpha_{1,0}$; the composition in the other order is similar.

(b) One example: $\alpha_{1,1} \circ \alpha_{0,1} \neq \alpha_{0,1} \circ \alpha_{1,1}$.

5.20.

$*$	a	b
a	a	b
b	b	a

5.21.

$*$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y

5.22. Let c denote the inverse of b, and let e denote the identity element. Then $a = a * e = a * (b * c) = (a * b) * c = b * c = e$.

5.23. $e * a = a$ for each $a \in G$ and $a * f = a$ for each $a \in G$. The proof also assumes the existence of at least one identity element. (Properties of equality are used also. See Section 9.)

5.24. For $\alpha, \beta \in G^S$, define $\alpha\beta$ by $(\alpha\beta)(x) = \alpha(x)\beta(x)$ for each $x \in S$.

Section 6

6.2. (a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

6.4. (a) (1)(2 6)(3 4 5) or (2 6)(3 4 5)

(b) (1 2 3 4) (c) (1 2) (d) (1 4)(2 5 3)

6.5. (a) (1), (1 2), (1 3), (1 4), (2 3), (2 4), (3 4), (1 2)(3 4), (1 3)(2 4), (1 4)(2 3), (1 2 3), (1 3 2), (1 2 4), (1 4 2), (1 3 4), (1 4 3), (2 3 4), (2 4 3), (1 2 3 4), (1 2 4 3), (1 3 2 4), (1 3 4 2), (1 4 2 3), (1 4 3 2)

(b) The first ten elements, as listed in part (a).

6.6. (a) 2 (b) $(n-1)!$

6.7. (a) $(a_k a_{k-1} \cdots a_2 a_1)$ (b) Only $k = 1$ and $k = 2$.

6.8. $\alpha = (2\ 3)$, $\beta = (1\ 3)$, $\beta\alpha = (1\ 3\ 2)$, and $\alpha\beta = (1\ 2\ 3)$.

6.9. Use the suggestion and the fact that every element is a product of cycles.

6.10. ...for some $a, b \in G$.

6.11. Let $S = \{x, y, z, \cdots\}$, and define α and β in Sym(S) by $\alpha(x) = x$, $\alpha(y) = z$, $\alpha(z) = y$, $\beta(x) = z$, $\beta(y) = y$, $\beta(z) = x$, and $\alpha(s) = \beta(s) = s$ for all other $s \in S$. Then $\alpha \circ \beta \neq \beta \circ \alpha$.

6.12. $M(S)$ contains all four mappings from S to S, while Sym(S) contains only the two invertible mappings.

6.13. $\alpha = (1\ 2\ \cdots\ k)$ for small k will reveal the idea.

6.14. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

6.15. $n(n-1)(n-2)/3$

6.16. (a) a_{k+1} for $k < s$, and a_1 for $k = s$.

(b) b_{k+1} for $k < t$, and b_1 for $k = t$.

(c) m (d) $\alpha \circ \beta = \beta \circ \alpha$.

Section 7

7.2. (a), (b), and (d) are subgroups, (c) is not.

7.4. (a) $G_T = G_{(T)} = \{(1), (2\ 3\ 4), (2\ 4\ 3), (2\ 3), (2\ 4), (3\ 4)\}$

(b) $G_T = \{(1)\}$. $G_{(T)} = \{(1), (1\ 2\ 3), (1\ 3\ 2), (1\ 2), (1\ 3), (2\ 3)\}$

7.6. This group consists of all translations.

7.7. θ is one-to-one: if $\theta(a) = \theta(b)$, then $a(1\ 2) = b(1\ 2)$ so $a = b$ by right cancellation. If b is odd, then $b(1\ 2) \in A_n$ and $\theta(b(1\ 2)) = b$.

7.8. $\{(1), (1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3),$

$(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$

7.9. Only (a) and (c) are subgroups.

7.10. Straightforward, using Theorem 7.1.

7.11. (a) Straightforward, using Theorem 7.1.

(b) For example, use all the 2×2 matrices with even integers as entries.

7.12. $\{n/2 : n \in \mathbb{Z}\}$ would do.

7.13. Use Theorem 7.1. (Modify the proof of Theorem 15.1.)

7.14. $(1\ 2)(1\ 2\ 3) = (2\ 3) \notin H \cup K$

7.15. Follow the suggestion at the end of the proof of Theorem 7.2. (Replace "$(\alpha \circ \beta)(t) = t$ for every $t \in T$" by "$(\alpha \circ \beta)(T) = T$", and so on.)

7.16. $|T| = 0$ or $|T| = 1$.

7.17. Since $G = S_n$, $G_{(T)}$ consists precisely of all the permutations that can be written as $\beta \circ \gamma$, when β is a permutation of T (leaving the elements of T' fixed) and γ is a permutation of T' (leaving the elements of T fixed). The symmetry of this characterization shows that $G_{(T)} = G_{(T')}$.

7.18. $n = 1$ and $T = \phi$ or $T = S$, or $n = 2$ and $|T| = 1$.

7.19. $|T| = n$ or $|T| = n - 1$.

7.20. (a) $(n-k)!$ (b) $k!(n-k)!$ (See 7.17.)

7.21. If $x * x = x$, then $x^{-1} * (x * x) = x^{-1} * x = e$, and therefore $(x^{-1} * x) * x = e * x = x = e$.

7.22. Assume first that H is a subgroup. Then (a) here is true by (a) of Theorem 7.1. Moreover, if $a, b \in H$, then $b^{-1} \in H$ by (c) of Theorem 7.1, and then $a * b^{-1} \in H$ by (b) of Theorem 7.1; therefore (b) here is true. Now assume that (a) and (b) here are true; it suffices to show that (a), (b), and (c) of Theorem 7.1 must be true as a consequence. By (a) here, condition (a) of Theorem 7.1 is true. If $a, b \in H$, then $e = a * a^{-1} \in H$ by condition (b) here; therefore $b^{-1} = e * b^{-1} \in H$ by condition (b) here; $a * b = a * (b^{-1})^{-1} \in H$ by condition (b) here; this gives condition (b) of Theorem 7.1. The preceding argument also shows why (c) of Theorem 7.1 is true.

7.23. Use Theorem 7.1. First, $a * e = e * a$, so $e \in C(a)$ and $C(a) \neq \phi$. Second, if $x, y \in C(a)$, then $a * (x * y) = (a * x) * y = (x * a) * y = x * (a * y) = x * (y * a) = (x * y) * a$, so $x * y \in C(a)$. Finally, if $x \in C(a)$, then $a * x = x * a$, so $x^{-1} * (a * x) * x^{-1} = x^{-1} * (x * a) * x^{-1}$, $x^{-1} * a = a * x^{-1}$, and $x^{-1} \in C(a)$.

7.24. Similar to Problem 7.23.

7.25. (a) Let $S = \mathbb{Z}$, $T = \mathbb{N}$, and define α by $\alpha(n) = n + 1$ for each $n \in S$.

(b) Let $S = \mathbb{Z}$, $T = \mathbb{N}$, and $G = \mathrm{Sym}(S)$. With α defined as in part (a), $\alpha \in G_{[T]}$ but $\alpha^{-1} \notin G_{[T]}$, so $G_{[T]}$ is not a subgroup of G.

Section 8

8.2. Similar to 8.1. (Table 8.1 provides a check.)

8.4. The group contains only the identity and reflection through the bisector of the odd angle (the one unequal to the other two).

8.5. There will be five rotations (including the identity) and five reflections.

8.6.

$\mu_0 \mapsto (a)$	$\mu_H \mapsto (a\ d)(b\ c)$
$\mu_{90} \mapsto (a\ b\ c\ d)$	$\mu_V \mapsto (a\ b)(c\ d)$
$\mu_{180} \mapsto (a\ c)(b\ d)$	$\mu_1 \mapsto (b\ d)$
$\mu_{270} \mapsto (a\ d\ c\ b)$	$\mu_2 \mapsto (a\ c)$

8.7. Denote $(a\ b)(c)(d)$ by α, and use the notation of Theorem 8.1. Then $d(\alpha(a), \alpha(c)) = d(b, c) \neq d(a, c)$.

8.8. No. For example, any two concentric circles would have identical symmetry groups.

8.9. $\{\mu_0,\ \mu_{180},\ \rho_1,\ \rho_2\}$