Chapter 1

Problem 1.1

Given that "0" is transmitted (in which case the output of the modulator is α), the demodulator will produce the wrong bit ("1") if and only if the value of the noise exceeds α. The probability of that to happen is

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\alpha}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy = \frac{1}{\sqrt{2\pi}} \int_{\frac{\alpha}{\sigma}}^{\infty} e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{t^2}{2}} dt - \frac{1}{\sqrt{2\pi}} \int_{0}^{\frac{\alpha}{\sigma}} e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{2} - \operatorname{erf}(\alpha/\sigma).$$

The same result will be obtained if we assume that "1" is transmitted.

The memoryless property of the BSC follows from the fact that the random variables ν_j and ν_ℓ are statistically independent for $j \neq \ell$.

Problem 1.2

Without loss of generality we can assume that $\bar{x}$, $\bar{y}$, and $\bar{z}$ take the form

$$\begin{aligned}
\bar{x} &= \underbrace{00\ldots0}_{t}\ \underbrace{00\ldots0}_{t}\ \underbrace{00\ldots0}_{t}\ \underbrace{00\ldots0}_{n-3t}\\
\bar{y} &= 11\ldots1\ 11\ldots1\ 00\ldots0\ 00\ldots0\\
\bar{z} &= 11\ldots1\ 00\ldots0\ 11\ldots1\ 00\ldots0
\end{aligned}$$

Now, $d(\bar{x},\bar{v}) = t \Rightarrow w(\bar{v}) = t$, and

$d(\bar{y},\bar{v}) = d(\bar{z},\bar{v}) = t \Rightarrow$ the support of $\bar{v}$ is contained in those of $\bar{y}$ and $\bar{z}$.

Therefore, the only possible choice for $\bar{v}$ is

$$\bar{v} = \underbrace{11\ldots1}_{t}\ \underbrace{00\ldots0\ 00\ldots0\ 00\ldots0}_{n-t}$$

Problem 1.3

Clearly, $\text{rank}(A-B) \geq 0$, with equality if and only if $A=B$.

Secondly, $\text{rank}(A-B) = \text{rank}(-(A-B)) = \text{rank}(B-A)$, thereby proving symmetry.

Thirdly, $\text{rank}(A-C) = \text{rank}(A-B+B-C)$
$$\leq \text{rank}(A-B) + \text{rank}(B-C),$$
thus obtaining transitivity.

Problem 1.4

First, the addition of an overall parity bit will not decrease the minimum distance.

Now, if $\bar{c}_1$ and $\bar{c}_2$ are codewords of the original code such that $d(\bar{c}_1, \bar{c}_2)$ equals the (odd) minimum distance d, then exactly one of the weights, $w(\bar{c}_1)$ and $w(\bar{c}_2)$, is odd. This means that these codewords differ in the added bit.

Problem 1.5

The two codewords are complement of one another; therefore, Perr equals the probability of having three errors or more. Namely,

$$P_{err} = \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5 = 0.00856.$$

Problem 1.6

1. $R = \dfrac{\log_2 16}{7} = 4/7.$
2. If the minimum distance were greater than 3, there would be a codeword in F^7 at Hamming distance at least 2 from any codeword. And if the minimum distance were less than 3, there would be a word in F^7 at distance ≤ 1 from at least two codewords.
3. $1 - (1-p)^7 - \binom{7}{1} p (1-p)^6 \simeq 0.02.$
4. The same value as in part 3.
5. When there is no coding, the error probability is $1 - (1-p)^4 \simeq 0.039.$

Problem 1.7

The computation is very similar to that in Example 1.6, except that here,

$$\text{Prob}\{\bar{y} \text{ received} \mid \bar{c} \text{ transmitted}\}$$
$$= \left(p/(q-1)\right)^{d(\bar{y},\bar{c})} (1-p)^{n-d(\bar{y},\bar{c})}$$
$$= (1-p)^n \left(\frac{p}{(1-p)(q-1)}\right)^{d(\bar{y},\bar{c})},$$

and $\frac{p}{(1-p)(q-1)} < 1$ whenever $p < 1 - \frac{1}{q}$.

Problem 1.8

This follows from the equality

$$\frac{1}{2}\sum_{j=1}^{n} (-1)^{c_j} \mu(y_j) = \log_2 \prod_{j=1}^{n} \text{Prob}(y_j|c_j) - \sum_{j=1}^{n} \log_2 \sqrt{\text{Prob}(y_j|0)\,\text{Prob}(y_j|1)}$$

(and the last sum does not depend on the codeword).

Problem 1.9

1. This follows directly from the definition of P_{err}.

2. Since $\mathcal{D}$ is a MLD, then

$$\mathrm{Prob}(\bar{y}|\bar{c}) \le \mathrm{Prob}(\bar{y}|\bar{c}') \quad \text{for every } \bar{y} \in \mathcal{Y}(\bar{c}').$$

Now combine this inequality with part 1.

3. Since the channel is memoryless,

$$\sum_{\bar{y} \in \Phi^n} \sqrt{\mathrm{Prob}(\bar{y}|\bar{c})\,\mathrm{Prob}(\bar{y}|\bar{c}')}$$

$$= \sum_{y_1 \in \Phi} \sum_{y_2 \in \Phi} \cdots \sum_{y_n \in \Phi} \prod_{j=1}^{n} \sqrt{\mathrm{Prob}(y_j|c_j)\,\mathrm{Prob}(y_j|c_j')}$$

$$= \prod_{j=1}^{n} \sum_{y \in \Phi} \sqrt{\mathrm{Prob}(y|c_j)\,\mathrm{Prob}(y|c_j')}\,.$$

Next notice that when $c_j = c_j'$,

$$\sum_{y \in \Phi} \sqrt{\mathrm{Prob}(y|c_j)\,\mathrm{Prob}(y|c_j')} = \sum_{y \in \Phi} \mathrm{Prob}(y|c_j)$$

$$= 1.$$

4. By the Cauchy-Schwartz inequality,

$$\sum_{y \in \Phi} \sqrt{\mathrm{Prob}(y|c)\,\mathrm{Prob}(y|c')}$$

$$\le \sqrt{\sum_{y \in \Phi} \mathrm{Prob}(y|c)} \cdot \sqrt{\sum_{y \in \Phi} \mathrm{Prob}(y|c')} = 1,$$

(Problem 1.9 – Continued)

with equality if and only if

$\text{Prob}(y|c) = \text{Prob}(y|c')$ for every $y \in \Phi$.

5. This follows from part 3, where now

$$\prod_{j : c_j \neq c_j'} \sum_{y \in \Phi} \sqrt{\text{Prob}(y|c_j)\,\text{Prob}(y|c_j')}$$

$$= \prod_{j : c_j \neq c_j'} \sum_{y \in \Phi} \sqrt{\text{Prob}(y|0)\,\text{Prob}(y|1)}$$

$$= \prod_{j : c_j \neq c_j'} \gamma = \gamma^{d(\bar{c},\bar{c}')} .$$

6. By following the hint, for every $c \neq c'$:

$$\sqrt{\text{Prob}(y|c)\,\text{Prob}(y|c')} = \begin{cases} \sqrt{\frac{(1-p)p}{q-1}} & \text{if } y \in \{c, c'\} \\ \frac{p}{q-1} & \text{otherwise} \end{cases}$$

Therefore,

$$\sum_{y \in \Phi} \text{Prob}(y|c)\,\text{Prob}(y|c') = 2\sqrt{\frac{p(1-p)}{q-1}} + (q-2)\cdot\frac{p}{q-1}$$

Next use part 3.

7. By part 4 we get that $\gamma \le 1$. Therefore,

$$P_{\text{err}}(\bar{c}) \le \sum_{\bar{c}' \in \mathcal{C} \setminus \{\bar{c}\}} \gamma^{d(\bar{c},\bar{c}')} \le (M-1)\gamma^{d} .$$

Problem 1.10

Let $\bar{c}_1$ and $\bar{c}_2$ be two codewords such that $d(\bar{c}_1, \bar{c}_2) = d$. There is a (not necessarily unique) "half way" word $\bar{y} \in F^n$ such that

$$d(\bar{y}, \bar{c}_1) \leq \frac{d+1}{2} \quad \text{and} \quad d(\bar{y}, \bar{c}_2) \leq \frac{d+1}{2}.$$

Clearly, either $\mathcal{D}(\bar{y}) \neq \bar{c}_1$ or $\mathcal{D}(\bar{y}) \neq \bar{c}_2$.

Problem 1.11

1. $R = \frac{\log_2 16}{8} = \frac{1}{2}$.
2. By the triangle inequality:
$$4 \leq d(\bar{c}, \bar{c}') \leq d(\bar{y}, \bar{c}) + d(\bar{y}, \bar{c}').$$
Hence, $d(\bar{y}, \bar{c}) \leq 1 \Longrightarrow d(\bar{y}, \bar{c}') \geq 3$.
3. $\binom{8}{2} p^2 (1-p)^6 \simeq 2.6 \times 10^{-3}$.
4. $1 - (1-p)^8 - \binom{8}{1} p (1-p)^7 - \binom{8}{2} p^2 (1-p)^6 \simeq 5.4 \times 10^{-5}$.
5. This is the sum of the probabilities in parts 3 and 4.
6. If there are two errors then $\mathcal{D}$ necessarily produces "e" (if there are less than two errors, then $\mathcal{D}$ necessarily produces the correct codeword).

Problem 1.12

The probability equals that of having two erasures or more, regardless of the transmitted codeword. This probability is

$$1-(1-p)^4-4p(1-p)^3 = 0.0523.$$

Problem 1.13

Case 1: the transmitted codeword is 0000. Here $\mathcal{D}$ produces "e" if and only if the erasures cover the "1"s in at least one of the other codewords; i.e., when the erasure patterns are

0???, ?0??, ??0?, or ????

The probability is then

$$3p^3(1-p)+p^4 = 0.028.$$

Case 2: the transmitted codeword is (say) 0111. Here $\mathcal{D}$ produces "e" if and only if the erasure patterns are

0???, ????, ??11, ???1, ??1?, ?1?1, ?1??. The probability is then

$$2p^2(1-p)^2+4p^3(1-p)+p^4 = 0.0199.$$