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CHAPTER 1

Solution 1.1

(a) For $k = 2$, $(A+B)^2 = A^2 + AB + BA + B^2$. If $AB = BA$, then $(A+B)^2 = A^2 + 2AB + B^2$. In general if $AB = BA$, then the k-fold product $(A+B)^k$ can be written as a sum of terms of the form A^jB^{k-j}, $j = 0, \ldots, k$. The number of terms that can be written as A^jB^{k-j} is given by the binomial coefficient $\begin{pmatrix} k \\ j \end{pmatrix}$. Therefore $AB = BA$ implies

$$(A+B)^k = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} A^jB^{k-j}$$

(b) Write

$$\det\left[\lambda I - A(t)\right] = \lambda^n + a_{n-1}(t)\lambda^{n-1} + \cdots + a_1(t)\lambda + a_0(t)$$

where invertibility of $A(t)$ implies $a_0(t) \neq 0$. The Cayley-Hamilton theorem implies

$$A^n(t) + a_{n-1}(t)A^{n-1}(t) + \cdots + a_0(t)I = 0$$

for all t. Multiplying through by $A^{-1}(t)$ yields

$$A^{-1}(t) = \frac{-a_1(t)I - \cdots - a_{n-1}(t)A^{n-2}(t) - A^{n-1}(t)}{a_0(t)}$$

for all t. Since $a_0(t) = \det\left[-A(t)\right]$, $|a_0(t)| = |\det A(t)|$. Assume $\varepsilon > 0$ is such that $|\det A(t)| \geq \varepsilon$ for all t. Since $\|A(t)\| \leq \alpha$ we have $|a_{ij}(t)| \leq \alpha$, and thus there exists a γ such that $|a_j(t)| \leq \gamma$ for all t. Then, for all t,

$$\|A^{-1}(t)\| = \frac{\|a_1(t)I + \cdots + A^{n-1}(t)\|}{|\det A(t)|}$$

$$\leq \frac{\gamma + \gamma\alpha + \cdots + \alpha^{n-1}}{\varepsilon} \triangleq \beta$$

Solution 1.2

(a) If λ is an eigenvalue of A, then recursive use of $Ap = \lambda p$ shows that λ^k is an eigenvalue of A^k. However to show multiplicities are preserved is more difficult, and apparently requires Jordan form, or at least results on similarity to upper triangular form.
(b) If λ is an eigenvalue of invertible A, then λ is nonzero and $Ap = \lambda p$ implies $A^{-1}p = (1/\lambda)p$. As in **(a)**, addressing preservation of multiplicities is more difficult.
(c) A^T has eigenvalues $\lambda_1, \ldots, \lambda_n$ since $\det(\lambda I - A^T) = \det(\lambda I - A)^T = \det(\lambda I - A)$.
(d) A^H has eigenvalues $\bar{\lambda}_1, \ldots, \bar{\lambda}_n$ using **(c)** and the fact that the determinant (sum of products) of a conjugate is the conjugate of the determinant. That is

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$$\det(\lambda I - A^H) = \det(\bar{\lambda} I - A)^H = \overline{\det(\bar{\lambda} I - A)}$$

(e) αA has eigenvalues $\alpha\lambda_1, \ldots, \alpha\lambda_n$ since $Ap = \lambda p$ implies $(\alpha A)p = (\alpha\lambda)p$.
(f) Eigenvalues of A^TA are not nicely related to eigenvalues of A. Consider the example

$$A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}$$

where the eigenvalues of A are both zero, and the eigenvalues of A^TA are $0, \alpha$. (If A is symmetric, then **(a)** applies.)

Solution 1.3

(a) If the eigenvalues of A are all zero, then $\det(\lambda I - A) = \lambda^n$ and the Cayley-Hamilton theorem shows that A is nilpotent. On the other hand if one eigenvalue, say λ_1 is nonzero, let p be a corresponding eigenvector. Then $A^kp = \lambda_1^kp \neq 0$ for all $k \geq 0$, and A cannot be nilpotent.
(b) Suppose Q is real and symmetric, and λ is an eigenvalue of Q. Then $\bar{\lambda}$ also is an eigenvalue. From the eigenvalue/eigenvector equation $Qp = \lambda p$ we get $p^HQp = \lambda p^Hp$. Also $Q\bar{p} = \bar{\lambda}\bar{p}$, and transposing gives $p^HQp = \bar{\lambda}p^Hp$. Subtracting the two results gives $(\lambda - \bar{\lambda})p^Hp = 0$. Since $p \neq 0$, this gives $\lambda = \bar{\lambda}$, that is, λ is real.
(c) If A is upper triangular, then $\lambda I - A$ is upper triangular. Recursive Laplace expansion of the determinant about the first column gives

$$\det(\lambda I - A) = (\lambda - a_{11}) \cdots (\lambda - a_{nn})$$

which implies the eigenvalues of A are the diagonal entries $a_{11}, \ldots, a_{nn}$.

Solution 1.4

(a)

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ implies } A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ implies } \|A\| = 1$$

(b)

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \text{ implies } A^TA = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Then

$$\det(\lambda I - A^TA) = (\lambda - 16)(\lambda - 4)$$

which implies $\|A\| = 4$.
(c)

$$A = \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix} \text{ implies } A^HA = \begin{bmatrix} (1+i)(1-i) & 0 \\ 0 & (1-i)(1+i) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

This gives $\|A\| = \sqrt{2}$.

Solution 1.5

Let

$$A = \begin{bmatrix} 1/\alpha & \alpha \\ 0 & 1/\alpha \end{bmatrix}, \quad \alpha > 1$$

Then the eigenvalues are $1/\alpha$ and, using an inequality on text page 7,

$$\|A\| \geq \max_{1 \leq i, j \leq 2} |a_{ij}| = \alpha$$

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Solution 1.6 By definition of the spectral norm, for any $\alpha \neq 0$ we can write

$$\begin{aligned} \|A\| &= \max_{\|x\|=1} \|Ax\| = \max_{\|x\|=1} \frac{\|Ax\|}{\|x\|} \\ &= \max_{\|\alpha x\|=1} \frac{\|A\alpha x\|}{\|\alpha x\|} = \max_{\|x\|=1/|\alpha|} \frac{|\alpha|\,\|Ax\|}{|\alpha|\,\|x\|} \end{aligned}$$

Since this holds for any $\alpha \neq 0$,

$$\|A\| = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Therefore

$$\|A\| \geq \frac{\|Ax\|}{\|x\|}$$

for any $x \neq 0$, which gives

$$\|Ax\| \leq \|A\|\,\|x\|$$

Solution 1.7 By definition of the spectral norm,

$$\begin{aligned} \|AB\| &= \max_{\|x\|=1} \|(AB)x\| = \max_{\|x\|=1} \|A(Bx)\| \\ &\leq \max_{\|x\|=1} \{\|A\|\,\|Bx\|\}\,, \quad \text{by Exercise 1.6} \\ &= \|A\| \max_{\|x\|=1} \|Bx\| = \|A\|\,\|B\| \end{aligned}$$

If A is invertible, then $AA^{-1} = I$ and the obvious $\|I\| = 1$ give

$$1 = \|AA^{-1}\| \leq \|A\|\,\|A^{-1}\|$$

Therefore

$$\|A^{-1}\| \geq \frac{1}{\|A\|}$$

Solution 1.8 We use the following easily verified facts about partitioned vectors:

$$\left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| \geq \|x_1\|,\ \|x_2\|\ ; \quad \left\| \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right\| = \|x_1\|\,, \quad \left\| \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \right\| = \|x_2\|$$

Write

$$Ax = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{bmatrix}$$

Then for A_{11}, for example,

$$\begin{aligned} \|A\| &= \max_{\|x\|=1} \|Ax\| \geq \max_{\|x\|=1} \|A_{11}x_1 + A_{12}x_2\| \\ &\geq \max_{\|x_1\|=1} \|A_{11}x_1\| = \|A_{11}\| \end{aligned}$$

The other partitions are handled similarly. The last part is easy from the definition of induced norm. For example if

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$$A = \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix}$$

then partitioning the vector x similarly we see that

$$\max_{\|x\| = 1} \|A\,x\| = \max_{\|x_2\| = 1} \|A_{12}x_2\| = \|A_{12}\|$$

Solution 1.9 By the Cauchy-Schwarz inequality, and $\|x^T\| = \|x\|$,

$$\begin{aligned} |x^TA\,x| &\le \|x^TA\|\,\|x\| = \|A^Tx\|\,\|x\| \\ &\le \|A^T\|\,\|x\|^2 = \|A\|\,\|x\|^2 \end{aligned}$$

This immediately gives

$$x^TA\,x \ge -\|A\|\,\|x\|^2$$

If λ is an eigenvalue of A and x is a corresponding unity-norm eigenvector, then

$$|\lambda| = |\lambda|\,\|x\| = \|\lambda x\| = \|A\,x\| \le \|A\|\,\|x\| = \|A\|$$

Solution 1.10 Since $Q = Q^T$, $Q^TQ = Q^2$, and the eigenvalues of Q^2 are $\lambda_1^2, \ldots, \lambda_n^2$. Therefore

$$\|Q\| = \sqrt{\lambda_{\max}(Q^2)} = \max_{1 \le i \le n} |\lambda_i|$$

For the other equality Cauchy-Schwarz gives

$$\begin{aligned} |x^TQx| &\le \|x^TQ\|\,\|x\| = \|Qx\|\,\|x\| \\ &\le \|Q\|\,\|x\|^2 = [\max_{1 \le i \le n} |\lambda_i|\,]\,x^Tx \end{aligned}$$

Therefore $|x^TQx| \le \|Q\|$ for all unity-norm x. Choosing x_a as a unity-norm eigenvector of Q corresponding to the eigenvalue that yields $\max_{1 \le i \le n} |\lambda_i|$ gives

$$|x_a^TQx_a| = x_a^T\,[\max_{1 \le i \le n} |\lambda_i|\,]\,x_a = \max_{1 \le i \le n} |\lambda_i|$$

Thus $\max_{\|x\| = 1} |x^TQx| = \|Q\|$.

Solution 1.11 Since $\|A\,x\| = \sqrt{(A\,x)^T(A\,x)} = \sqrt{x^TA^TA\,x}$,

$$\begin{aligned} \|A\| &= \max_{\|x\| = 1} \sqrt{x^TA^TA\,x} \\ &= \left[\max_{\|x\| = 1} x^TA^TA\,x\right]^{1/2} \end{aligned}$$

The Rayleigh-Ritz inequality gives, for all unity-norm x,

$$x^TA^TA\,x \le \lambda_{\max}(A^TA)\,x^Tx = \lambda_{\max}(A^TA)$$

and since $A^TA \ge 0$, $\lambda_{\max}(A^TA) \ge 0$. Choosing x_a to be a unity-norm eigenvector corresponding to $\lambda_{\max}(A^TA)$ gives

$$x_a^TA^TA\,x_a = \lambda_{\max}(A^TA)$$

Thus

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$$\max_{\|x\| = 1} x^T A^T A x = \lambda_{\max}(A^T A)$$

so we have $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$.

Solution 1.12 Since $A^T A > 0$ we have $\lambda_i(A^T A) > 0$, $i = 1, \ldots, n$, and $(A^T A)^{-1} > 0$. Then by Exercise 1.11,

$$\|A^{-1}\|^2 = \lambda_{\max}((A^T A)^{-1}) = \frac{1}{\lambda_{\min}(A^T A)}$$

$$= \frac{\prod_{i=1}^{n} \lambda_i(A^T A)}{\lambda_{\min}(A^T A) \cdot \det(A^T A)} \leq \frac{[\lambda_{\max}(A^T A)]^{n-1}}{(\det A)^2}$$

$$= \frac{\|A\|^{2(n-1)}}{(\det A)^2}$$

Therefore

$$\|A^{-1}\| \leq \frac{\|A\|^{n-1}}{|\det A|}$$

Solution 1.13 Assume $A \neq 0$, for the zero case is trivial. For any unity-norm x and y,

$$|y^T A x| \leq \|y^T\| \, \|A x\|$$
$$\leq \|y\| \, \|A\| \, \|x\| = \|A\|$$

Therefore

$$\max_{\|x\|, \|y\| = 1} |y^T A x| \leq \|A\|$$

Now let unity-norm x_a be such that $\|A x_a\| = \|A\|$, and let

$$y_a = \frac{A x_a}{\|A\|}$$

Then $\|y_a\| = 1$ and

$$|y_a^T A x_a| = \frac{|x_a^T A^T A x_a|}{\|A\|} = \frac{\|A x_a\|^2}{\|A\|} = \frac{\|A\|^2}{\|A\|} = \|A\|$$

Therefore

$$\max_{\|x\|, \|y\| = 1} |y^T A x| = \|A\|$$

Solution 1.14 The coefficients of the characteristic polynomial of a matrix are continuous functions of matrix entries, since determinant is a continuous function of the entries (sum of products). Also the roots of a polynomial are continuous functions of the coefficients. *(A proof is given in Appendix A.4 of* E.D. Sontag, *Mathematical Control Theory*, Springer-Verlag, New York, 1990.) Since a composition of continuous functions is a continuous function, the pointwise-in-t eigenvalues of $A(t)$ are continuous in t.
This argument gives that the (nonnegative) eigenvalues of $A^T(t)A(t)$ are continuous in t. Then the maximum at each t is continuous in t — plot two eigenvalues and consider their pointwise maximum to see this. Finally since square root is a continuous function of nonnegative arguments, we conclude $\|A(t)\|$ is continuous in t.
However for continuously-differentiable $A(t)$, $\|A(t)\|$ need not be continuously differentiable in t. Consider the