Chapter 1

Solutions for Chapter 1

Problem 1.1

A cylindrical pipe with diameter $d = 1.0\ m$ is subject to internal pressure $p = 5.0\ MPa$ but it is free to expand along the length, so that only hoop stress $\sigma = pd/2t$ is developed in the wall.

(a) What is the optimum orientation for the fibers?, and why?

(b) What is the required wall thickness and material type that yields the minimum thickness?

(c) What is the required wall thickness and material type for minimum weight?

(d) What is the required wall thickness and material type for minimum cost? Assume the following costs (\$/kg)[1] for the materials (from left to right) in Table 1.1: 2.60, 3.20, 3.50, Table 1.2: 1.00, 2.60, 2.80, Table 1.3: 9.60, 10.80, 9.80, 12.50, and Table 1.4: 10.20, 17.00, 16.00, 16.00.

Use Tables 1.1–1.4, a resistance factor $\phi = 0.25$, and a load factor $\alpha = 1.0$ (the internal pressure has no variability). Do not consider those materials for which the required data are not available.

Solution to Problem 1.1

The hoop stress is

$$\sigma_H = \frac{pd}{2t}$$

and the stress along the length of the pipe is zero. Al limit load αp and resistance ϕF_{1t} we have the design equation

$$\phi F_{1t} = \frac{\alpha pd}{2t}$$

(a) The optimum orientation is in the hoop direction because $F_{2t} = 0$.

(b) From the design equation, the thickness can be calculated as

$$t = \frac{\alpha pd}{2\phi F_{1t}}$$

[1]These are not current market prices.

and the minimum thickness would be obtained using the material with highest strength, which from Tables 1.1–1.4 is for T800/3900-2, thus

$$t = \frac{1 \times 5 \times 1}{2 \times 0.25 \times 2,698} = 3.7\ mm$$

(c) The weight per unit length can be calculated as

$$\frac{W}{L} = \rho \pi d t = \frac{\pi \alpha p d^2}{2} \frac{\rho}{\phi F_{1t}}$$

which can be minimized by choosing the lowest ρ/F_{1t}. From Tables 1.1–1.4, taking into account that the density of some composites are not available, the lowest W/L is for is for IM7/8552 so that W/L is calculated to be $W/L = 20.9$ kg/m, and thickness $t = 4.3$ mm.

(d) The cost per unit length is

$$\frac{\text{cost}}{L} = \frac{W}{L}(\text{cost/W}) = \frac{\pi \alpha p d^2}{2} \frac{\rho(\text{cost/W})}{\phi F_{1t}}$$

The lowest is for E-Glass/Polyester at $94.6/m. Then, t=11 mm.

Problem 1.2

Compare the weight per unit length of the minimum weight pipe of Problem 1.1 with a pipe made out of 6061-T6 aluminum with yield strength 270 MPa and density $\rho = 2.70\ g/cc$. Use the same load and resistance factors.

Solution to Problem 1.2

For 6061-T6 Aluminum, $F_{1t} = 270\,MPa$, and density $\rho = 2.7\,g/cc$

$$t = \frac{\alpha p d}{2\phi F_{1t}} = \frac{1 \times 5 \times 1}{2 \times 0.25 \times 270} = 37.0 \text{ mm}$$

$$\frac{W}{L} = \rho \pi d t = 2.7 \times \frac{10^{-3}}{10^{-6}} \times 1 \times 37 \times 10^{-3} = 314 \text{ kg/m}$$

Problem 1.3

(paper only) A thin film is bonded to a very thick and rigid substrate, but during bonding a circular debonded area remains between the film and substrate thus creating a disk-shaped crack with radius a. To probe the toughness of the adhesive, the film is pulled with load P at the center of the disk until it starts to peel off. The film has elastic modulus E, Poisson's ratio ν, and thickness t. The edge of the disk can be assumed to be fixed (no rotation) at $r = a$. Derive an expression for the fracture toughness in crack-opening mode I. Note: For a circular disk with radius a, thickness t, clamped at $r = a$, under concentrated load P at the center, the center deflection is [1, Table 11.2.10b]

$$\Delta = \frac{P\,a^2}{16\pi D} \qquad ; \qquad D = \frac{E\,t^3}{12(1-\nu^2)}$$

where E, ν, are the elastic modulus and Poisson's ratio of the film, respectively.

Solution to Problem 1.3

From Section 1.4, from a concentrated load and deflection Δ under the load P, the potential energy is

$$\Pi = -\frac{P\Delta}{2}$$

For a circular disk with radius a, thickness t, clamped at $r = a$, under concentrated load P at the center, the center deflection is [1, Table 11.2.17]

$$\Delta = \frac{P\,a^2}{16\pi D} \qquad ; \qquad D = \frac{E\,t^3}{12(1-\nu^2)}$$

where E, ν, are the elastic modulus and Poisson's ratio of the film, respectively. Then,

$$\Pi = -\frac{P^2 a^2}{32\pi D}$$

and

$$\frac{d\Pi}{da} = -\frac{P^2 a}{16\pi D}$$

An increment of crack radius da produces an increment of crack area $dA = 2\pi a\,da$. Then, from (1.3)

$$G_I = -\frac{d\Pi}{dA} = -\frac{1}{2\pi a}\frac{d\Pi}{ds} = \frac{P^2}{32\pi^2 D}$$

Finally, the fracture toughness G_{Ic} is reached when $P = P_c$ is the load at which the crack begins to grow, that is

$$G_{Ic} = \frac{3P_c^2(1+\nu^2)}{8\pi^2 E t^3}$$

Problem 1.4

(paper only) Consider a material with measured strength values $\mathbf{F}$ given in Table 1.7. Assume the expected stress values $\boldsymbol{\sigma}$ to be one-half of the values in Table 1.7. Calculate

(a) the safety factor R, and

(b) and the reliability Q of the system.

Solution to Problem 1.4

From Table 1.2, and assuming a *Normal* distribution, the strength has a mean $\mu_F = 65.4$, a standard deviation $s_F = 14.47$, and COV $\varkappa_F = 0.22$.

Taking $\frac{1}{2}$ of every value in Table 1.2, the applied stress has mean $\mu_\sigma = 32.7$, standard deviation $s_\sigma = 7.235$, and COV $\varkappa_\sigma = 0.22$. Note the COV values for both stress and strength are the same (see also Problems 1.8–1.9).

(a) Assuming that the safety factor is defined as a ratio between the mean values:

$$R = \frac{\mu_F}{\mu_\sigma} = 2$$

but note that in more sophisticated analysis, the safety factor may be defined as a ratio between *characteristic, minimum, or guaranteed* values.

(b) Given random variables F, σ, the margin of safety is a new random variable defined as $m = F - \sigma$. If F, σ, are Normally distributed, so is m. To calculate the reliability, first calculate the mean and standard deviation of m as follows

$$\mu_m = \mu_F - \mu_\sigma = 32.7$$

$$s_m = \sqrt{s_F^2 + s_\sigma^2} = \sqrt{14.47^2 + 7.325^2} = 16.178$$

For $m = 0$, the standard normal variable z is calculated as

$$z = -\frac{\mu_m}{s_m} = -\frac{32.7}{16.178} = -2.021$$

The reliability Q is then

$$Q = 1 - CFD(-\mu_m/s_m)$$

which can be calculated with MATLAB as `Q=1-cdf('norm',-2.021,0,1)=0.9784`

$$Q = 0.9784 = 97.84\%$$

Problem 1.5

A component of a composite rocket motor case is subject to a state of stress $\sigma_x = 10\,MPa$, $\sigma_y = 5\,MPa$, $\sigma_{xy} = 2.5\,MPa$. Use the material data for Carbon/Epoxy (AS4–3501-6) in Table 1.3. Using "principal stress design," compute:

(a) The optimum fiber orientation.

(b) The largest load factor α that can be tolerated for the lamina with optimum orientation, considering all possible modes of failure and a resistance factor $\phi = 1.0$.

(c) The lowest resistance factor ϕ that can be tolerated for the lamina with optimum orientation, considering all possible modes of failure and a load factor $\alpha = 1.0$.

Solution to Problem 1.5

The state of stress (Figure 1.1) is $\sigma_x = 10$ MPa,$\sigma_y = 5$ MPa,$\sigma_{xy} = 2.5$ MPa.

Calculate the principal stresses and orientation.

$$p = \frac{\sigma_x + \sigma_y}{2} = \frac{10 + 5}{2} = 7.5 \text{ MPa}$$

$$q = \frac{\sigma_x - \sigma_y}{2} = \frac{10 - 5}{2} = 2.5 \text{ MPa}$$

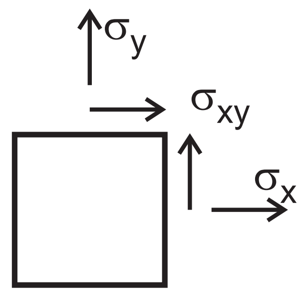

Figure 1.1: Exercise 1.3 (a)

$$r = \sigma_{xy} = 2.5$$

$$R = \sqrt{q^2 + r^2} = \sqrt{2.5^2 + 2.5^2} = 3.536$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{r}{q}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2.5}{2.5}\right) = 22.5°$$

$$\sigma_I = p + R = 7.5 + 3.536 = 11.04 \text{ MPa}$$

$$\sigma_{II} = p - R = 7.5 - 3.536 = 3.964 \text{ MPa}$$

(a) Optimum Fiber Orientation (Figure 1.2):

$$\theta = 22.5°$$

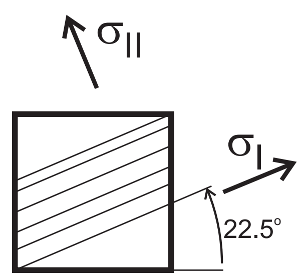

Figure 1.2: Exercise 1.3 (b)

(b) From Table 1.3, $F_{1t} = 1,950$ MPa and $F_{2t} = 48$ MPa. To get the load factor α, the following must be satisfied:

In the 1-direction:

$$\phi F_{1t} > \alpha \sigma_I$$

Therefore, with $\phi = 1$

$$\alpha < \frac{\phi F_{1t}}{\sigma_I} = \frac{1,950}{11.04} = 176.6$$

and in the 2-direction:

$$\alpha < \frac{\phi F_{2t}}{\sigma_{II}} = \frac{48 \text{ MPa}}{3.964\text{MPa}} = 12.11$$

The actual load factor that can be tolerated is the smallest of the two calculated, i.e., $\alpha < 12.11$, because at that point the structure breaks due to σ_{II} reaching the transverse tensile strength F_{2t}. Even if the fiber direction would tolerate 166 times the nominal load that produced the given stress state $\sigma = (10, 5, 25)$ MPa, the transverse direction can tolerate only 12.11 times the nominal load.

(c) From Table 1.3, $F_{1t} = 1,950$ MPa and $F_{2t} = 48$ MPa. To get the resistance factor ϕ, the following must be satisfied:

$$\phi F > \alpha\sigma$$

Therefore, with $\alpha = 1$

$$\phi > \frac{\sigma_I}{F_{1t}} = \frac{11.04}{1,950} = 0.0057$$

$$\phi > \frac{\sigma_{II}}{F_{2t}} = \frac{3.964}{48} = 0.0826$$

The lowest resistance factor that can be tolerated is the largest of the two calculated, i.e., $\phi > 0.0826$, because anything less would be insufficient to carry the transverse stress σ_{II}. That means that, given a fixed value F_{2t}, the material variability has to be small enough to yield $\phi > 0.0826$. Higher variability leads to smaller ϕ. The longitudinal direction would tolerate more variability, thus a smaller ϕ.

Problem 1.6

Using *single safety factor design, i.e., allowable stress design (ASD)* (1.17), calculate the % change needed on the mean stress μ_σ in Problem 1.4 to achieve a reliability of 99.5%.

Solution to Problem 1.6

To achieve $Q = 0.995$, in Table 1.6, $z = -2.576$ (adding the column plus row values for z corresponding to table entry 0.995). Then, using (1.17) and $\varkappa_F = 0.22$, we need a safety factor

$$R = \frac{1 + \sqrt{1 - (1 - 2.576^2 \times 0.22^2)(1 - 2.576^2 \times 0.22^2)}}{(1 - 2.576^2 \times 0.22^2)} = 2.555$$

In Problem 1.4, R=2, so the mean value of stress has to be lowered to get a safety factor R=2.555. Therefore, the mean value of stress must be reduced from 32.7 to

$$\mu_\sigma = 32.7 \times 2/2.555 = 25.6$$

The mean stress can be reduced by either reducing the load applied to the system or by increasing the geometric dimensions of the system. This is what a mechanical/structural designer does, i.e., to select materials and then change the geometry of the system to safely carry the applied loads.

Problem 1.7

Using *multiple safety factor design*, i.e., LRFD (1.24), calculate the % change needed on the stress in Problem 1.4 to achieve a reliability of 99.5%.

Solution to Problem 1.7

To achieve $Q = 0.995$, in Table 1.1, $z = -2.576$, then using (1.24) and $\varkappa_F = 0.22$ we need a safety factor

$$R = \frac{\alpha}{\phi} = \frac{1 - z\varkappa_\sigma}{1 + z\varkappa_F} = \frac{1 + 2.576 \times 0.22}{1 - 2.576 \times 0.22} = \frac{1.5667}{0.4333} = 3.6158$$

where the load factor is

$$\alpha = 1 - z\varkappa_\sigma = 1.5667$$

and the resistance factor is

$$\phi = 1 + z\varkappa_F = 0.4333$$

In Problem 1.4, R=2, so the mean value of stress has to be lowered to get R=3.6158. That means that the mean value of stress must be reduced from 32.7 to

$$\mu_\sigma = 32.7 \times 2/3.6158 = 18.09$$

By comparing with Problem 1.6, we note that *multiple factor design (LRFD)* (1.24) is more conservative than *single safety factor design, i.e., allowable stress design (ASD)* (1.17). Conservative is OK in design as long as the cost is not too high, where cost could be price, weight, etc.

Problem 1.8

Consider a material with strength mean value of 65.4 MPa and COV 22%, subjected to stress with a mean value 32.7 MPa and COV 11%.

(a) Calculate the load and resistance factors for a reliability of 99.5%.

(b) Calculate the resulting safety factor.

(c) Compare the result to the safety factor in Problem 1.7 and comment on your finding.

Solution to Problem 1.8

With $Q = 0.995$, Table 1.1 yields $z = -2.576$. Then

$$\alpha = 1 + 2.576 \times 0.11 = 1.2834$$

$$\phi = 1 - 2.576 \times 0.22 = 0.4333$$

$$R = \frac{\alpha}{\phi} = \frac{1.234}{0.433} = 2.9619 < 3.6158 \text{ in Problem 1.7}$$

The safety factor is smaller than in Problem 1.7 because the variability of the load is less, the load factor is less. Less variability means that not so much safety factor needs to be used to cover for the uncertainty on the value of the actual load in service.

Problem 1.9

Consider a material with strength mean value of 65.4 MPa and COV 11%, subjected to stress with a mean value 32.7 MPa and COV 22%.

(a) Calculate the load and resistance factors for a reliability of 99.5%.

(b) Calculate the resulting safety factor.

(c) Compare the result to the safety factor in Problem 1.7 and comment on your finding.

Solution to Problem 1.9

With $Q = 0.995$, Table 1.1 yields $z = -2.576$. Then

$$\alpha = 1 + 2.576 \times 0.2 = 1.5667$$

$$\phi = 1 - 2.576 \times 0.11 = 0.7166$$

$$R = \frac{\alpha}{\phi} = \frac{1.5667}{0.7166} = 2.1862 < 3.6158 \text{ in Problem 1.7}$$

The safety factor is smaller than in Problem 1.7 because the variability of the strength is less, the resistance factor is less. Less variability means that not so much safety factor needs to be used to cover for the uncertainty on the value of the actual strength of the material.

Problem 1.10

Given the strength data F shown below, and assuming that a Normal distribution represents the data well, calculate the A-, B-, and C-basis values. Use Table 1.8.

F = [443.2, 444.1, 468.8, 444.6, 447.8] MPa

Solution to Problem 1.10

First we calculate the mean $\bar{f}$ and standard deviation s:

$$\bar{f} = 449.7\ MPa \tag{1.1}$$

$$s = 10.817\ MPa \tag{1.2}$$

Next, the number of data points is $n = 5$, so the k-values are obtained from Table 1.8 as 5.7411, 3.4066, and 4.2027, for A-, B-, and C-basis, respectively. Then, the basis values are:

$$f_A = \bar{f} - k\ s = 449.7 - 5.7411 \times 10.817 = 387.6\ MPa \tag{1.3}$$

$$f_B = \bar{f} - k\ s = 449.7 - 3.4066 \times 10.817 = 412.9\ MPa \tag{1.4}$$

$$f_A = \bar{f} - k\ s = 449.7 - 4.2027 \times 10.817 = 404.2\ MPa \tag{1.5}$$

Scilab code for this problem is available on the instructor's Website [2].

```
// Problem 1.10
clc;mode(0);format(7);
// table 1.8
table1_8 = [[2 , 37.094 , 20.581 , 26.26 , 150.89 , 74.913 , 98.081 ];
            [3 , 10.553 , 6.1553 , 7.6559 , 23.377 , 15.458 , 17.884 ];
            [4 , 7.0424 , 4.1619 , 5.1439 , 25.078 , 13.13 , 16.748 ];
            [5 , 5.7411 , 3.4066 , 4.2027 , 19.846 , 10.509 , 13.333 ];
            [6 , 5.062 , 3.0063 , 3.7077 , 17.133 , 9.1436 , 11.558 ];
            [7 , 4.6417 , 2.7554 , 3.3995 , 15.463 , 8.2957 , 10.463 ];
            [8 , 4.3539 , 2.5819 , 3.1873 , 14.324 , 7.7234 , 9.7151 ];
            [9 , 4.143 , 2.4538 , 3.0312 , 13.493 , 7.3003 , 9.1688 ];
            [10 , 3.9811 , 2.3546 , 2.911 , 12.857 , 6.9767 , 8.7508 ];
            [15 , 3.5201 , 2.0684 , 2.566 , 11.06 , 6.0604 , 7.5664 ];
            [20 , 3.2952 , 1.926 , 2.396 , 10.194 , 5.6177 , 6.9949 ];
            [30 , 3.0639 , 1.7773 , 2.2198 , 9.3137 , 5.1672 , 6.4137 ];
            [50 , 2.8624 , 1.6456 , 2.065 , 8.5593 , 4.7808 , 5.9151 ];
            [75 , 2.7481 , 1.5697 , 1.9764 , 8.1376 , 4.5647 , 5.6364 ];
            [100 , 2.684 , 1.5267 , 1.9265 , 7.9037 , 4.4449 , 5.4817 ]];
// data set
F = [443.2, 444.1, 468.8, 444.6, 447.8];
f_ = mean(F) //MPa
s_  = stdev(F) //MPa
n = length(F) // number of specimens in the sample data set
// from Table 1.8 with n=5
// A-basis
kQC = table1_8(n-1,2);//A-basis
fA = f_ - kQC * s_
// B-basis
kQC = table1_8(n-1,3);//B-basis
fA = f_ - kQC * s_
// C-basis
kQC = table1_8(n-1,4);//C-basis
fA = f_ - kQC * s_
```

Problem 1.11

(paper only) Given the strength data F shown below, and assuming that a Weibull distribution represents the data well, calculate the A-, B-, and C-basis values. Use Table 1.8.

F = [443.2, 444.1, 468.8, 444.6, 447.8] MPa

Solution to Problem 1.11

First we calculate the maximum likely estimators (mle) parameters for that data F using MATLAB:

```
mleParams = mle(F,'distribution','Weibull');
```

$$\hat{l} = 454.9 \quad ; \quad \hat{k} = 40.32$$

Then from Table 1.8, for n=5, A-basis (Q = .99, C = .95) we get $V_{QC} = 19.846$ and

$$f_{QC} = \hat{\lambda}[-\log Q]^{1/\hat{k}} \exp\left(-\frac{V_{QC}}{\hat{k}\sqrt{n}}\right) = 325.7\ MPa$$

and for B-basis (Q = .90, C = .95) we get $V_{QC} = 10.51$ and

$$f_{QC} = \hat{\lambda}[-\log Q]^{1/\hat{k}} \exp\left(-\frac{V_{QC}}{\hat{k}\sqrt{n}}\right) = 382.9\ MPa$$

and for V-basis (Q = .90, C = .90) we get $V_{QC} = 13.33$ and

$$f_{QC} = \hat{\lambda}[-\log Q]^{1/\hat{k}} \exp\left(-\frac{V_{QC}}{\hat{k}\sqrt{n}}\right) = 364.6\ MPa$$

MATLAB code for this problem is available on the instructor's Website [2].

```
% Problem 1.11
F = [443.2, 444.1, 468.8, 444.6, 447.8]
n = length(F)
warning('off','stats:mle:ChangedParameters');% suppress warnings
mleParams = mle(F,'distribution','Weibull');
lhat = mleParams(1)
khat = mleParams(2)
% table 1.8
table1_8 = [[2 , 37.094 , 20.581 , 26.26 , 150.89 , 74.913 , 98.081 ];
[3 , 10.553 , 6.1553 , 7.6559 , 23.377 , 15.458 , 17.884 ];
[4 , 7.0424 , 4.1619 , 5.1439 , 25.078 , 13.13 , 16.748 ];
[5 , 5.7411 , 3.4066 , 4.2027 , 19.846 , 10.509 , 13.333 ];
[6 , 5.062 , 3.0063 , 3.7077 , 17.133 , 9.1436 , 11.558 ];
[7 , 4.6417 , 2.7554 , 3.3995 , 15.463 , 8.2957 , 10.463 ];
[8 , 4.3539 , 2.5819 , 3.1873 , 14.324 , 7.7234 , 9.7151 ];
[9 , 4.143 , 2.4538 , 3.0312 , 13.493 , 7.3003 , 9.1688 ];
[10 , 3.9811 , 2.3546 , 2.911 , 12.857 , 6.9767 , 8.7508 ];
[15 , 3.5201 , 2.0684 , 2.566 , 11.06 , 6.0604 , 7.5664 ];
[20 , 3.2952 , 1.926 , 2.396 , 10.194 , 5.6177 , 6.9949 ];
[30 , 3.0639 , 1.7773 , 2.2198 , 9.3137 , 5.1672 , 6.4137 ];
[50 , 2.8624 , 1.6456 , 2.065 , 8.5593 , 4.7808 , 5.9151 ];
[75 , 2.7481 , 1.5697 , 1.9764 , 8.1376 , 4.5647 , 5.6364 ];
[100 , 2.684 , 1.5267 , 1.9265 , 7.9037 , 4.4449 , 5.4817 ]];
```

```
% A-basis
Q = .99; C = .95;
V_QC = table1_8(n-1,5)
f_QC = lhat*(-log(Q))^(1/khat) * exp(-V_QC/khat/sqrt(n))
% B-basis
Q = .90; C = .95;
V_QC = table1_8(n-1,6)
f_QC = lhat*(-log(Q))^(1/khat) * exp(-V_QC/khat/sqrt(n))
% C-basis
Q = .95; C = .95;
V_QC = table1_8(n-1,7)
f_QC = lhat*(-log(Q))^(1/khat) * exp(-V_QC/khat/sqrt(n))
```