

Chapter 3, Section 1

24 July 2001

Exercise 3.1

$$M = 20 \text{ kg}, \omega_{nat} = 100 \text{ rad/s} \Rightarrow k = M \omega_{nat}^2 = 2(10^5) \text{ N/m}$$

$$q = 0.02 \cos(110t - 1.5) \text{ meter}$$

$$= \text{Re} [0.02 e^{i(\omega t - 1.5)}] \quad \omega = 110 \text{ rad/s}$$

$$Q(t) = M \ddot{q} + C \dot{q} + Kq$$

$$= \text{Re} [(-M\omega^2 + Ci\omega + K)(0.02) e^{i(\omega t - 1.5)}]$$

$$\text{But } \frac{K}{M} = \omega_{nat}^2 \text{ \& } C/M = 2\gamma \omega_{nat}, \text{ so}$$

$$Q(t) = M \text{Re} [(-\omega^2 + 2i\gamma \omega_{nat} \omega + \omega_{nat}^2)(0.02) e^{i(\omega t - 1.5)}]$$

$$\text{For } \gamma = 0 \text{ \& } \omega = 110:$$

$$Q = 20 \text{Re} [-42 e^{i(110t - 1.5)}]$$

$$\Rightarrow = -840 \cos(110t - 1.5) \text{ newton}$$

$$\text{For } \gamma = 0.4 \text{ \& } \omega = 110 \text{ rad/s:}$$

$$Q = 20 \text{Re} [(-210.0 + 8800i)(0.02) e^{i(110t - 1.5)}]$$

$$= \text{Re} [(-840 + 3520i) e^{i(110t - 1.5)}]$$

$$= \text{Re} [3619 e^{i1.8051} e^{i(110t - 1.5)}]$$

$$\Rightarrow = 3619 \cos(110t + 0.3051) \text{ newton}$$

Exercise 3.2

$$m\ddot{q} + c\dot{q} + kq = F\cos(\omega t)$$

$$m = 8 \text{ kg}, \quad \gamma = 0.25, \quad \omega_d = 10\pi \text{ rad/s}, \quad F/k = 0.002 \text{ meter}$$

$$\omega_{nat} = \frac{\omega_d}{(1-\gamma^2)^{1/2}} = 32.146 \text{ rad/s}$$

$$r = \frac{5.2(2\pi)}{\omega_{nat}} = 1.006976$$

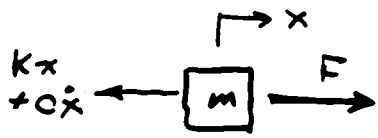
$$q = \frac{F}{k} |D(r, \gamma)| \cos(\omega t - \phi)$$

$$|D(r, \gamma)| = \frac{1}{[(1-r^2)^2 + 4\gamma^2 r^2]^{1/2}} = 1.9854$$

$$\phi = \tan^{-1} \frac{2\gamma r}{(1-r^2)} = -1.5430 + \pi = 1.5986 \text{ rad} = 91.593^\circ$$

$$\text{Thus } q = 0.003971 \cos(10.4\pi t - 1.5986) \text{ meter}$$

Exercise 3.3



$$M\ddot{x} + C\dot{x} + Kx = \text{Re}\{F e^{i\omega t}\}$$

$$M = 80 \text{ kg}, \omega_{nat} = 12.2(2\pi) \text{ rad/s}$$

$$K = M\omega_{nat}^2 = 4.701(10^5) \text{ N/m}$$

$$\omega = 12(2\pi) \text{ rad/s} \Rightarrow r = \frac{12}{12.2}$$

$$X = \frac{F}{K} \frac{1}{1 - r^2 + 2i\gamma r}$$

$$\text{Then } |\ddot{x}| = \omega^2 |x| = \frac{|F|}{K} \frac{\omega^2}{|1 - r^2 + 2i\gamma r|} < 15g$$

$$\text{so } |F| < \frac{15g K}{\omega^2} |1 - r^2 + 2i\gamma r|$$

$$\gamma = 0 \Rightarrow |F| < 395.6 \text{ N}$$

$$\gamma = 0.05 \Rightarrow F < |395.6 + 1196.5i| = 1260.2 \text{ N}$$

Exercise 3.4

Given $K = 5000 \text{ N/m}$, $M = 2 \text{ kg}$, $F = 20 \sin(\omega t) \text{ N}$ gives

$$q = -0.010 \cos(\omega t) \text{ meter.}$$

Find γ & ω

Solution: Write the force & response in complex form;

$$F = \text{Re}\{F e^{i\omega t}\}, \quad q = \text{Re}\{X e^{i\omega t}\}$$

$$\text{where } F = \frac{20}{i} \text{ \& } X = -0.010$$

$$\text{but } X = \frac{F}{k} D(r, \gamma) \Rightarrow D(r, \gamma) = \frac{kX}{F} = -2.5i = 2.5e^{-i\pi/2}$$

$$\text{but } D(r, \gamma) = |D| e^{-i\phi} \text{ so}$$

$$|D| = 2.5 \text{ \& } \phi = \pi/2$$

Whenever $\phi = \pi/2$, it must be that $r = 1$

$$\text{Thus } \omega = \omega_{nat} = \left(\frac{K}{M}\right)^{1/2} = 50 \text{ rad/s} \quad \Leftarrow$$

$$\text{Then, at } r = 1, |D| = \frac{1}{2\gamma} \Rightarrow \gamma = \frac{1}{2(2.5)} = 0.20 \quad \Leftarrow$$

Exercise 3.5

Given $m = 10 \text{ kg}$, $\omega_{nat} = 1000(2\pi) \text{ rad/s}$,
 $Q = 1200 \sin(\omega t) \text{ newton}$, $|q| = 0.0024 \text{ meter}$ at $\omega = \omega_{nat}$.
 Find $q(t)$ when $\omega = 0.95 \omega_{nat}$ & $\omega = 1.05 \omega_{nat}$

Solution: Steady state response in general is

$$Q(t) = \text{Re} \{ F e^{i\omega t} \} \Rightarrow q = \text{Re} \left\{ \frac{F}{k} |D| e^{i(\omega t - \phi)} \right\}$$

The given force is $Q = 1200 \sin(\omega t) = \text{Re} \left\{ \frac{1200}{i} e^{i\omega t} \right\}$

Thus $F = \frac{1200}{i}$

To find k , use $\omega_{nat} = \left(\frac{k}{m} \right)^{1/2} \Rightarrow k = \omega_{nat}^2 m = 3.948(10^8) \text{ N/m}$

Also when $\omega = \omega_{nat}$, $r = 1 \Rightarrow D = \frac{1}{1 - r^2 + 2i\zeta r} = \frac{1}{2i\zeta}$

Thus the amplitude at $\omega = \omega_{nat}$ is

$$|q| = 0.0024 = \frac{|F|}{k} |D| = \frac{1200}{3.948(10^8)} \frac{1}{2\zeta}$$

Solve: $\zeta = 6.333(10^{-4})$

At any r : $|D| = \frac{1}{[(1-r^2)^2 + 4\zeta^2 r^2]^{1/2}}$ & $\phi = \tan^{-1} \left(\frac{2\zeta r}{1-r^2} \right)$

When $\omega = 0.95 \omega_{nat} \Rightarrow r = 0.95$

$$|D| = 10.256 \quad \& \quad \phi = 0.01234 \text{ rad} = 0.707^\circ$$

$$q = \text{Re} \left\{ \frac{1200/i}{k} (10.256) e^{i(\omega t - \phi)} \right\}$$

$$= \text{Re} \left\{ \frac{3.117(10^{-5})}{i} e^{i(\omega t - 0.1234)} \right\}$$

$$= 3.117(10^{-5}) \sin(190\pi t - 0.01234) \text{ meter} \Leftarrow$$

When $\omega = 1050(2\pi) \text{ rad/s} \Rightarrow r = 1.05$

$$|D| = 9.755 \quad \& \quad \phi = 3.129 \text{ rad} = 179.26^\circ \quad (\text{note } 0 < \phi < \pi \text{ rad})$$

$$q = \text{Re} \left\{ \frac{1200(9.755)}{i k} e^{i(\omega t - 3.129)} \right\}$$

$$= 2.965(10^{-5}) \sin(210\pi t - 3.129) \text{ meter} \Leftarrow$$

Exercise 3.6

Given $F = 60 \sin(\omega t) + 80 \cos(\omega t) \Rightarrow |q| = 0.05 \text{ meter} @ \omega = 0$,

$\omega_{nat} = 20(2\pi) \text{ rad/s}$, $q = X \sin(\omega t)$ at $\omega = 18(2\pi) \text{ rad/s}$

Find X

Express the force as a complex variable;

$$F(t) = \text{Re} \left\{ \left(\frac{60}{i} + 80 \right) e^{i\omega t} \right\} = \text{Re} (100 e^{-0.6435i} e^{i\omega t})$$

$$\text{Then } x = \text{Re} (X e^{i\omega t})$$

where $X = \frac{F}{k} D(r, \gamma)$, $r = \omega / \omega_{nat}$

$$\text{At } r = \frac{18}{20}, \quad q = X \sin(\omega t) = \text{Re} \left(\frac{X}{i} e^{i\omega t} \right)$$

$$\text{At } r = 0, \quad |q| = 0.05 = \frac{|F|}{k} |D| \text{ but } |D| = 1 \text{ at } r = 0$$

$$\text{Thus } k = \frac{|F|}{0.05} = 2000 \text{ N/m}$$

Then $r = 0.9$ gives

$$\frac{X}{i} = \frac{100 e^{-0.6435i}}{2000} \quad D(r, \gamma) = 0.05 e^{-0.6435i} |D| e^{-i\phi}$$

Match the arguments of the terms on either side,

$$\text{using } \frac{1}{i} = e^{-i\pi/2} \Rightarrow -\frac{\pi}{2} = -0.6433 - \phi$$

$$\text{so } \phi = \frac{\pi}{2} - 0.6433 = 0.9275$$

$$\text{But } \tan \phi = \frac{2\gamma r}{1-r^2} \Rightarrow \gamma = \left[\frac{1-0.9^2}{2(0.9)} \right] \tan \phi = 0.1408$$

$$\text{Then } |D| = \frac{1}{[(1-0.9^2)^2 + 4\gamma^2(0.9^2)]^{1/2}} = 3.157$$

$$|X| = 0.05 (3.157) = 0.1579 \text{ meter} \quad \leftarrow$$

Exercise 3.7

Given $Q(t) = 400 \cos(\omega t)$, steady-state properties

1. $\omega = 40(2\pi) \text{ rad/s} \Rightarrow \dot{q} = \beta Q = \beta(400) \cos(80\pi t)$

2. $|q| = 0.016 \text{ meter @ } \omega = 25(2\pi) \text{ rad/s}$

3. $\omega = 42(2\pi) \text{ rad/s} \Rightarrow \text{graph of } q \text{ vs } t$

Find ω_{nat} , γ , k , C , $\max q$ in graph, $\dot{q}$ @ $\omega = 40(2\pi) \text{ rad/s}$

Solution:

From 1: Integrate $q = \frac{\beta(400)}{80\pi} \sin(80\pi t)$

The phase lag is $\phi = \pi/2$ because

$$Q = \text{Re}\{400 e^{i80\pi t}\}$$

$$q = \text{Re}\left\{\frac{\beta 400}{80\pi} \frac{1}{t} e^{i80\pi t}\right\}$$

But $\phi = \pi/2$ means $\omega = \omega_{nat}$ for any γ .

Thus $\omega_{nat} = 80\pi \text{ rad/s} = 40 \text{ Hz}$ $\Leftarrow$

From 2: $r = \frac{25}{\omega_{nat}} = \frac{25}{40} = 0.625$

$$q = \text{Re}\left\{\frac{F}{k} D(r, \gamma) e^{i\omega t}\right\}$$

$$|q| = 0.016 = \frac{|F|}{k} |D(r, \gamma)| = \frac{400}{k} \frac{1}{\{(1-r^2)^2 + 4\gamma^2 r^2\}^{1/2}}$$

From 3: $r = \frac{42}{\omega_{nat}} = \frac{42}{40} = 1.05$

In the graph let $q = |q| \cos(\omega t - \phi)$

Set $q = 0$ @ $t = 0.0025 \text{ sec} \Rightarrow \omega t - \phi = \pm \pi/2, \pm 3\pi/2$

Because $\dot{q} = -\omega |q| \sin(\omega t - \phi) > 0$, $\omega t - \phi = -\pi/2$ or $3\pi/2$

This gives $\phi = 84\pi(0.0025) + \frac{\pi}{2}$ or $84\pi(0.0025) - 3\pi/2$

However, $0 < \phi < \pi \Rightarrow \phi = 2.231$

but $\tan(\phi) = \frac{2\gamma r}{1-r^2} \Rightarrow \tan(2.231) = \frac{2\gamma(1.05)}{1-1.05^2}$

Solve: $\gamma = 0.06290$ $\Leftarrow$

To find K go back to the 2nd fact

$$0.016 = \frac{400}{K} \frac{1}{[(1-r^2)^2 + 4\zeta^2 r^2]^{1/2}} \text{ with } r=0.625$$

Thus

$$K = \frac{400}{0.016} \frac{1}{[(1-0.625^2)^2 + 4\zeta^2]^{1/2}}$$

$$= 4.069(10^4) \text{ N/m} \quad \Leftarrow$$

Next evaluate M using $\omega_{nat}^2 = \frac{K}{M}$,

$$\text{so } M = \frac{K}{(80\pi)^2} = 0.6442 \text{ kg} \quad \Leftarrow$$

Then $\frac{C}{M} = 2\zeta\omega_{nat}$ gives

$$C = 0.6442(2)(0.6290)(80\pi) = 26.37 \text{ N-m/s} \quad \Leftarrow$$

To find $|q|$ in the graph, set $r = \frac{\omega}{\omega_{nat}} = 1.05$

$$|q| = \frac{|F|}{K} |D(r, \zeta)|$$

$$= \frac{400}{4.069(10^4)} \frac{1}{[(1-1.05^2)^2 + 4\zeta^2(1.05^2)]^{1/2}}$$

$$= 0.00588 \text{ meter} \quad \Leftarrow$$

To find $\dot{q}$ at $r=1$ ($\omega = \omega_{nat}$)

$$q = \text{Re} \left\{ \frac{F}{K} D(r, \zeta) e^{i\omega t} \right\}$$

$$\text{so } \dot{q} = \text{Re} \left\{ i\omega \frac{F}{K} D(r, \zeta) e^{i\omega t} \right\}$$

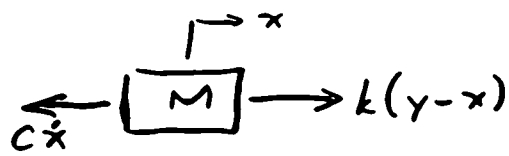
$$\text{Set } r=1 \Rightarrow D(1, \zeta) = \frac{1}{2\zeta i}$$

$$\dot{q} = \text{Re} \left\{ \omega \frac{F}{K} \frac{1}{2\zeta} e^{i\omega t} \right\}$$

$$= (80\pi) \frac{400}{K} \frac{1}{2\zeta} \text{Re} \{ e^{i\omega t} \}$$

$$= 19.63 \cos(80\pi t) \text{ m/s} \quad \Leftarrow$$

Exercise 3.8



$$M\ddot{x} + c\dot{x} + kx = ky$$

$$y = \operatorname{Re}\left(\frac{Y}{i} \exp(i\omega t)\right)$$

$$\text{Let } x = \operatorname{Re}\left(\frac{X}{i} \exp(i\omega t)\right)$$

$$\text{so } X = \frac{kY}{k + i\omega c - \omega^2 M}$$

$$x = \operatorname{Re}\left[\frac{Y}{i} \frac{k}{k + i\omega c - \omega^2 M} \exp(i\omega t)\right]$$

$$\begin{aligned} F = k(y - x) &= \operatorname{Re}\left[\frac{Y}{i} \left(1 - \frac{k}{k + i\omega c - \omega^2 M}\right) \exp(i\omega t)\right] \\ &= \operatorname{Re}\left[\frac{Y}{i} \frac{i\omega c - \omega^2 M}{k + i\omega c - \omega^2 M} \exp(i\omega t)\right] \end{aligned}$$

Rewrite x as

$$x = \operatorname{Re}\left[\frac{Y}{i} \frac{k/M}{k/M + i\omega c/M - \omega^2} \exp(i\omega t)\right]$$

$$= \operatorname{Re}\left[\frac{Y}{i} \frac{1}{1 + 2i\zeta r - r^2} \exp(i\omega t)\right], \quad r = 1.1$$

$$\text{Thus } |X| = |Y| \frac{1}{|1 + 2i\zeta r - r^2|}$$

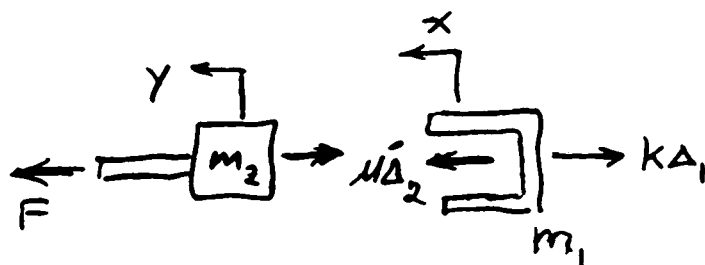
$$\text{Set } r = 1.10 \quad \zeta = 0.2 \Rightarrow |X| = 2.051 Y$$

$$\arg(X) = \arg(Y) - \phi$$

$$= \arg(Y) - \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

$$= \arg(Y) - 2.935 \text{ rad}$$

Exercise 3.9



Given $y = A \sin \omega t$,
 $m_1 = 0.5 \text{ kg}$,
 $m_2 = 1.0 \text{ kg}$,
 $k = 3.2(10^3) \text{ N/m}$,
 $\mu = 40 \text{ N-s/m}$,
 $A = 0.02 \text{ meter}$

Find amplitude & phase of F relative to y for $\omega = 150\pi$ & 170π .

Solution: Eqs of motion: $\Delta_1 = x$, $\Delta_2 = y - x$

Piston: $F - \mu(\dot{y} - \dot{x}) = m_2 \ddot{y}$

Tube: $\mu(\dot{y} - \dot{x}) - kx = m_1 \ddot{x}$

It is given that $y = A \sin(\omega t) = \text{Re} \left[\frac{A}{i} e^{i\omega t} \right]$

Harmonic variation $\Rightarrow$ Assume $x = \text{Re} \left[\frac{X}{i} e^{i\omega t} \right]$, $F = \text{Re} \left[\frac{\hat{F}}{i} e^{i\omega t} \right]$

Note: Using $\frac{1}{i}$ as a factor for x & F means that the polar angles of $\hat{F}$ and X are phase angles relative to y .

Substitute into eqs of motion & cancel $e^{i\omega t}$:

$$\hat{F} - \mu(i\omega)(A - X) = -m_2 \omega^2 A \quad (1)$$

$$\mu(i\omega)(A - X) - kX = -m_1 \omega^2 X \quad (2)$$

These are two simultaneous equations for F and X at any ω because $A = 0.02$:

$$\text{From (2): } X = \frac{i\omega \mu}{k + i\omega \mu - m_1 \omega^2} A$$

From (1):

$$\hat{F} = [(i\omega \mu - m_2 \omega^2) + \frac{\mu^2 \omega^2}{k + i\omega \mu - m_1 \omega^2}] A$$

For $\omega = 75 \text{ rad/s}$:

$$F = -104.88 + 0.99i = 104.88 e^{i3.132}$$

$$\text{or } |F| = 104.88 \text{ N @ } \phi = 3.132 \frac{180^\circ}{\pi} = 179.46^\circ \text{ ahead of } y \Leftarrow$$

For $\omega = 85 \text{ rad/s}$:

$$F = -152.63 + 0.99i = 152.63 e^{i3.135}$$

$$\text{or } |F| = 152.63 \text{ N @ } \phi = 3.135 \frac{180^\circ}{\pi} = 179.63^\circ \text{ ahead of } y \Leftarrow$$

Chapter 3, Section 2-3

24 July 2001

Exercise 3.10

Given plot of $|\theta|$ vs ω , $I_0 = 4 \text{ kg-m}^2$

Estimate ω_{nat} & ζ . Then find K , C , & $|T|$ for system.

Solution; Because rotation $\Rightarrow \theta = \varphi$, the eq of motion is

$$M\ddot{\theta} + C\dot{\theta} + K\theta = Re(Te^{i\omega t})$$

where $M = I_0$ for rotation.

Estimate ω_{nat} as the value of ω at the peak:

$$\omega_{nat} = 35 \text{ Hz} = 70\pi \text{ rad/s}$$

Two ways to estimate ζ :

$$\left. \begin{array}{l} \text{(a) At } \omega = 0, |\theta| = \frac{|F|}{K} \approx 0.30 \left(\frac{\pi}{180}\right) \\ \text{at } \omega = \omega_{nat}, |\theta| = \frac{|F|}{K} \frac{1}{2\zeta} \approx 2.70 \left(\frac{\pi}{180}\right) \end{array} \right\} \zeta = \frac{0.30}{2(2.70)} = 0.056$$

(b) Bandwidth method:

Half power points $\omega_1 = 33(2\pi)$, $\omega_2 = 37(2\pi) \text{ rad/s}$

$$\text{So } \Delta\omega = \omega_2 - \omega_1 = 4(2\pi) \text{ rad/s}$$

$$QF = \frac{\omega_{nat}}{\Delta\omega} = \frac{35}{4} = \frac{1}{2\zeta} \Rightarrow \zeta = \frac{4}{2(35)} = 0.057$$

Both values agree. Use $\zeta = 0.057$

$$\text{Then } \omega_{nat}^2 = \frac{K}{I_0} \Rightarrow K = I_0 \omega_{nat}^2 = 1.934(105) \text{ N/m} \Leftarrow$$

$$\zeta = \frac{C}{2(KI_0)^{1/2}} \Rightarrow C = 2\zeta(KI_0)^{1/2} = 100.3 \text{ N-s/m} \Leftarrow$$

$$\text{At } \omega = 0: \frac{|F|}{K} = 0.30 \left(\frac{\pi}{180}\right) \text{ \& } |F| \text{ is the torque } |T| \text{ (rotation)}$$

$$\text{So } |T| = 0.30 \left(\frac{\pi}{180}\right) K = 1012.9 \text{ N-m} \Leftarrow$$

Exercise 3.11

Given $Q(t) = F \cos(\omega t)$, $x = 4 \sin(\omega t)$ at $\omega = 100(2\pi)$ rad/s

$$|q| = 2.83 \text{ mm at } \omega = 105(2\pi) \text{ rad/s}$$

Find ϕ at $\omega = 105(2\pi)$, $|q|$ & ϕ at $\omega = 110(2\pi)$ rad/s

Solution: At $\omega = 200\pi$, $x = 4 \sin(\omega t) \equiv 4 \cos(\omega t - \frac{\pi}{2})$

$$\text{so } \phi = \frac{\pi}{2} \text{ at } \omega = 200\pi \Rightarrow \omega = \omega_{nat} = 200\pi \text{ rad/s}$$

$$\text{In general } |q| = \frac{F}{k} |D(r, \gamma)|$$

$$\text{At } \omega = \omega_{nat}, |D| = \frac{1}{2\gamma} \Rightarrow 4 = \frac{F}{k} \frac{1}{2\gamma}$$

$$\text{At } \omega = 105(2\pi), r = \frac{105}{100} \text{ \& } |q| = 2.83 = \frac{F}{k} |D(1.05, \gamma)|$$

$$\text{Thus } \frac{|x(r=1.05)|}{|x(r=1)|} = \frac{2.83}{4} = 0.7075 \approx \frac{1}{\sqrt{2}}$$

so $\omega = 105(2\pi)$ is the upper half power point

$$\text{Bandwidth } \Delta\omega = 2(210\pi - 200\pi) = 20\pi$$

$$QF = \frac{\omega_{nat}}{\Delta\omega} = \frac{200\pi}{20\pi} = 10 = \frac{1}{2\gamma} \Rightarrow \gamma = \frac{1}{20} = 0.05$$

$$\text{Then } 4 = \frac{F}{k} \frac{1}{2\gamma} \Rightarrow \frac{F}{k} = \frac{4}{10} = 0.4$$

$$|q| = \frac{F}{k} |D(r, \gamma)| \text{ \& } \phi = \tan^{-1}\left(\frac{2\gamma r}{1-r^2}\right)$$

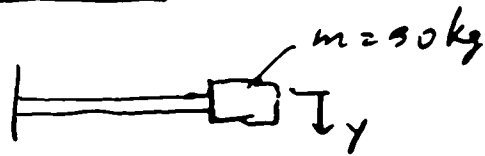
$$\text{For } r = 1.05, \phi = -0.7974 \text{ rad} + \pi = 134.3^\circ \quad \Leftarrow$$

For $\omega = 110(2\pi)$, $r = 1.1$:

$$|D| = 4.218 \text{ \& } \phi = -0.4295 + \pi = 2.713 \text{ rad} = 152.4^\circ \quad \Leftarrow$$

$$\text{so } |q| = \frac{F}{k} |D| = 0.4(4.218) = 1.687 \text{ mm}$$

Exercise 3.12



$$F = mg \Rightarrow y = 0.004 \text{ meter}$$

$$Q(t) = F_0 \sin(\omega t)$$

$$\omega = 0 \Rightarrow |y| = 0.02 \text{ meter}, \omega = \omega_{nat} \Rightarrow |y| = 0.4 \text{ meter}$$

$$\text{Static } y = \frac{mg}{k} = 0.004 \Rightarrow \omega_{nat}^2 = \frac{k}{m} = \frac{g}{0.004}$$

$$\omega_{nat} = \left(\frac{9.807}{0.004} \right)^{1/2} = 49.32 \text{ rad/s}$$

$$\text{Then } \omega \approx 0 \Rightarrow |y| = 0.02 = \frac{F_0}{k}$$

$$\omega = \omega_{nat} \Rightarrow |y| = 0.4 = \frac{F_0}{k} \frac{1}{2\gamma} \Rightarrow 0.4 = 0.02 \left(\frac{1}{2\gamma} \right)$$

$$\gamma = 0.025$$

$$\text{Assume viscous damping} \Rightarrow E_{dis} = \pi \omega C |y|^2$$

$$(P_{dis})_{av} = \frac{E_{dis}}{T} = \frac{1}{2} \omega^2 C |y|^2 \quad \& \quad C = 2\gamma \omega_{nat} M$$

$$(P_{dis})_{av} = \gamma \omega_{nat} \omega^2 M |y|^2$$

$$\text{At any } \omega: |y| = \frac{F_0}{k} |D(r, \gamma)|$$

$$(P_{dis})_{av} = \gamma \omega_{nat} \omega^2 M \left(\frac{F_0}{k} \right)^2 |D(r, \gamma)|^2$$

$$\text{Set } \omega = r \omega_{nat} \Rightarrow (P_{dis})_{av} = \gamma \omega_{nat}^3 M \left(\frac{F_0}{k} \right)^2 [r D(r, \gamma)]^2$$

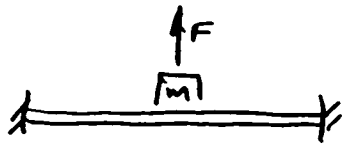
$$P_{dis} = 0.025 (49.32)^3 (50) (0.02)^2 [r D(r, \gamma)]^2$$

$$\Rightarrow = 60.70 \frac{r^2}{(1-r^2)^2 + 4(0.025)^2 r^2} \text{ watts}$$

$$\Rightarrow QF = \frac{\omega_{nat}}{\Delta \omega} = \frac{1}{2\gamma} \Rightarrow \Delta \omega = 2\gamma \omega_{nat} = 2.476 \text{ rad/s} = 0.39 \text{ Hz}$$

$$\Rightarrow \text{At } r = 0.9, (P_{dis})_{av} = 1290 \text{ watt}$$

Exercise 3.13



$$m = 40 \text{ kg}, \quad \eta_{\text{static}} = \frac{m g}{k}$$

$$\therefore \frac{k}{m} = \omega_{\text{nat}}^2 = \frac{g}{\eta_{\text{static}}} = \frac{9.807}{0.005}$$

$$\omega_{\text{nat}} = 44.200 \text{ rad/s}$$

(a) Structural damping

$$F = \text{Re}\{F_0 \exp(i\omega t)\} \quad \& \quad x = \text{Re}\{X \exp(i\omega t)\}$$

$$X = \frac{F}{k} \frac{1}{1 - \left(\frac{\omega}{\omega_{\text{nat}}}\right)^2 + i\gamma} \Rightarrow |X| = \frac{F}{k} \frac{1}{\left\{ \left[1 - \left(\frac{\omega}{\omega_{\text{nat}}}\right)^2\right]^2 + \gamma^2 \right\}^{1/2}}$$

$$\text{Max } |X| = 0.06 \text{ meter} \Rightarrow \omega = \omega_{\text{nat}}$$

$$0.06 = \frac{|F|}{k} \frac{1}{\gamma} \quad \text{but } k = m \omega_{\text{nat}}^2 = 78456 \text{ N/m}$$

$$\text{Set } |F| = 240 \Rightarrow \gamma = \frac{240}{0.06 k} = 0.05098 \quad \Leftarrow$$

$$\text{For } \omega = 9(2\pi) \text{ rad/s} \Rightarrow \omega/\omega_{\text{nat}} = 1.2768 = r$$

$$|X| = \frac{24}{k} \frac{1}{\left[(1-r^2)^2 + \gamma^2 \right]^{1/2}} = 0.00484 = 4.84 \text{ mm} \quad \Leftarrow$$

(b) Viscous damping:

$$\text{When } \omega/\omega_{\text{nat}} = 1 \Rightarrow |X| = \frac{|F|}{k} \frac{1}{2\gamma}$$

$$\text{Thus } \gamma = \frac{240}{(0.06 k)(2)} = 0.02549$$

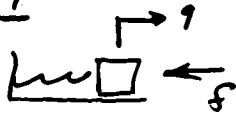
$$\text{For } \omega/\omega_{\text{nat}} = 1.2768 = r; |X| = \frac{240}{k} \frac{1}{\left[(1-r^2)^2 + 4\gamma^2 r^2 \right]^{1/2}}$$

$$|X| = 0.00483 = 4.83 \text{ mm}$$

Both types of damping give similar amplitudes

at $r = 1.2768$ because $|D| \approx 1/(1-r^2)$ away from $r = 1$ $\Leftarrow$

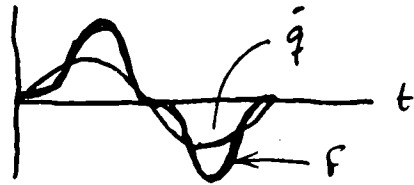
Exercise 3.17



$$f = \beta |\dot{q}|^2 \sin(\dot{q}) = \beta |\dot{q}|^2 \left(\frac{\dot{q}}{|\dot{q}|} \right) = \beta |\dot{q}| \dot{q}$$

Define delayed time: $t' = t - \phi/\omega$

Then $q = X \cos(\omega t') \Rightarrow \dot{q} = -X\omega \sin(\omega t')$



$$\begin{aligned} E_{dis} &= \int_0^{2\pi/\omega} f \dot{q} dt \\ &= 2 \int_0^{\pi/\omega} f \dot{q} dt \\ &= 2\beta X^3 \omega^3 \int_0^{\pi/\omega} \sin(\omega t')^3 dt' \\ E_{dis} &= \frac{8}{3} \beta \omega^2 X^3 \end{aligned}$$

$\Rightarrow$

$$\Rightarrow \text{Set } E_{dis} = \pi \omega C_{eq} X^2 \Rightarrow C_{eq} = \frac{8}{3\pi} \beta \omega X$$

$$\Rightarrow \gamma_{eq} = \frac{\alpha}{\pi k} = \frac{\omega C_{eq}}{k} = \frac{8}{3\pi} \frac{\beta \omega^2}{k} X$$

From eq (3.2.25)

$$X = \frac{F}{k} \frac{1}{(1-r^2 + i\gamma_{eq})}$$

$$\text{Set } k = M \omega_{nat}^2 \Rightarrow \gamma_{eq} = \frac{8}{3\pi} \frac{\beta}{M} r^2 X$$

$$\Rightarrow \frac{kX}{F} \left[1 - r^2 + i \left(\frac{8}{3\pi} \frac{\beta}{M} r^2 X \right) \right] = 1 \quad ; \text{quadratic eq for } X$$

Complex root is OK

Exercise 3.14

$$(P_{dis})_{av} = 1000 \text{ watt over } 100 < \frac{\omega}{2\pi} < 1000 \text{ Hz} \Rightarrow |X| = 0.008 \text{ m}$$

$$E_{dis} = P_{dis}(T) = \frac{2000\pi}{\omega} = \pi \omega C_{eq} |X|^2$$

$$\text{so } C_{eq} = \frac{2000}{\omega^2 |X|^2}$$

$$\text{Then } C_{eq} = \frac{\alpha}{\pi \omega} \text{ \& } \gamma_{eq} = \frac{\alpha}{\pi k} \Rightarrow \gamma_{eq} = \frac{\omega C_{eq}}{k} = \frac{2000}{\omega k |X|^2} = \frac{3.125(10^7)}{\omega k}$$

$$90^\circ \text{ phase lag} \Rightarrow 200 \text{ Hz} = \frac{\omega_{nat}}{2\pi} \Rightarrow \omega_{nat} = 400\pi$$

$$\text{Then } k = M \omega_{nat}^2 = 7.896(10^5) \text{ N/m}$$

$$\text{Thus } \gamma_{eq} = \frac{39.58}{\omega} = \frac{39.58}{r \omega_{nat}} = \frac{0.03150}{r} \quad \Leftarrow$$

$$\text{Set } |X| = \frac{|F|}{k} \frac{1}{[(1-r^2)^2 + \gamma_{eq}^2]^{1/2}}$$

$$\text{so } |F| = k |X| [(1-r^2)^2 + \gamma_{eq}^2]^{1/2} \\ = 7.896(10^5)(0.008) [(1-r^2)^2 + \frac{0.0315^2}{r^2}]^{1/2}$$

$$\omega = 100(2\pi) \Rightarrow r = 0.5 \Rightarrow |F| = 4754 \text{ N}$$

$$\omega = 200(2\pi) \Rightarrow r = 1 \Rightarrow |F| = 198.94 \text{ N}$$

$$\omega = 1000(2\pi) \Rightarrow r = 5 \Rightarrow |F| = 151.6 \text{ kN}$$

$$N_{\max}(|F|) \Rightarrow \frac{d|F|}{dr} = 0$$

$$\text{Let } U = (1-r^2)^2 + \frac{0.0315^2}{r^2} \Rightarrow |F| = 6317 U^{1/2}$$

$$\frac{d|F|}{dr} = 6317 \left(\frac{1}{2}\right) U^{-1/2} \frac{dU}{dr} = 0 \Rightarrow \frac{dU}{dr} = 0$$

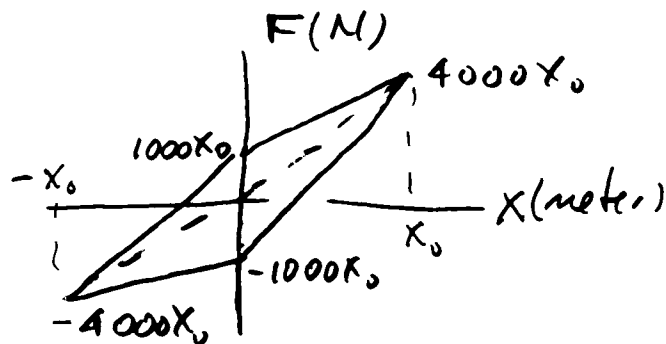
$$\text{i.e. } 2(1-r^2)(-2r) + 0.0315^2 \left(-\frac{2}{r^3}\right) = 0$$

$$\text{But } 100 \text{ Hz} < \omega < 1 \text{ kHz} \Leftrightarrow 0.5 < r < 5$$

The last term is very small in this range, so

$$\frac{d|F|}{dr} = 0 \Rightarrow r \approx 1 \Rightarrow \omega = \omega_{nat} \quad \Leftarrow$$

Exercise 3.15



Given suspension force
 vs x indep of ω , $M=10\text{kg}$.
 Find (a) equivalent
 linear K & C ,
 (b) x_{ss} when
 $F = 400 \sin(50t)$
 $+ 300 \cos(50t) \text{ N}$

Solution: The diagonal line represents the mean value
 which is the contribution of the spring, so

$$K_{eq} = 4000 \text{ N/m} \quad \Leftarrow$$

The area enclosed by the curve is E_{dis} for a cycle:

$$E_{dis} = 2 \left[\frac{1}{2} (2000x_0)(x_0) \right] \text{ (two triangles)}$$

$$\text{Equate } E_{dis} = \pi \omega C_{eq} X_0^2 \Rightarrow C_{eq} = \frac{2000}{\pi \omega} \quad \Leftarrow$$

Thus at $\omega = 50 \text{ rad/s}$:

$$M \ddot{x} + C_{eq} \dot{x} + K_{eq} x = 400 \sin(50t) + 300 \cos(50t)$$

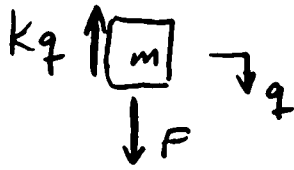
$$10 \ddot{x} + 12.733 \dot{x} + 4000 x = \text{Re} \left[\left(\frac{400}{i} + 300 \right) e^{i50t} \right]$$

$$\text{Set } x = \text{Re}(X e^{i50t})$$

$$[-10(50^2) + 12.733(50i) + 4000]X = \left(\frac{400}{i} + 300 \right)$$

$$X = -0.01485 + 0.01860i \text{ meter} \quad \Leftarrow$$

Exercise 3.16



When $q = \text{Re}[Y \exp(i\omega t)]$ then
 $M\ddot{q} + K(\omega)q = F = \text{Re}[F_0 \exp(i\omega t)]$
 give $[K(\omega) - \omega^2 M]Y = F_0$

$$Y = \frac{F_0}{K(\omega) - \omega^2 M}$$

Static deflection ($\omega = 0$) $\delta = \frac{Mg}{K(0)}$

Rewrite
$$Y = \frac{F_0}{K(0)} \frac{1}{\left[\frac{K(\omega)}{K(0)} - \omega^2 \frac{M}{K(0)} \right]}$$

but $\frac{M}{K(0)} = \frac{\delta}{g} \equiv \frac{1}{\omega_{nat}^2}$

Define $r = \frac{\omega}{\omega_{nat}}$, $\alpha = \frac{K(\omega)}{K(0)}$

so
$$Y = \frac{F_0}{mg} \delta \frac{1}{\alpha - r^2}$$

To evaluate, note that $E(0) = E_0$, so

$$\alpha = 1 + \left(\frac{E_{\infty}}{E_0} - 1 \right) \frac{\omega \tau}{\omega^2 \tau^2 + 1} (\omega \tau + i)$$

$$= 1 + \left(\frac{E_{\infty}}{E_0} - 1 \right) \frac{r \omega_{nat} \tau}{r^2 (\omega_{nat} \tau)^2 + 1} [r (\omega_{nat} \tau) + i]$$

Plot $|Y|$ vs r for $0.75 \leq r \leq 1.25$ with $\omega_{nat} \tau$ fixed

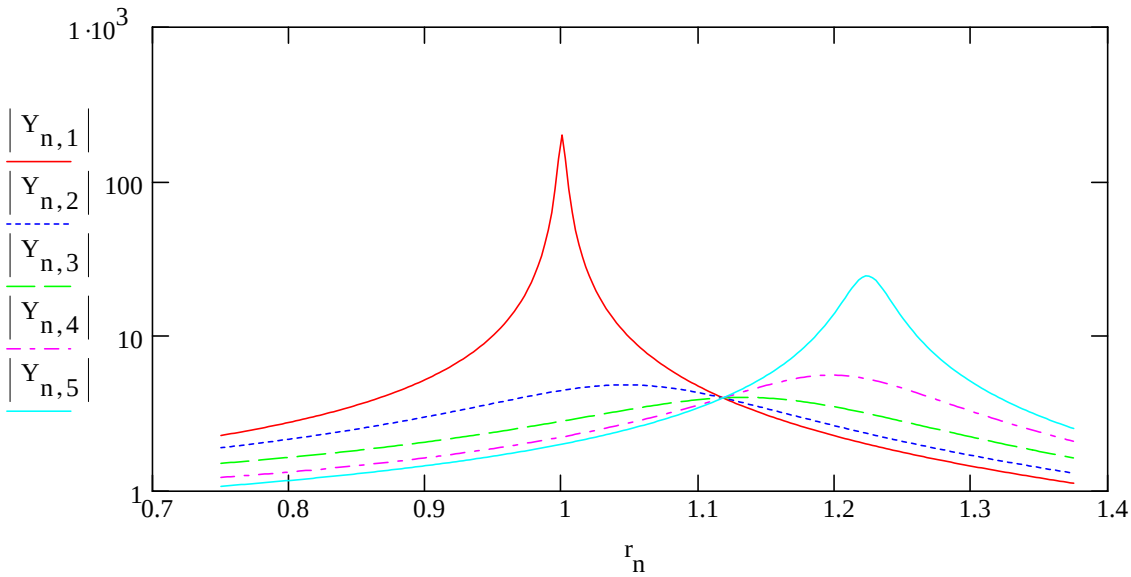
$$\alpha(r, \tau \omega_{nat}) := 1 + (1.5 - 1) \cdot \frac{r \cdot \tau \omega_{nat}}{(r \cdot \tau \omega_{nat})^2 + 1} \cdot (r \cdot \tau \omega_{nat} + i)$$

$$N := 251 \qquad n := 1..N \qquad r_n := 0.75 + (n - 1) \cdot \frac{0.50}{200}$$

$$\tau \omega_{nat_1} := 0.01 \qquad \tau \omega_{nat_2} := 0.5 \qquad \tau \omega_{nat_3} := 1.0 \qquad \tau \omega_{nat_4} := 2.0 \qquad \tau \omega_{nat_5} := 10.0$$

$$j := 1..5$$

$$Y_{n,j} := \frac{1}{\alpha(r_n, \tau \omega_{nat_j}) - (r_n)^2}$$



Third case, $\omega_{nat}\tau = 1$, gives the smallest peak value of $|Y|$

Quality factors:

Frequency for peaks: $\omega_{peak_1} := 1$ $\omega_{peak_2} := 1.05$ $\omega_{peak_3} := 1.14$

$$\omega_{peak_4} := 1.21 \qquad \omega_{peak_5} := 1.23$$

$$Y_{peak_j} := \max(|Y^{<j>}|)$$

Find half-power points by inspection:

$0.71 \cdot Y_{\text{peak}_1} = 142.007$

$\overrightarrow{\left|Y^{<1>T}\right|} =$

	95	96	97	98	99	100	101	102	103	104
1	33.1	39.4	48.7	63.3	89.4	141	200	141.9	89.5	63.2

$$\Delta \omega_1 := r_{102} - r_{100}$$

$0.71 \cdot Y_{\text{peak}_2} = 3.451$

$\overrightarrow{\left|Y^{<2>T}\right|} =$

	70	71	72	73	74	75	76	77	78	79
0	3.327	3.361	3.396	3.431	3.467	3.503	3.539	3.576	3.613	3.651

$$\Delta \omega_2 := r_{159} - r_{74}$$

$0.71 \cdot Y_{\text{peak}_3} = 2.862$

$\overrightarrow{\left|Y^{<3>T}\right|} =$

	96	97	98	99	100	101	102	103	104	105
1	2.709	2.732	2.756	2.78	2.804	2.828	2.853	2.879	2.904	2.93

$$\Delta \omega_3 := r_{199} - r_{101}$$

$0.71 \cdot Y_{\text{peak}_4} = 3.99$

$\overrightarrow{\left|Y^{<4>T}\right|} =$

	204	205	206	207	208	209	210	211	212	213
1	4.392	4.319	4.247	4.175	4.104	4.034	3.965	3.897	3.83	3.764

$$\Delta \omega_4 := r_{210} - r_{148}$$

$0.71 \cdot Y_{\text{peak}_5} = 17.451$

$\overrightarrow{\left|Y^{<5>T}\right|} =$

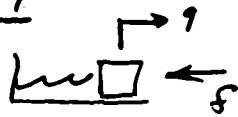
	180	181	182	183	184	185	186	187	188	189
0	14.27	15.32	16.48	17.74	19.1	20.51	21.89	23.13	24.07	24.58

$$\Delta \omega_5 := r_{183} - r_{149}$$

$$QF_j := \frac{\omega_{\text{peak}_j}}{\Delta \omega_j}$$

$$QF^T = (200 \quad 4.941 \quad 4.653 \quad 7.806 \quad 14.471)$$

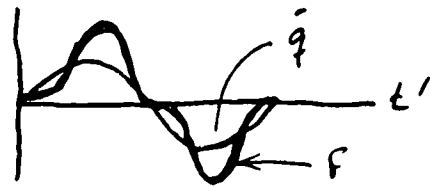
Exercise 3.17



$$f = \beta |\dot{q}|^2 \sin(\dot{q}) = \beta |\dot{q}|^2 \left(\frac{\dot{q}}{|\dot{q}|} \right) = \beta |\dot{q}| \dot{q}$$

Define delayed time: $t' = t - \phi/\omega$

Then $q = X \cos(\omega t')$ $\Rightarrow \dot{q} = -X\omega \sin(\omega t')$



$$\begin{aligned} E_{dis} &= \int_0^{2\pi/\omega} f \dot{q} dt \\ &= 2 \int_0^{\pi/\omega} f \dot{q} dt \\ &= 2\beta X^3 \omega^3 \int_0^{\pi/\omega} \sin^3(\omega t') dt' \end{aligned}$$

$$E_{dis} = \frac{8}{3} \beta \omega^2 X^3 \quad \Leftarrow$$

Set $E_{dis} = \pi \omega C_{eq} X^2 \Rightarrow C_{eq} = \frac{8}{3\pi} \beta \omega X \quad \Leftarrow$

$$\gamma_{eq} = \frac{\alpha}{\pi k} = \frac{\omega C_{eq}}{k} = \frac{8}{3\pi} \frac{\beta \omega^2}{k} X \quad \Leftarrow$$

From eq (3.2.23)

$$X = \frac{F}{k} \frac{1}{(1 - r^2 + i\gamma_{eq})}$$

Set $k = M \omega_{nat}^2 \Rightarrow \gamma_{eq} = \frac{8}{3\pi} \frac{\beta}{M} r^2 X$

$$\frac{kX}{F} \left[1 - r^2 + i \left(\frac{8}{3\pi} \frac{\beta}{M} r^2 X \right) \right] = 1 \quad \Leftarrow$$

quadratic eq for X, Complex root is OK

Exercise 3.18

$$M\ddot{q} + Kq = F \cos(\omega_{nat} t), \quad q(0) = \dot{q}(0) = 0$$

$$M = 100 \text{ kg} \quad \& \quad \frac{Mg}{K} = 0.005 \text{ meter} \Rightarrow K = \frac{100(9.807)}{0.005}$$

$$\omega_{nat} = \left(\frac{K}{M}\right)^{1/2} = 44.29 \text{ rad/s}$$

$$\frac{F}{K} = 0.0004 \text{ meter} \Rightarrow F = 78.46 \text{ N}$$

To use eq (3.3.4), $Q(t) = F \cos(\omega_{nat} t) \Rightarrow \psi = 0$

$$\text{Thus } q = \frac{F}{2M\omega_{nat}} t \sin(\omega_{nat} t) + C_1 \cos(\omega_{nat} t) + C_2 \sin(\omega_{nat} t)$$

Satisfy i.c. $q(0) = C_1 = 0$

$$\dot{q}(0) = \omega_{nat} C_2 = 0$$

$$\text{Thus } q = 0.008858 t \sin(44.29 t) \text{ meter} \quad \leftarrow$$

A conservative estimate for $|q| < 0.020$ meters is based on the envelope function:

$$0.008858 t < 0.02$$

$$t < 2.258 \text{ sec} \quad \leftarrow$$

If the force were $F \sin(\omega_{nat} t)$, the particular solution would be

$$q_p = -\frac{F}{2M\omega_{nat}} t \cos(\omega_{nat} t)$$

In that case q_c would not be zero, so setting

$|q_p + q_c| < 0.02$ would lead to a different result