Chapter 1

The Wave Function

Problem 1.1

(a)

$$\langle j\rangle^2 = 21^2 = \boxed{441.}$$

$$\langle j^2\rangle = \frac{1}{N}\sum j^2 N(j) = \frac{1}{14}\left[(14^2)+(15^2)+3(16^2)+2(22^2)+2(24^2)+5(25^2)\right]$$
$$= \frac{1}{14}(196+225+768+968+1152+3125) = \frac{6434}{14} = \boxed{459.571.}$$

(b)

j	$\Delta j = j - \langle j\rangle$
14	$14-21=-7$
15	$15-21=-6$
16	$16-21=-5$
22	$22-21=1$
24	$24-21=3$
25	$25-21=4$

$$\sigma^2 = \frac{1}{N}\sum(\Delta j)^2 N(j) = \frac{1}{14}\left[(-7)^2+(-6)^2+(-5)^2\cdot 3+(1)^2\cdot 2+(3)^2\cdot 2+(4)^2\cdot 5\right]$$
$$= \frac{1}{14}(49+36+75+2+18+80) = \frac{260}{14} = \boxed{18.571.}$$

$$\sigma = \sqrt{18.571} = \boxed{4.309.}$$

(c)

$$\langle j^2\rangle - \langle j\rangle^2 = 459.571 - 441 = 18.571. \quad \text{[Agrees with (b).]}$$

Problem 1.2

(a)

$$\langle x^2\rangle = \int_0^h x^2 \frac{1}{2\sqrt{hx}}dx = \frac{1}{2\sqrt{h}}\left(\frac{2}{5}x^{5/2}\right)\Big|_0^h = \frac{h^2}{5}.$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{h^2}{5} - \left(\frac{h}{3}\right)^2 = \frac{4}{45}h^2 \;\Rightarrow\; \sigma = \boxed{\frac{2h}{3\sqrt{5}} = 0.2981h.}$$

(b)

$$P = 1 - \int_{x_-}^{x_+} \frac{1}{2\sqrt{hx}}dx = 1 - \frac{1}{2\sqrt{h}}(2\sqrt{x})\Big|_{x_-}^{x_+} = 1 - \frac{1}{\sqrt{h}}\left(\sqrt{x_+} - \sqrt{x_-}\right).$$

$$x_+ \equiv \langle x\rangle + \sigma = 0.3333h + 0.2981h = 0.6315h; \quad x_- \equiv \langle x\rangle - \sigma = 0.3333h - 0.2981h = 0.0352h.$$

$$P = 1 - \sqrt{0.6315} + \sqrt{0.0352} = \boxed{0.393.}$$

Problem 1.3

(a)

$$1 = \int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}dx. \qquad \text{Let } u \equiv x - a,\ du = dx,\ u : -\infty \to \infty.$$

$$1 = A\int_{-\infty}^{\infty} e^{-\lambda u^2}du = A\sqrt{\frac{\pi}{\lambda}} \quad\Rightarrow\quad \boxed{A = \sqrt{\frac{\lambda}{\pi}}.}$$

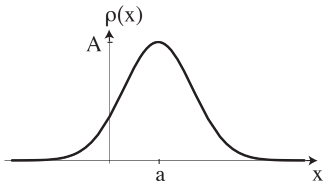
(b)

$$\langle x\rangle = A\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}dx = A\int_{-\infty}^{\infty}(u+a)e^{-\lambda u^2}du$$
$$= A\left[\int_{-\infty}^{\infty} ue^{-\lambda u^2}du + a\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right] = A\left(0 + a\sqrt{\frac{\pi}{\lambda}}\right) = \boxed{a.}$$

$$\langle x^2\rangle = A\int_{-\infty}^{\infty} x^2e^{-\lambda(x-a)^2}dx$$
$$= A\left\{\int_{-\infty}^{\infty} u^2e^{-\lambda u^2}du + 2a\int_{-\infty}^{\infty} ue^{-\lambda u^2}du + a^2\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right\}$$
$$= A\left[\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + 0 + a^2\sqrt{\frac{\pi}{\lambda}}\right] = \boxed{a^2 + \frac{1}{2\lambda}.}$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2\lambda}}.}$$

(c)



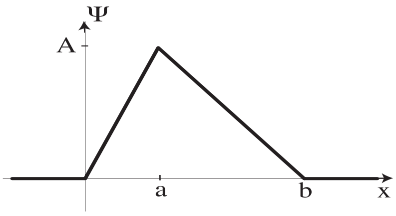
Problem 1.4

(a)

$$1 = \frac{|A|^2}{a^2}\int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2}\int_a^b (b-x)^2 dx = |A|^2 \left\{ \frac{1}{a^2}\left(\frac{x^3}{3}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(-\frac{(b-x)^3}{3}\right)\Big|_a^b \right\}$$

$$= |A|^2\left[\frac{a}{3} + \frac{b-a}{3}\right] = |A|^2\frac{b}{3} \Rightarrow \boxed{A = \sqrt{\frac{3}{b}}.}$$

(b)



(c) At $\boxed{x = a.}$

(d)

$$P = \int_0^a |\Psi|^2 dx = \frac{|A|^2}{a^2}\int_0^a x^2 dx = |A|^2\frac{a}{3} = \boxed{\frac{a}{b}.}\begin{cases} P = 1 & \text{if } \ b = a, \ \checkmark \\ P = 1/2 & \text{if } \ b = 2a. \ \checkmark \end{cases}$$

(e)

$$\langle x \rangle = \int x|\Psi|^2 dx = |A|^2\left\{\frac{1}{a^2}\int_0^a x^3 dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2 dx\right\}$$

$$= \frac{3}{b}\left\{\frac{1}{a^2}\left(\frac{x^4}{4}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(b^2\frac{x^2}{2} - 2b\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_a^b\right\}$$

$$= \frac{3}{4b(b-a)^2}\left[a^2(b-a)^2 + 2b^4 - 8b^4/3 + b^4 - 2a^2b^2 + 8a^3b/3 - a^4\right]$$

$$= \frac{3}{4b(b-a)^2}\left(\frac{b^4}{3} - a^2b^2 + \frac{2}{3}a^3b\right) = \frac{1}{4(b-a)^2}(b^3 - 3a^2b + 2a^3) = \boxed{\frac{2a+b}{4}.}$$

Problem 1.5

(a)

$$1 = \int |\Psi|^2 dx = 2|A|^2 \int_0^\infty e^{-2\lambda x} dx = 2|A|^2 \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\Bigg|_0^\infty = \frac{|A|^2}{\lambda}; \quad \boxed{A = \sqrt{\lambda}.}$$

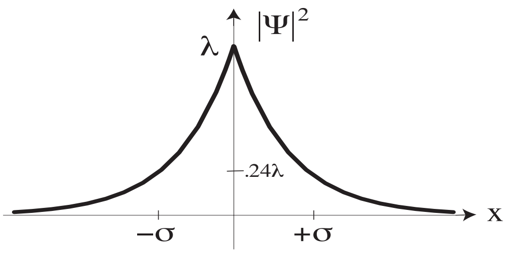
(b)

$$\langle x \rangle = \int x|\Psi|^2 dx = |A|^2 \int_{-\infty}^\infty x e^{-2\lambda|x|} dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2 \rangle = 2|A|^2 \int_0^\infty x^2 e^{-2\lambda x} dx = 2\lambda \left[\frac{2}{(2\lambda)^3} \right] = \boxed{\frac{1}{2\lambda^2}.}$$

(c)

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2}\lambda}.} \qquad |\Psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda/\sqrt{2}\lambda} = \lambda e^{-\sqrt{2}} = 0.2431\lambda.$$



Probability outside:

$$2\int_\sigma^\infty |\Psi|^2 dx = 2|A|^2 \int_\sigma^\infty e^{-2\lambda x} dx = 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\Bigg|_\sigma^\infty = e^{-2\lambda\sigma} = \boxed{e^{-\sqrt{2}} = 0.2431.}$$

Problem 1.6

For integration by parts, the differentiation has to be with respect to the *integration* variable – in this case the differentiation is with respect to t, but the integration variable is x. It's true that

$$\frac{\partial}{\partial t}(x|\Psi|^2) = \frac{\partial x}{\partial t}|\Psi|^2 + x\frac{\partial}{\partial t}|\Psi|^2 = x\frac{\partial}{\partial t}|\Psi|^2,$$

but this does *not* allow us to perform the integration:

$$\int_a^b x\frac{\partial}{\partial t}|\Psi|^2 dx = \int_a^b \frac{\partial}{\partial t}(x|\Psi|^2)dx \neq (x|\Psi|^2)\Big|_a^b.$$

Problem 1.7

From Eq. 1.33, $\frac{d\langle p\rangle}{dt} = -i\hbar\int \frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx$. But, noting that $\frac{\partial^2\Psi}{\partial x\partial t} = \frac{\partial^2\Psi}{\partial t\partial x}$ and using Eqs. 1.23-1.24:

$$\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial t}\right) = \left[-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right]\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]$$
$$= \frac{i\hbar}{2m}\left[\Psi^*\frac{\partial^3\Psi}{\partial x^3} - \frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\right] + \frac{i}{\hbar}\left[V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right]$$

The first term integrates to zero, using integration by parts twice, and the second term can be simplified to $V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*V\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial V}{\partial x}\Psi = -|\Psi|^2\frac{\partial V}{\partial x}$. So

$$\frac{d\langle p\rangle}{dt} = -i\hbar\left(\frac{i}{\hbar}\right)\int -|\Psi|^2\frac{\partial V}{\partial x}dx = \langle -\frac{\partial V}{\partial x}\rangle. \qquad \text{QED}$$

Problem 1.8

Suppose Ψ satisfies the Schrödinger equation *without* V_0: $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$. We want to find the solution Ψ_0 *with* V_0: $i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0$.

Claim: $\Psi_0 = \Psi e^{-iV_0t/\hbar}$.

Proof: $i\hbar\frac{\partial\Psi_0}{\partial t} = i\hbar\frac{\partial\Psi}{\partial t}e^{-iV_0t/\hbar} + i\hbar\Psi\left(-\frac{iV_0}{\hbar}\right)e^{-iV_0t/\hbar} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right]e^{-iV_0t/\hbar} + V_0\Psi e^{-iV_0t/\hbar}$
$= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0.$ QED

This has *no* effect on the expectation value of a dynamical variable, since the extra phase factor, being independent of x, cancels out in Eq. 1.36.

Problem 1.9

(a)

$$1 = 2|A|^2\int_0^\infty e^{-2amx^2/\hbar}dx = 2|A|^2\frac{1}{2}\sqrt{\frac{\pi}{(2am/\hbar)}} = |A|^2\sqrt{\frac{\pi\hbar}{2am}}; \quad \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.}$$

(b)

$$\frac{\partial\Psi}{\partial t} = -ia\Psi; \quad \frac{\partial\Psi}{\partial x} = -\frac{2amx}{\hbar}\Psi; \quad \frac{\partial^2\Psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) = -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi.$$

Plug these into the Schrödinger equation, $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$:

$$V\Psi = i\hbar(-ia)\Psi + \frac{\hbar^2}{2m}\left(-\frac{2am}{\hbar}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\Psi$$
$$= \left[\hbar a - \hbar a\left(1 - \frac{2amx^2}{\hbar}\right)\right]\Psi = 2a^2mx^2\Psi, \quad \text{so} \quad \boxed{V(x) = 2ma^2x^2.}$$

(c)

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2\rangle = 2|A|^2\int_0^{\infty} x^2 e^{-2amx^2/\hbar}dx = 2|A|^2\frac{1}{2^2(2am/\hbar)}\sqrt{\frac{\pi\hbar}{2am}} = \boxed{\frac{\hbar}{4am}.}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.}$$

$$\langle p^2\rangle = \int \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \Psi dx = -\hbar^2\int \Psi^*\frac{\partial^2\Psi}{\partial x^2}dx$$

$$= -\hbar^2\int \Psi^*\left[-\frac{2am}{\hbar}\left(1-\frac{2amx^2}{\hbar}\right)\Psi\right]dx = 2am\hbar\left\{\int|\Psi|^2dx - \frac{2am}{\hbar}\int x^2|\Psi|^2dx\right\}$$

$$= 2am\hbar\left(1-\frac{2am}{\hbar}\langle x^2\rangle\right) = 2am\hbar\left(1-\frac{2am}{\hbar}\frac{\hbar}{4am}\right) = 2am\hbar\left(\frac{1}{2}\right) = \boxed{am\hbar.}$$

(d)

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{4am} \Longrightarrow \boxed{\sigma_x = \sqrt{\frac{\hbar}{4am}}}; \quad \sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = am\hbar \Longrightarrow \boxed{\sigma_p = \sqrt{am\hbar}.}$$

$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{4am}}\sqrt{am\hbar} = \frac{\hbar}{2}$. This *is* (just barely) consistent with the uncertainty principle.

Problem 1.10

From Math Tables: $\pi = 3.141592653589793238462643\cdots$

(a)

$P(0)=0$	$P(1)=2/25$	$P(2)=3/25$	$P(3)=5/25$	$P(4)=3/25$
$P(5)=3/25$	$P(6)=3/25$	$P(7)=1/25$	$P(8)=2/25$	$P(9)=3/25$

In general, $P(j) = \frac{N(j)}{N}$.

(b) *Most probable*: $\boxed{3.}$ *Median*: 13 are ≤ 4, 12 are ≥ 5, so median is $\boxed{4.}$

Average: $\langle j\rangle = \frac{1}{25}[0\cdot 0 + 1\cdot 2 + 2\cdot 3 + 3\cdot 5 + 4\cdot 3 + 5\cdot 3 + 6\cdot 3 + 7\cdot 1 + 8\cdot 2 + 9\cdot 3]$

$= \frac{1}{25}[0+2+6+15+12+15+18+7+16+27] = \frac{118}{25} = \boxed{4.72.}$

(c) $\langle j^2\rangle = \frac{1}{25}[0 + 1^2\cdot 2 + 2^2\cdot 3 + 3^2\cdot 5 + 4^2\cdot 3 + 5^2\cdot 3 + 6^2\cdot 3 + 7^2\cdot 1 + 8^2\cdot 2 + 9^2\cdot 3]$

$= \frac{1}{25}[0+2+12+45+48+75+108+49+128+243] = \frac{710}{25} = \boxed{28.4.}$

$\sigma^2 = \langle j^2\rangle - \langle j\rangle^2 = 28.4 - 4.72^2 = 28.4 - 22.2784 = 6.1216;\quad \sigma = \sqrt{6.1216} = \boxed{2.474.}$

Problem 1.11

(a)

$$\frac{1}{2}mv^2 + V = E \quad \rightarrow \quad \boxed{v(x) = \sqrt{\frac{2}{m}\left(E - V(x)\right)}.}$$

(b)

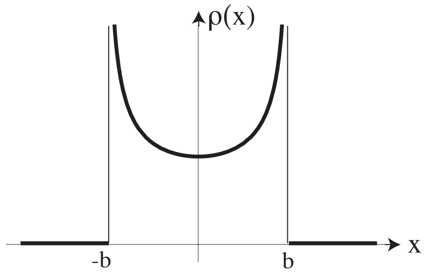
$$T = \int_a^b \frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx^2\right)}}\,dx = \sqrt{\frac{m}{k}}\int_a^b \frac{1}{\sqrt{(2E/k) - x^2}}\,dx.$$

Turning points: $v = 0 \Rightarrow E = V = \frac{1}{2}kb^2 \Rightarrow b = \sqrt{2E/k}$; $\quad a = -b$.

$$T = 2\sqrt{\frac{m}{k}}\int_0^b \frac{1}{\sqrt{b^2 - x^2}}\,dx = 2\sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{x}{b}\right)\Big|_0^b = 2\sqrt{\frac{m}{k}}\sin^{-1}(1)$$
$$= 2\sqrt{\frac{m}{k}}\left(\frac{\pi}{2}\right) = \pi\sqrt{\frac{m}{k}}.$$

$$\rho(x) = \frac{1}{\pi\sqrt{\frac{m}{k}}\sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx^21\right)}} = \boxed{\frac{1}{\pi\sqrt{b^2 - x^2}}.}$$

$$\int_a^b \rho(x)\,dx = \frac{2}{\pi}\int_0^b \frac{1}{\sqrt{b^2 - x^2}}\,dx = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1.\ \checkmark$$



(c) $\boxed{\langle x\rangle = 0.}$

$$\langle x^2\rangle = \frac{1}{\pi}\int_{-b}^b \frac{x^2}{\sqrt{b^2 - x^2}}\,dx = \frac{2}{\pi}\int_0^b \frac{x^2}{\sqrt{b^2 - x^2}}\,dx$$
$$= \frac{2}{\pi}\left[-\frac{x}{2}\sqrt{b^2 - x^2} + \frac{b^2}{2}\sin^{-1}\left(\frac{x}{b}\right)\right]\Big|_0^b = \frac{b^2}{\pi}\sin^{-1}(1) = \frac{b^2}{\pi}\frac{\pi}{2} = \frac{b^2}{2} = \boxed{\frac{E}{k}.}$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\langle x^2\rangle} = \frac{b}{\sqrt{2}} = \boxed{\sqrt{\frac{E}{k}}.}$$

Problem 1.12

(a)

$$\rho(p)\,dp = \frac{dt}{T} = \frac{|dt/dp|\,dp}{T}$$

where dt is now the time it spends with momentum in the range dp (dt is intrinsically positive, but $dp/dt = F = -kx$ runs negative—hence the absolute value). Now

$$\frac{p^2}{2m} + \frac{1}{2}kx^2 = E \Rightarrow x = \pm\sqrt{\frac{2}{k}\left(E - \frac{p^2}{2m}\right)},$$

so

$$\rho(p) = \frac{1}{\pi\sqrt{\frac{m}{k}}\,k\sqrt{\frac{2}{k}\left(E-\frac{p^2}{2m}\right)}} = \boxed{\frac{1}{\pi\sqrt{2mE-p^2}}} = \frac{1}{\pi\sqrt{c^2-p^2}},$$

where $c \equiv \sqrt{2mE}$. This is the same as $\rho(x)$ (Problem 1.11(b)), with c in place of b (and, of course, p in place of x).

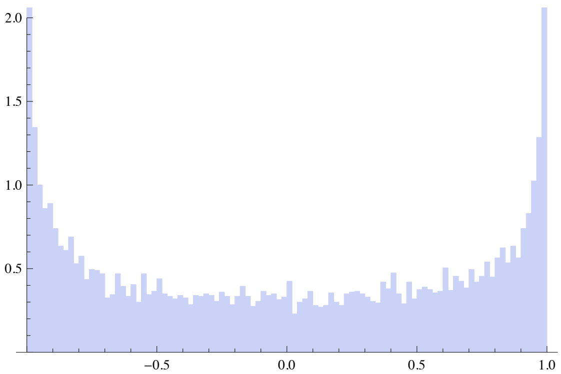
(b) From Problem 1.11(c), $\boxed{\langle p\rangle = 0,\ \langle p^2\rangle = \frac{c^2}{2},\ \sigma_p = \frac{c}{\sqrt{2}} = \sqrt{mE}\,.}$

(c) $\sigma_x\sigma_p = \sqrt{\frac{E}{k}}\sqrt{mE} = \boxed{\sqrt{\frac{m}{k}}\,E} = \frac{E}{\omega}$. If $E \geq \frac{1}{2}\hbar\omega$, then $\sigma_x\sigma_p \geq \frac{1}{2}\hbar$, which is precisely the Heisenberg uncertainty principle!

Problem 1.13

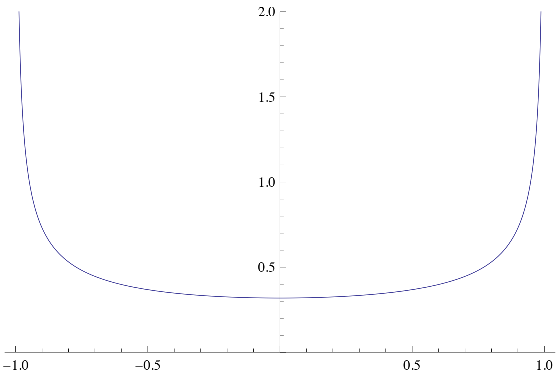
```
x[t_] := Cos[t]
snapshots = Table[x[π RandomReal[j]], {j, 10000}]
```

```
Histogram[snapshots, 100, "PDF", PlotRange → {0, 2}]
```

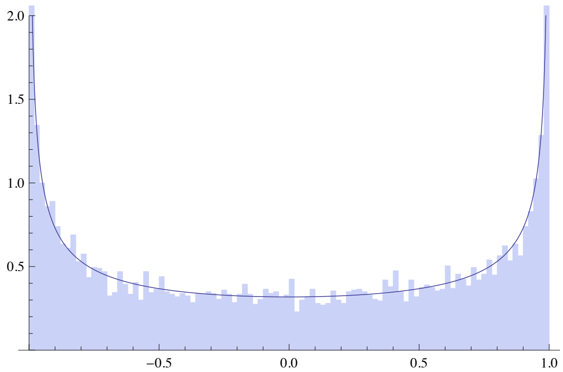


```
r[x_] := 1/(π Sqrt[1 - x^2])

Plot[ r[x], {x, -1, 1}, PlotRange -> {0, 2}]
```




```
Show[Histogram[snapshots, 100, "PDF", PlotRange → {0, 2}],
  Plot[ r[x], {x, -1, 1}, PlotRange -> {0, 2}]]
```



Problem 1.14

(a) $P_{ab}(t) = \int_a^b |\Psi(x,t)|^2 dx$, so $\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial}{\partial t}|\Psi|^2 dx$. But (Eq. 1.25):

$$\frac{\partial|\Psi|^2}{\partial t} = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right] = -\frac{\partial}{\partial x}J(x,t).$$

$$\therefore \frac{dP_{ab}}{dt} = -\int_a^b \frac{\partial}{\partial x}J(x,t)dx = -\left[J(x,t)\right]\Big|_a^b = J(a,t) - J(b,t). \qquad \text{QED}$$

Probability is dimensionless, so J has the dimensions 1/time, and units $\boxed{\text{seconds}^{-1}.}$

(b) Here $\Psi(x,t) = f(x)e^{-iat}$, where $f(x) \equiv Ae^{-amx^2/\hbar}$, so $\Psi\frac{\partial\Psi^*}{\partial x} = fe^{-iat}\frac{df}{dx}e^{iat} = f\frac{df}{dx}$, and $\Psi^*\frac{\partial\Psi}{\partial x} = f\frac{df}{dx}$ too, so $\boxed{J(x,t) = 0.}$

Problem 1.15

Use Eqs. [1.23] and [1.24], and integration by parts:

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx &= \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(\Psi_1^*\Psi_2\right)dx = \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial t}\Psi_2+\Psi_1^*\frac{\partial\Psi_2}{\partial t}\right)dx\\ &= \int_{-\infty}^{\infty}\left[\left(\frac{-i\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2}+\frac{i}{\hbar}V\Psi_1^*\right)\Psi_2+\Psi_1^*\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2}-\frac{i}{\hbar}V\Psi_2\right)\right]dx\\ &= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2-\Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2}\right)dx\\ &= -\frac{i\hbar}{2m}\left[\frac{\partial\Psi_1^*}{\partial x}\Psi_2\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial\Psi_1^*}{\partial x}\frac{\partial\Psi_2}{\partial x}dx-\Psi_1^*\frac{\partial\Psi_2}{\partial x}\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{\partial\Psi_1^*}{\partial x}\frac{\partial\Psi_2}{\partial x}dx\right]=0.\ \text{QED}\end{aligned}$$

Problem 1.16

(a)

$$\begin{aligned}1 &= |A|^2\int_{-a}^{a}(a^2-x^2)^2dx = 2|A|^2\int_0^a(a^4-2a^2x^2+x^4)dx = 2|A|^2\left[a^4x-2a^2\frac{x^3}{3}+\frac{x^5}{5}\right]\Big|_0^a\\ &= 2|A|^2a^5\left(1-\frac{2}{3}+\frac{1}{5}\right)=\frac{16}{15}a^5|A|^2,\ \text{so}\ \boxed{A=\sqrt{\frac{15}{16a^5}}.}\end{aligned}$$

(b)

$$\langle x\rangle = \int_{-a}^{a}x|\Psi|^2\,dx = \boxed{0.}\quad\text{(Odd integrand.)}$$

(c)

$$\langle p\rangle = \frac{\hbar}{i}A^2\int_{-a}^{a}(a^2-x^2)\underbrace{\frac{d}{dx}(a^2-x^2)}_{-2x}dx = \boxed{0.}\quad\text{(Odd integrand.)}$$

Since we only know $\langle x\rangle$ at $t=0$ we cannot calculate $d\langle x\rangle/dt$ directly.

(d)

$$\begin{aligned}\langle x^2\rangle &= A^2\int_{-a}^{a}x^2(a^2-x^2)^2dx = 2A^2\int_0^a(a^4x^2-2a^2x^4+x^6)dx\\ &= 2\frac{15}{16a^5}\left[a^4\frac{x^3}{3}-2a^2\frac{x^5}{5}+\frac{x^7}{7}\right]\Big|_0^a = \frac{15}{8a^5}(a^7)\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right)\\ &= \frac{\cancel{15}a^2}{8}\left(\frac{35-42+15}{\cancel{3}\cdot\cancel{5}\cdot 7}\right)=\frac{a^2}{8}\cdot\frac{8}{7}=\boxed{\frac{a^2}{7}.}\end{aligned}$$

(e)

$$\langle p^2\rangle = -A^2\hbar^2\int_{-a}^{a}(a^2-x^2)\underbrace{\frac{d^2}{dx^2}(a^2-x^2)}_{-2}dx = 2A^2\hbar^2 2\int_0^a (a^2-x^2)dx$$

$$= 4\cdot\frac{15}{16a^5}\hbar^2\left(a^2x-\frac{x^3}{3}\right)\Big|_0^a = \frac{15\hbar^2}{4a^5}\left(a^3-\frac{a^3}{3}\right) = \frac{15\hbar^2}{4a^2}\cdot\frac{2}{3} = \boxed{\frac{5}{2}\frac{\hbar^2}{a^2}.}$$

(f)

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{7}a^2} = \boxed{\frac{a}{\sqrt{7}}.}$$

(g)

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{5}{2}\frac{\hbar^2}{a^2}} = \boxed{\sqrt{\frac{5}{2}}\frac{\hbar}{a}.}$$

(h)

$$\sigma_x\sigma_p = \frac{a}{\sqrt{7}}\cdot\sqrt{\frac{5}{2}}\frac{\hbar}{a} = \sqrt{\frac{5}{14}}\hbar = \sqrt{\frac{10}{7}}\frac{\hbar}{2} > \frac{\hbar}{2}.\ \checkmark$$

Problem 1.17

(a) Eq. 1.24 now reads $\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V^*\Psi^*$, and Eq. 1.25 picks up an extra term:

$$\frac{\partial}{\partial t}|\Psi|^2 = \cdots + \frac{i}{\hbar}|\Psi|^2(V^*-V) = \cdots + \frac{i}{\hbar}|\Psi|^2(V_0+i\Gamma-V_0+i\Gamma) = \cdots - \frac{2\Gamma}{\hbar}|\Psi|^2,$$

and Eq. 1.27 becomes $\frac{dP}{dt} = -\frac{2\Gamma}{\hbar}\int_{-\infty}^{\infty}|\Psi|^2dx = -\frac{2\Gamma}{\hbar}P.$ QED

(b)

$$\frac{dP}{P} = -\frac{2\Gamma}{\hbar}dt \Longrightarrow \ln P = -\frac{2\Gamma}{\hbar}t + \text{constant} \Longrightarrow \boxed{P(t) = P(0)e^{-2\Gamma t/\hbar},}\ \text{so}\ \boxed{\tau = \frac{\hbar}{2\Gamma}.}$$

Problem 1.18

$$\frac{h}{\sqrt{3mk_BT}} > d \Rightarrow T < \frac{h^2}{3mk_Bd^2}.$$

(a) Electrons ($m = 9.1\times10^{-31}$ kg):

$$T < \frac{(6.6\times10^{-34})^2}{3(9.1\times10^{-31})(1.4\times10^{-23})(3\times10^{-10})^2} = \boxed{1.3\times10^5\ \text{K.}}$$

Silicon nuclei ($m = 28m_p = 28(1.7\times10^{-27}) = 4.8\times10^{-26}$ kg):

$$T < \frac{(6.6\times10^{-34})^2}{3(4.8\times10^{-26})(1.4\times10^{-23})(3\times10^{-10})^2} = \boxed{2.4\ \text{K.}}$$

(b) $PV = Nk_BT$; volume occupied by one molecule $(N = 1,\ V = d^3) \Rightarrow d = (k_BT/P)^{1/3}$.

$$T < \frac{h^2}{3mk_B}\left(\frac{P}{k_BT}\right)^{2/3} \Rightarrow T^{5/3} < \frac{h^2}{3m}\frac{P^{2/3}}{k_B^{5/3}} \Rightarrow T < \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}.$$

For helium $(m = 4m_p = 6.8 \times 10^{-27}$ kg) at 1 atm $= 1.0 \times 10^5$ N/m^2:

$$T < \frac{1}{(1.4 \times 10^{-23})}\left(\frac{(6.6 \times 10^{-34})^2}{3(6.8 \times 10^{-27})}\right)^{3/5} (1.0 \times 10^5)^{2/5} = \boxed{2.8 \text{ K.}}$$

For atomic hydrogen $(m = m_p = 1.7 \times 10^{-27}$ kg) with $d = 0.01$ m:

$$T < \frac{(6.6 \times 10^{-34})^2}{3(1.7 \times 10^{-27})(1.4 \times 10^{-23})(10^{-2})^2} = \boxed{6.2 \times 10^{-14} \text{ K.}}$$

At 3 K it is definitely in the classical regime.