Lecture 1

State-Space Linear Systems

Exercises

1.1 (Block diagram decomposition). Consider a system P_1 that maps each input u to the solutions y of

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Represent this system in terms of a block diagram consisting only of

- *integrator* systems, represented by the symbol $\boxed{\int}$, that map their input $u(\cdot) \in \mathbb{R}$ to the solution $y(\cdot) \in \mathbb{R}$ of $\dot{y} = u$;

- *summation* blocks, represented by the symbol $\boxed{\sum}$, that map their input vector $u(\cdot) \in \mathbb{R}^k$ to the scalar output $y(t) = \sum_{i=1}^k u_i(t)$, $\forall t \geqslant 0$; and

- *gain* memoryless systems, represented by the symbol $\boxed{g}$, that map their input $u(\cdot) \in \mathbb{R}$ to the output $y(t) = g u(t) \in \mathbb{R}$, $\forall t \geqslant 0$ for some $g \in \mathbb{R}$. □
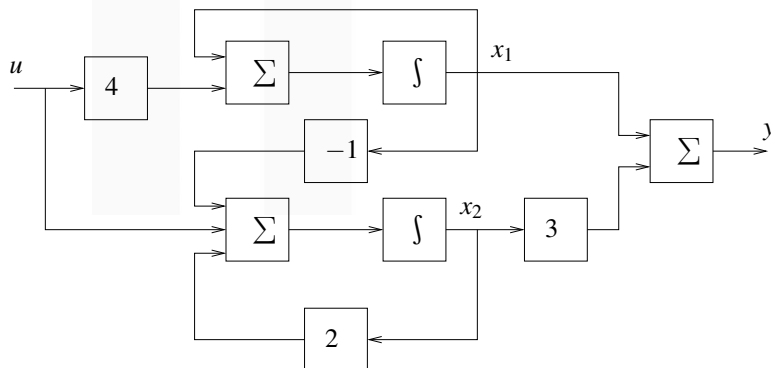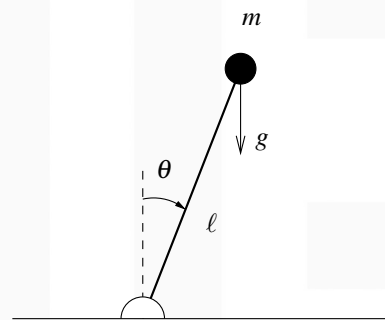

Figure 1.1. Solution to Exercise 1.1

Lecture 2

Linearization

Exercises

m

g

θ

ℓ

From Newton's law:

$$m\ell^2\ddot{\theta} = mg\ell\sin\theta - b\dot{\theta} + T,$$

where T denotes a torque applied at the base, and g is the gravitational acceleration.

Figure 2.1. Inverted pendulum.

2.3 (Local linearization around equilibrium: saturated inverted pendulum). Consider the inverted pendulum in Figure 2.1 and assume the input and output to the system are the signals u and y defined as

$$T = \text{sat}(u), \qquad\qquad y = \theta,$$

where "sat" denotes the unit-slope saturation function that truncates u at $+1$ and -1.

(a) Linearize this system around the equilibrium point for which $\theta = 0$.

(b) Linearize this system around the equilibrium point for which $\theta = \pi$ (assume that the pendulum is free to rotate all the way to this configuration without hitting the table).

(c) Linearize this system around the equilibrium point for which $\theta = \frac{\pi}{4}$.

Does such an equilibrium point always exist?

(d) Assume that $b = 1/2$ and $mg\ell = 1/4$. Compute the torque $T(t)$ needed for the pendulum to fall from $\theta(0) = 0$ with constant velocity $\dot{\theta}(t) = 1$, $\forall t \geqslant 0$. Linearize the system around this trajectory. □

2

João P. Hespanha

Solution to Exercise 2.3. Setting $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}'$, the system dynamics are given by

$$\dot{x} = f(x,u) := \begin{bmatrix} x_2 \\ \frac{g}{\ell} \sin x_1 - \frac{b}{m\ell^2} x_2 + \frac{1}{m\ell^2} \operatorname{sat}(u) \end{bmatrix}$$

$$y = g(x,u) := x_1.$$

(a) The linearized dynamics around an arbitrary equilibrium point $x^{\mathrm{eq}}, u^{\mathrm{eq}}$ are then given by

$$\delta x = \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x^{\mathrm{eq}} \\ u=u^{\mathrm{eq}}}} \delta x + \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x^{\mathrm{eq}} \\ u=u^{\mathrm{eq}}}} \delta u$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} \cos x_1^{\mathrm{eq}} & -\frac{b}{m\ell^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \frac{d \operatorname{sat}(u^{\mathrm{eq}})}{du} \end{bmatrix} \delta u$$

$$\delta y = \frac{\partial g(x,u)}{\partial x}\Big|_{\substack{x=x^{\mathrm{eq}} \\ u=u^{\mathrm{eq}}}} \delta x + \frac{\partial g(x,u)}{\partial u}\Big|_{\substack{x=x^{\mathrm{eq}} \\ u=u^{\mathrm{eq}}}} \delta u$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x,$$

where $\delta x := x - x^{\mathrm{eq}}$ and $\delta u = u - u^{\mathrm{eq}}$.

The only equilibrium point consistent with $x_1^{\mathrm{eq}} = \theta = 0$ is $x_2^{\mathrm{eq}} = \dot{\theta} = 0$ and $u^{\mathrm{eq}} = 0$ and the linearization is given by

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} \delta u \qquad \delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x.$$

Note that the derivative of sat around zero is equal to one.

(b) The only equilibrium point consistent with $x_1^{\mathrm{eq}} = \theta = \pi$ is $x_2^{\mathrm{eq}} = \dot{\theta} = 0$ and $u^{\mathrm{eq}} = 0$ and the linearization linearization is given by

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} \delta u \qquad \delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x.$$

(c) For $x_1^{\mathrm{eq}} = \theta = \pi/4$ to be an equilibrium point, we need

$$0 = mg\ell \sin \frac{\pi}{4} + \operatorname{sat}(u^{\mathrm{eq}})$$

to have a solution. This means that we must have

$$mg\ell \frac{\sqrt{2}}{2} \in [-1, 1].$$

Except for the extreme case of $|u^{\mathrm{eq}}| = 1$, we will always have $\frac{d \operatorname{sat}(u^{\mathrm{eq}})}{du} = 1$ and therefore the linearization is given by

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} \frac{\sqrt{2}}{2} & -\frac{b}{m\ell^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} \delta u \qquad \delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x.$$

(d) We are seeking for a solution of the form

$$\theta(t) = t, \qquad \dot{\theta}(t) = 1, \qquad \ddot{\theta}(t) = 0, \qquad \forall t \geqslant 0.$$

3

Lecture 2

Substituting θ and $\dot{\theta}$ in Newton's law we obtain

$$0 = mg\ell \sin t - b + T(t) \quad \Leftrightarrow \quad T(t) = b - mg\ell \sin t = \frac{1}{2} - \frac{\sin t}{4} \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad \forall t \geq 0.$$

Attention! Note the time-varying nature of the linearized system.

The linearized dynamics around this trajectory are then given by

$$\delta \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} \cos t & -\frac{b}{m\ell^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} \delta u \qquad \delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x. \qquad \blacksquare$$

2.4 (Local linearization around equilibrium: pendulum). The following equation models the motion of a frictionless pendulum:

$$\ddot{\theta} + k \sin \theta = \tau$$

where $\theta \in \mathbb{R}$ is the angle of the pendulum with the vertical, $\tau \in \mathbb{R}$ an applied torque, and k a positive constant.

(a) Compute a state-space model for the system when $u := \tau$ is viewed as the input and $y := \theta$ as the output. Write the model in the form

$$\dot{x} = f(x, u) \qquad y = g(x, u)$$

for appropriate functions f and g.

(b) Find the equilibrium points of this system corresponding to the constant input $\tau(t) = 0$, $t \geq 0$.

Hint: There are many.

(c) Compute the linearization of the system around the solution $\tau(t) = \theta(t) = \dot{\theta}(t) = 0$, $t \geq 0$.

Hint: Do not forget the output equations. $\qquad \square$

Solution to Exercise 2.4. (a) Defining $x := \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ we have

$$\dot{x} = f(x, u), \qquad y = g(x, u)$$

where

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u\right) := \begin{bmatrix} x_2 \\ -k \sin x_1 + u \end{bmatrix}, \qquad g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u\right) := x_1.$$

(b) Setting $f(x, u) = 0$, with $u = 0$, we obtain

$$\begin{cases} x_2 = 0 \\ -k \sin x_1 = 0 \end{cases} \quad \Leftrightarrow \quad x_1 = \pi m,$$

where m can be any integer in $\mathbb{Z}$. Therefore the equilibrium points are pairs $(x_{\text{eq}}, u_{\text{eq}})$ of the form

$$x_{\text{eq}} = \begin{bmatrix} \pi m \\ 0 \end{bmatrix}, \, m \in \mathbb{Z} \qquad u_{\text{eq}} = 0.$$

4

João P. Hespanha

(c)

$$\Delta x := x - x_0, \qquad \Delta u := u - u_0, \qquad \Delta y := y - y_0,$$

where $x_0(t) = 0$, $u_0(t) = 0$, $y_0(t) = 0$, $t \geq 0$ is the solution around which we are doing the linearization, we get

$$\dot{\Delta x} = A\Delta x + B\Delta u, \qquad \Delta y = C\Delta x$$

with

$$A := \frac{\partial f}{\partial x}(x_0, u_0) = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}, \quad B := \frac{\partial f}{\partial u}(x_0, u_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C := \frac{\partial g}{\partial x}(x_0, u_0) = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

∎

2.5 (Local linearization around equilibrium: nonlinear input). Consider the nonlinear system

$$\ddot{y} + \dot{y} + y = u^2 - 1.$$

(a) Compute a state-space representation for the system with input u and output y. Write the model in the form

$$\dot{x} = f(x, u) \qquad y = g(x, u)$$

for appropriate functions f and g.

(b) Linearize the system around the solution $y(t) = 0$, $u(t) = 1$, $\forall t \geq 0$. □

Solution to Exercise 2.5. Defining $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, we can write

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{y} - y + u^2 - 1 \end{bmatrix} = f(x, u),$$
$$y = g(x, u)$$

with

$$f(x, u) := \begin{bmatrix} x_2 \\ -x_1 - x_2 + u^2 - 1 \end{bmatrix}, \qquad g(x, u) = x_1$$

The linearized system around the equilibrium point $x = 0$, $u = 1$ is given by

$$\dot{\delta x} = A\,\delta x + b\,\delta u$$
$$\delta y = c\,\delta x + d\,\delta u$$

with

$$A = \frac{\partial f}{\partial x}\bigg|_{x=0, u=1} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad b = \frac{\partial f}{\partial u}\bigg|_{x=0, u=1} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$
$$c = \frac{\partial g}{\partial x}\bigg|_{x=0, u=1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad d = \frac{\partial g}{\partial u}\bigg|_{x=0, u=1} = \begin{bmatrix} 0 \end{bmatrix}.$$

∎

2.6 (Local linearization around equilibrium: one-link robot). Consider the one-link robot in Figure 2.2, where θ denotes the angle of the link with the horizontal, τ the torque applied at the base, (x, y) the position of the tip, ℓ the length of the link, I its moment of inertia, m the mass at the tip, g gravity's acceleration, and b a friction coefficient. This system evolves according to the following equation:

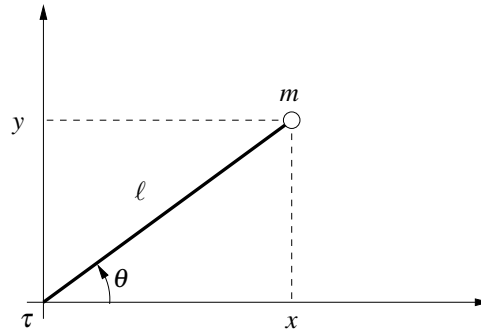$$I\ddot{\theta} = -b\dot{\theta} - gm\cos\theta + \tau.$$

Lecture 2

Figure 2.2. One-link robotic manipulator.

(a) Compute the state-space model for the system when $u = \tau$ is regarded as the input and the vertical position of the tip y is regarded as the output.

Please denote the state vector by z to avoid confusion with the horizontal position of the tip x, and write the model in the form

$$\dot{z} = f(z,u) \qquad\qquad y = g(z,u)$$

for appropriate functions f and g.

Hint: Do not forget the output equation!

(b) Show that $\theta(t) = \pi/2$, $\tau(t) = 0$, $\forall t \geqslant 0$ is a solution to the system and compute its linearization around this solution.

From your answer, can you predict if there will be problems when one wants to control the tip position close to this configuration just using feedback from y? □

Solution to Exercise 2.6. (a) Let us pick the state vector as $z := \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$. Then

$$\dot{z} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{b}{I}\dot{\theta} - \frac{gm}{I}\cos\theta + \frac{\tau}{I} \end{bmatrix} = f(z,u),$$

where

$$f(z,u) := \begin{bmatrix} z_2 \\ -\frac{b}{I}z_2 - \frac{gm}{I}\cos z_1 + \frac{u}{I} \end{bmatrix}.$$

Moreover,

$$y = \ell\sin\theta = g(z,u),$$

with

$$g(z,u) := \ell\sin z_1.$$

(b) For $z_0(t) = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}$ and $u_0(t) = 0, \geqslant 0$ we have

$$f(z_0,u) = \begin{bmatrix} 0 \\ -\frac{gm}{I}\cos\frac{\pi}{2} \end{bmatrix} = 0 = \dot{z}_0, \qquad\qquad \forall t \geqslant 0.$$

Therefore this pair z_0, u_0 satisfied the differential equation.

6

João P. Hespanha

The linearization around this solution is given by

$$\dot{\delta z} = A\delta z + B\delta u$$
$$\delta y = C\delta z + D\delta u,$$

where

$$A := \frac{\partial f}{\partial z}(z_0, u_0) = \begin{bmatrix} 0 & 1 \\ \frac{gm}{I}\sin\frac{\pi}{2} & -\frac{b}{I} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{gm}{I} & -\frac{b}{I} \end{bmatrix}, \qquad B := \frac{\partial f}{\partial z}(z_0, u_0) = \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix}$$

$$C := \frac{\partial g}{\partial z}(z_0, u_0) = \begin{bmatrix} \ell\cos\frac{\pi}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad D := \frac{\partial g}{\partial z}(z_0, u_0) = 0.$$

The fact that C and D are both zero means that around this configuration the torque has little affect on the output (i.e., the y position). In fact, it only affects y through second order effects that were neglected by the linearization. This indicates that it will be difficult to control y around this configuration. ∎

2.7 (Local linearization around trajectory: unicycle). A single-wheel cart (unicycle) moving on the plane with linear velocity v and angular velocity ω can be modeled by the nonlinear system

$$\dot{p}_x = v\cos\theta, \qquad\qquad \dot{p}_y = v\sin\theta, \qquad\qquad \dot{\theta} = \omega, \qquad (2.1)$$

where (p_x, p_y) denote the Cartesian coordinates of the wheel and θ its orientation. Regard this as a system with input $u := \begin{bmatrix} v & \omega \end{bmatrix}' \in \mathbb{R}^2$.

(a) Construct a state-space model for this system with state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} p_x\cos\theta + (p_y - 1)\sin\theta \\ -p_x\sin\theta + (p_y - 1)\cos\theta \\ \theta \end{bmatrix}$$

and output $y := \begin{bmatrix} x_1 & x_2 \end{bmatrix}' \in \mathbb{R}^2$.

(b) Compute a local linearization for this system around the equilibrium point $x^{\text{eq}} = 0$, $u^{\text{eq}} = 0$.

(c) Show that $\omega(t) = v(t) = 1$, $p_x(t) = \sin t$, $p_y(t) = 1 - \cos t$, $\theta(t) = t$, $\forall t \geq 0$ is a solution to the system.

(d) Show that a local linearization of the system around this trajectory results in an LTI system. □

Attention! Writing the system (2.1) in the carefully chosen coordinates x_1, x_2, x_3 is crucial to getting an LTI linearization. If one tried to linearize this system in the original coordinates p_x, p_y, θ, one would get an LTV system.

Solution to Exercise 2.7. (a)

$$\dot{x} = \begin{bmatrix} v + \omega x_2 \\ -\omega x_1 \\ \omega \end{bmatrix}$$

(b)

$$\dot{\delta x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\delta u, \qquad \delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\delta x.$$

Lecture 2

(c) In x coordinates the candidate solution is

$$x(t) = \begin{bmatrix} p_x\cos\theta + (p_y-1)\sin\theta \\ -p_x\sin\theta + (p_y-1)\cos\theta \\ \theta \end{bmatrix} = \begin{bmatrix} \sin t\cos t - \cos t\sin t \\ -\sin t\sin t - \cos t\cos t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix}$$

Therefore

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v+\omega x_2 \\ -\omega x_1 \\ \omega \end{bmatrix}.$$

(d)

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\delta u, \qquad \delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\delta x. \qquad ■$$

2.8 (Feedback linearization controller). Consider the inverted pendulum in Figure 2.1.

(a) Assume that you can directly control the system in torque (i.e., that the control input is $u = T$).

Design a feedback linearization controller to drive the pendulum to the upright position. Use the following values for the parameters: $\ell = 1$ m, $m = 1$ kg, $b = 0.1$ N m^{-1} s^{-1}, and $g = 9.8$ m s^{-2}. Verify the performance of your system in the presence of measurement noise using Simulink.

(b) Assume now that the pendulum is mounted on a cart and that you can control the cart's jerk, which is the derivative of its acceleration a. In this case,

$$T = -m\ell a\cos\theta, \qquad\qquad \dot{a} = u.$$

Design a feedback linearization controller for the new system.

What happens around $\theta = \pm\pi/2$? Note that, unfortunately, the pendulum needs to pass by one of these points for a swing-up, which is the motion from $\theta = \pi$ (pendulum down) to $\theta = 0$ (pendulum upright). □

Solution to Exercise 2.8. (a) Since the system is given by

$$m\ell^2\ddot{\theta} = mg\ell\sin\theta - b\dot{\theta} + u,$$

a proportional-derivative (PD) feedback linearization control is given by

$$u = -mg\ell\sin\theta + b\dot{\theta} + m\ell^2\left(-K_D\dot{\theta} - K_P\theta\right),$$

where the constants K_D and K_P should be chosen so as to achieve good performance for the following closed-loop system:

$$\ddot{\theta} = -K_D\dot{\theta} - K_P\theta.$$

(b) It is now convenient to write the system in the form

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta - \frac{b\dot{\theta}}{m\ell^2} - \frac{1}{\ell}a\cos\theta$$
$$\dot{a} = u.$$

8

João P. Hespanha

Defining

$$z := \frac{g \sin \theta}{\ell} - \frac{b \dot{\theta}}{m \ell^2} - \frac{a \cos \theta}{\ell},$$

we conclude that we can re-write the system as

$$\ddot{\theta} = z$$

$$\dot{z} = \frac{g \dot{\theta} \cos \theta}{\ell} - \frac{b z}{m \ell^2} + \frac{a \dot{\theta} \sin \theta}{\ell} - \frac{\cos \theta}{\ell} u$$

A feedback linearization control can then be given by

$$\frac{g \dot{\theta} \cos \theta}{\ell} - \frac{b z}{m \ell^2} + \frac{a \dot{\theta} \sin \theta}{\ell} - \frac{\cos \theta}{\ell} u = v \quad \Leftrightarrow \quad u = \frac{\ell}{\cos \theta} \left(\frac{g \dot{\theta} \cos \theta}{\ell} - \frac{b z}{m \ell^2} + \frac{a \dot{\theta} \sin \theta}{\ell} - v \right),$$

where v is chosen so as to achieve good performance for the following closed-loop system:

$$\ddot{\theta} = z$$

$$\dot{z} = v. \qquad \blacksquare$$

9