

## Solutions Manual

### Solutions to Chapter 1 Problems

#### S.1.1

The principal stresses are given directly by Eqs (1.11) and (1.12) in which  $\sigma_x = 80\text{N/mm}^2$ ,  $\sigma_y = 0$  (or vice versa) and  $\tau_{xy} = 45\text{N/mm}^2$ . Thus, from Eq. (1.11)

$$\sigma_I = \frac{80}{2} + \frac{1}{2}\sqrt{80^2 + 4 \times 45^2}$$

i.e.

$$\sigma_I = 100.2\text{N/mm}^2$$

From Eq. (1.12)

$$\sigma_{II} = \frac{80}{2} - \frac{1}{2}\sqrt{80^2 + 4 \times 45^2}$$

i.e.

$$\sigma_{II} = -20.2\text{N/mm}^2$$

The directions of the principal stresses are defined by the angle  $\theta$  in Fig. 1.8(b) in which  $\theta$  is given by Eq. (1.10). Hence

$$\tan 2\theta = \frac{2 \times 45}{80 - 0} = 1.125$$

which gives

$$\theta = 24^\circ 11' \quad \text{and} \quad \theta = 114^\circ 11'$$

It is clear from the derivation of Eqs (1.11) and (1.12) that the first value of  $\theta$  corresponds to  $\sigma_I$  while the second value corresponds to  $\sigma_{II}$ .

Finally, the maximum shear stress is obtained from either of Eqs (1.14) or (1.15). Hence from Eq. (1.15)

$$\tau_{\max} = \frac{100.2 - (-20.2)}{2} = 60.2\text{N/mm}^2$$

and will act on planes at  $45^\circ$  to the principal planes.

### S.1.2

The principal stresses are given directly by Eqs (1.11) and (1.12) in which  $\sigma_x = 50 \text{ N/mm}^2$ ,  $\sigma_y = -35 \text{ N/mm}^2$  and  $\tau_{xy} = 40 \text{ N/mm}^2$ . Thus, from Eq. (1.11)

$$\sigma_I = \frac{50 - 35}{2} + \frac{1}{2} \sqrt{(50 + 35)^2 + 4 \times 40^2}$$

i.e.

$$\sigma_I = 65.9 \text{ N/mm}^2$$

and from Eq. (1.12)

$$\sigma_{II} = \frac{50 - 35}{2} - \frac{1}{2} \sqrt{(50 + 35)^2 + 4 \times 40^2}$$

i.e.

$$\sigma_{II} = -50.9 \text{ N/mm}^2$$

From Fig. 1.8(b) and Eq. (1.10)

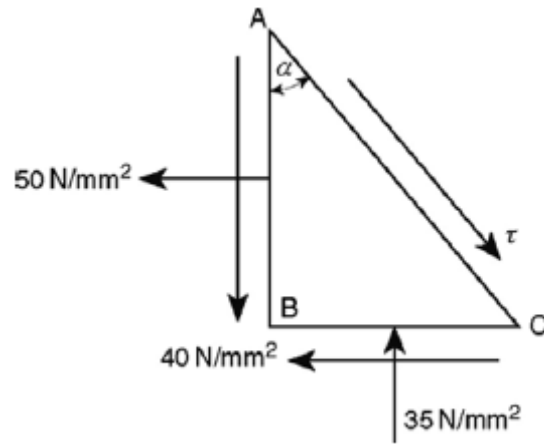
$$\tan 2\theta = \frac{2 \times 40}{50 + 35} = 0.941$$

which gives

$$\theta = 21^\circ 38' (\sigma_I) \quad \text{and} \quad \theta = 111^\circ 38' (\sigma_{II})$$

The planes on which there is no direct stress may be found by considering the triangular element of unit thickness shown in Fig. S.1.2 where the plane AC represents the plane on which there is no direct stress. For equilibrium of the element in a direction perpendicular to AC

$$0 = 50AB \cos \alpha - 35BC \sin \alpha + 40AB \sin \alpha + 40BC \cos \alpha \quad (i)$$



**Fig. S.1.2**

Dividing through Eq. (i) by  $AB$

$$0 = 50 \cos \alpha - 35 \tan \alpha \sin \alpha + 40 \sin \alpha + 40 \tan \alpha \cos \alpha$$

which, dividing through by  $\cos \alpha$ , simplifies to

$$0 = 50 - 35 \tan^2 \alpha + 80 \tan \alpha$$

from which

$$\tan \alpha = 2.797 \quad \text{or} \quad -0.511$$

Hence

$$\alpha = 70^\circ 21' \quad \text{or} \quad -27^\circ 5'$$

### S.1.3

The construction of Mohr's circle for each stress combination follows the procedure described in Section 1.8 and is shown in Figs S.1.3(a)–(d).

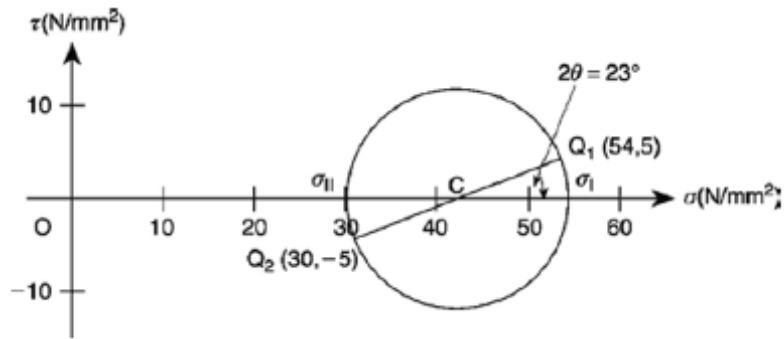


Fig. S.1.3(a)

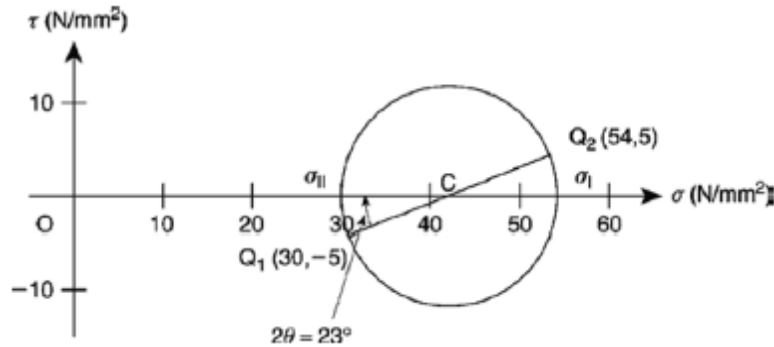


Fig. S.1.3(b)

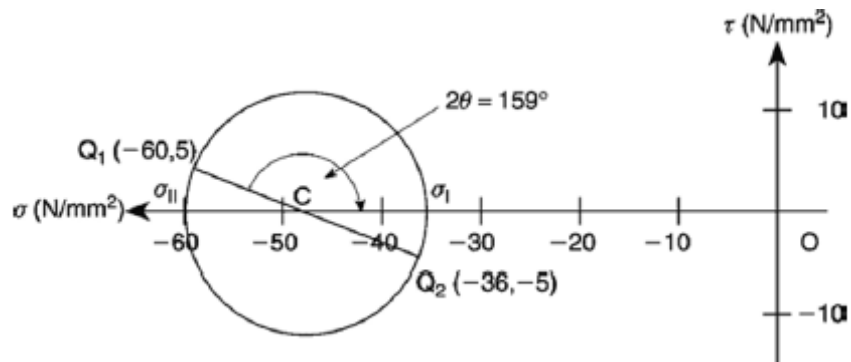


Fig. S.1.3(c)

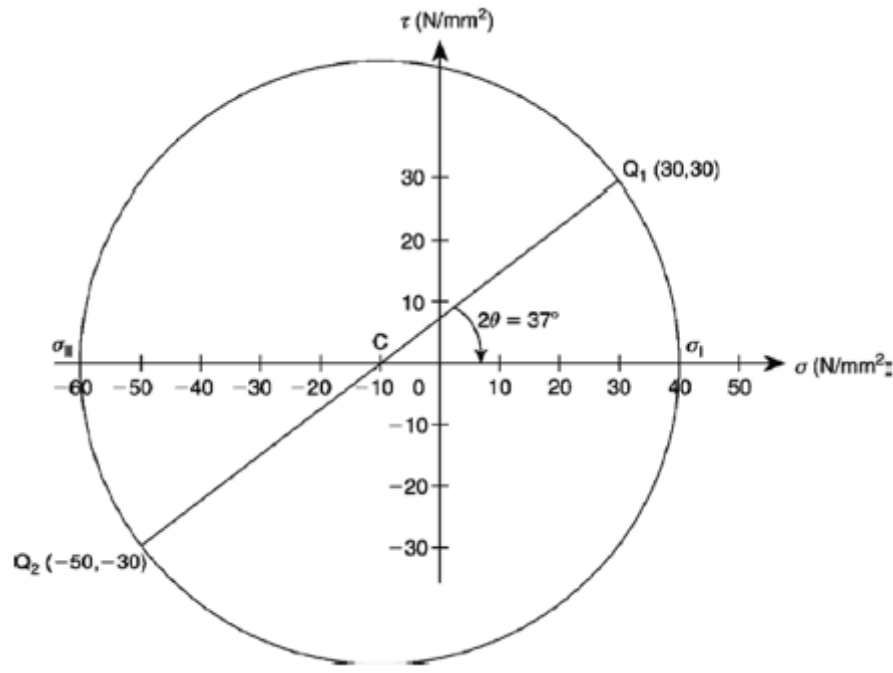


Fig. S.1.3(d)

#### S.1.4

The principal stresses at the point are determined, as indicated in the question, by transforming each state of stress into a  $\sigma_x, \sigma_y, \tau_{xy}$  stress system. Clearly, in the first case  $\sigma_x = 0$ ,  $\sigma_y = 10 \text{ N/mm}^2$ ,  $\tau_{xy} = 0$  (Fig. S.1.4(a)). The two remaining cases are transformed by considering the equilibrium of the triangular element ABC in Figs S.1.4(b), (c), (e) and (f). Thus, using the method described in Section 1.6 and the principle of superposition (see Section 5.9), the second stress system of Figs S.1.4(b) and (c) becomes the  $\sigma_x, \sigma_y, \tau_{xy}$  system shown in Fig. S.1.4(d) while

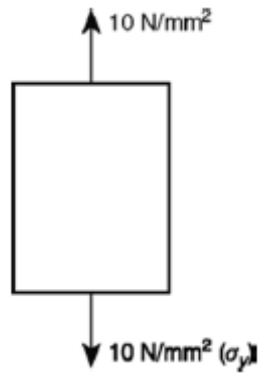


Fig. S.1.4(a)

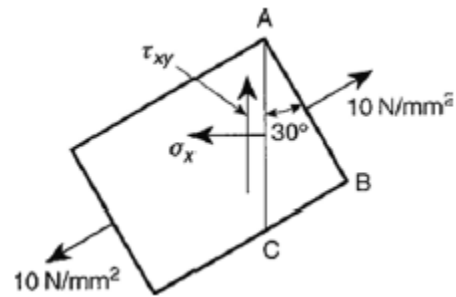


Fig. S.1.4(b)

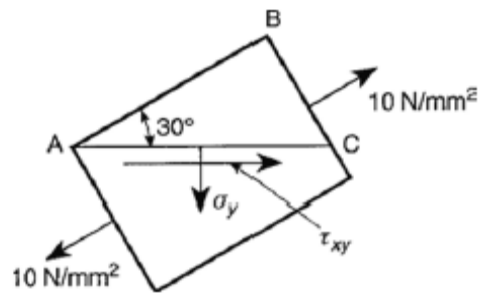


Fig. S.1.4(c)

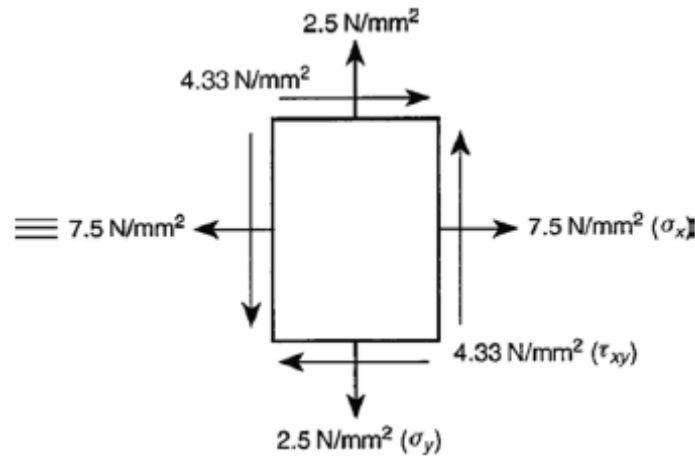


Fig. S.1.4(d)

the third stress system of Figs S.1.4(e) and (f) transforms into the  $\sigma_x$ ,  $\sigma_y$ ,  $\tau_{xy}$  system of Fig. S.1.4(g).

Finally, the states of stress shown in Figs S.1.4(a), (d) and (g) are superimposed to give the state of stress shown in Fig. S.1.4(h) from which it can be seen that  $\sigma_I = \sigma_{II} = 15 \text{ N/mm}^2$  and that the  $x$  and  $y$  planes are principal planes.

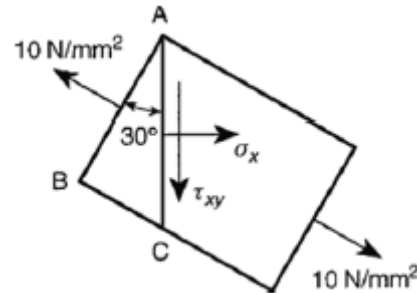


Fig. S.1.4(e)

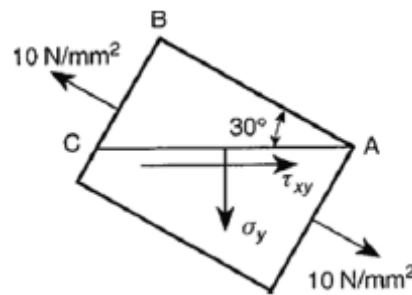


Fig. S.1.4(f)

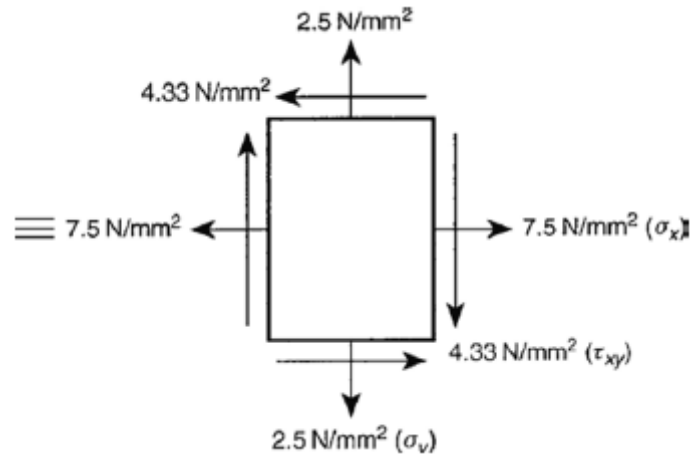


Fig. S.1.4(g)

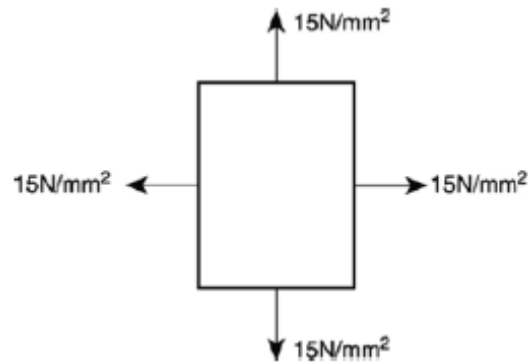


Fig. S.1.4(h)

### S.1.5

The geometry of Mohr's circle of stress is shown in Fig. S.1.5 in which the circle is constructed using the method described in Section 1.8.

From Fig. S.1.5

$$\sigma_x = OP_1 = OB - BC + CP_1 \quad (i)$$



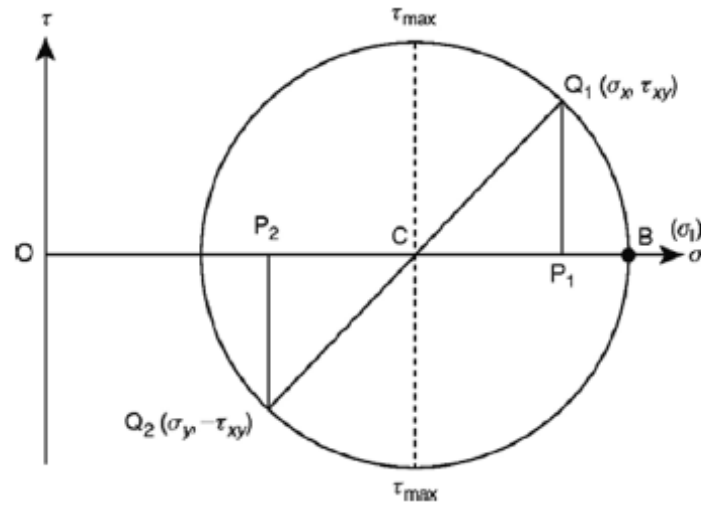


Fig. S.1.5

In Eq. (i)  $OB = \sigma_1$ ,  $BC$  is the radius of the circle which is equal to  $\tau_{\max}$  and  $CP_1 = \sqrt{CQ_1^2 - Q_1P_1^2} = \sqrt{\tau_{\max}^2 - \tau_{xy}^2}$ . Hence

$$\sigma_x = \sigma_1 - \tau_{\max} + \sqrt{\tau_{\max}^2 - \tau_{xy}^2}$$

Similarly

$$\sigma_y = OP_2 = OB - BC - CP_2 \text{ in which } CP_2 = CP_1$$

Thus

$$\sigma_y = \sigma_1 - \tau_{\max} - \sqrt{\tau_{\max}^2 - \tau_{xy}^2}$$

### S.1.6

From bending theory the direct stress due to bending on the upper surface of the shaft at a point in the vertical plane of symmetry is given by

$$\sigma_x = \frac{My}{I} = \frac{25 \times 10^6 \times 75}{\pi \times 150^4 / 64} = 75 \text{ N/mm}^2$$

From the theory of the torsion of circular section shafts the shear stress at the same point is

$$\tau_{xy} = \frac{Tr}{J} = \frac{50 \times 10^6 \times 75}{\pi \times 150^4 / 32} = 75 \text{ N/mm}^2$$

Substituting these values in Eqs (1.11) and (1.12) in turn and noting that  $\sigma_y = 0$

$$\sigma_I = \frac{75}{2} + \frac{1}{2} \sqrt{75^2 + 4 \times 75^2}$$

i.e.

$$\sigma_I = 121.4 \text{ N/mm}^2$$

$$\sigma_{II} = \frac{75}{2} - \frac{1}{2} \sqrt{75^2 + 4 \times 75^2}$$

i.e.

$$\sigma_{II} = -46.4 \text{ N/mm}^2$$

The corresponding directions as defined by  $\theta$  in Fig. 1.8(b) are given by Eq. (1.10)

i.e.

$$\tan 2\theta = \frac{2 \times 75}{75 - 0} = 2$$

Hence

$$\theta = 31^\circ 43' (\sigma_I)$$

and

$$\theta = 121^\circ 43' (\sigma_{II})$$

### S.1.7

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The direct strains are expressed in terms of the stresses using Eqs (1.42), i.e.

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \quad (\text{i})$$

$$\varepsilon_y = \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)] \quad (\text{ii})$$

$$\varepsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \quad (\text{iii})$$

Then

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1}{E} [\sigma_x + \sigma_y + \sigma_z - 2\nu(\sigma_x + \sigma_y + \sigma_z)]$$

i.e.

$$e = \frac{(1-2\nu)}{E}(\sigma_x + \sigma_y + \sigma_z)$$

whence

$$\sigma_y + \sigma_z = \frac{Ee}{(1-2\nu)} - \sigma_x$$

Substituting in Eq. (i)

$$\varepsilon_x = \frac{1}{E} \left[ \sigma_x - \nu \left( \frac{Ee}{1-2\nu} - \sigma_x \right) \right]$$

so that

$$E\varepsilon_x = \sigma_x(1+\nu) - \frac{\nu Ee}{1-2\nu}$$

Thus

$$\sigma_x = \frac{\nu Ee}{(1-2\nu)(1+\nu)} + \frac{E}{(1+\nu)} \varepsilon_x$$

or, since  $G = E/2(1+\nu)$  (see Section 1.15)

$$\sigma_x = \lambda e + 2G\varepsilon_x$$

Similarly

$$\sigma_y = \lambda e + 2G\varepsilon_y$$

and

$$\sigma_z = \lambda e + 2G\varepsilon_z$$

### S.1.8

The implication in this problem is that the condition of plane strain also describes the condition of plane stress. Hence, from Eqs (1.52)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad (i)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \quad (ii)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E} \tau_{xy} \quad (\text{see Section 1.15}) \quad (\text{iii})$$

The compatibility condition for plane strain is

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} \quad (\text{see Section 1.11}) \quad (\text{iv})$$

Substituting in Eq. (iv) for  $\varepsilon_x$ ,  $\varepsilon_y$  and  $\gamma_{xy}$  from Eqs (i)–(iii), respectively, gives

$$2(1+\nu) \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x^2} (\sigma_y - \nu \sigma_x) + \frac{\partial^2}{\partial y^2} (\sigma_x - \nu \sigma_y) \quad (\text{v})$$

Also, from Eqs (1.6) and assuming that the body forces  $X$  and  $Y$  are zero

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (\text{vi})$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \quad (\text{vii})$$

Differentiating Eq. (vi) with respect to  $x$  and Eq. (vii) with respect to  $y$  and adding gives

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial y \partial x} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0$$

or

$$2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = - \left( \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} \right)$$

Substituting in Eq. (v)

$$-(1+\nu) \left( \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} \right) = \frac{\partial^2}{\partial x^2} (\sigma_y - \nu \sigma_x) + \frac{\partial^2}{\partial y^2} (\sigma_x - \nu \sigma_y)$$

so that

$$-(1+\nu) \left( \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} \right) = \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} - \nu \left( \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} \right)$$

which simplifies to

$$\frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0$$

or

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = 0$$

### S.1.9

Suppose that the load in the steel bar is  $P_{st}$  and that in the aluminium bar is  $P_{al}$ . Then, from equilibrium

$$P_{st} + P_{al} = P \quad (i)$$

From Eq. (1.40)

$$\epsilon_{st} = \frac{P_{st}}{A_{st} E_{st}} \quad \epsilon_{al} = \frac{P_{al}}{A_{al} E_{al}}$$

Since the bars contract by the same amount

$$\frac{P_{st}}{A_{st} E_{st}} = \frac{P_{al}}{A_{al} E_{al}} \quad (ii)$$

Solving Eqs (i) and (ii)

$$P_{st} = \frac{A_{st} E_{st}}{A_{st} E_{st} + A_{al} E_{al}} P \quad P_{al} = \frac{A_{al} E_{al}}{A_{st} E_{st} + A_{al} E_{al}} P$$

from which the stresses are

$$\sigma_{st} = \frac{E_{st}}{A_{st} E_{st} + A_{al} E_{al}} P \quad \sigma_{al} = \frac{E_{al}}{A_{st} E_{st} + A_{al} E_{al}} P \quad (iii)$$

The areas of cross-section are

$$A_{st} = \frac{\pi \times 75^2}{4} = 4417.9 \text{ mm}^2 \quad A_{al} = \frac{\pi(100^2 - 75^2)}{4} = 3436.1 \text{ mm}^2$$

Substituting in Eq. (iii) we have

$$\sigma_{st} = \frac{10^6 \times 200000}{(4417.9 \times 200000 + 3436.1 \times 80000)} = 172.6 \text{ N/mm}^2 \text{ (compression)}$$

$$\sigma_{al} = \frac{10^6 \times 80000}{(4417.9 \times 200000 + 3436.1 \times 80000)} = 69.1 \text{ N/mm}^2 \text{ (compression)}$$

Due to the decrease in temperature in which no change in length is allowed the strain in the steel is  $\alpha_{st}T$  and that in the aluminium is  $\alpha_{al}T$ . Therefore due to the decrease in temperature

$$\sigma_{st} = E_{st} \alpha_{st} T = 200000 \times 0.000012 \times 150 = 360.0 \text{ N/mm}^2 \text{ (tension)}$$

$$\sigma_{al} = E_{al} \alpha_{al} T = 80000 \times 0.000005 \times 150 = 60.0 \text{ N/mm}^2 \text{ (tension)}$$

The final stresses in the steel and aluminium are then

$$\sigma_{st}(\text{total}) = 360.0 - 172.6 = 187.4 \text{ N/mm}^2 \text{ (tension)}$$

$$\sigma_{al}(\text{total}) = 60.0 - 69.1 = -9.1 \text{ N/mm}^2 \text{ (compression)}$$

### S.1.10

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The principal strains are given directly by Eqs (1.69) and (1.70). Thus

$$\varepsilon_1 = \frac{1}{2}(-0.002 + 0.002) + \frac{1}{\sqrt{2}} \sqrt{(-0.002 + 0.002)^2 + (+0.002 + 0.002)^2}$$

i.e.

$$\varepsilon_1 = +0.00283$$

Similarly

$$\varepsilon_{II} = -0.00283$$

The principal directions are given by Eq. (1.71), i.e.

$$\tan 2\theta = \frac{2(-0.002) + 0.002 - 0.002}{0.002 + 0.002} = -1$$

Hence

$$2\theta = -45^\circ \text{ or } +135^\circ$$

and

$$\theta = -22.5^\circ \text{ or } +67.5^\circ$$

### S.1.11

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The principal strains at the point **P** are determined using Eqs (1.69) and (1.70). Thus

$$\varepsilon_I = \left[ \frac{1}{2}(-222 + 45) + \frac{1}{\sqrt{2}} \sqrt{(-222 + 213)^2 + (-213 - 45)^2} \right] \times 10^{-6}$$

i.e.

$$\varepsilon_I = 94.0 \times 10^{-6}$$

Similarly

$$\varepsilon_{II} = -271.0 \times 10^{-6}$$

The principal stresses follow from Eqs (1.67) and (1.68). Hence

$$\sigma_I = \frac{31000}{1 - (0.2)^2} (94.0 - 0.2 \times 271.0) \times 10^{-6}$$

i.e.

$$\sigma_I = 1.29 \text{ N/mm}^2$$

Similarly

$$\sigma_{II} = -8.14 \text{ N/mm}^2$$

Since **P** lies on the neutral axis of the beam the direct stress due to bending is zero. Therefore, at **P**,  $\sigma_x = 7 \text{ N/mm}^2$  and  $\sigma_y = 0$ . Now subtracting Eq. (1.12) from (1.11)

$$\sigma_I - \sigma_{II} = \sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

$$1.29 + 8.14 = \sqrt{7^2 + 4\tau_{xy}^2}$$

from which  $\tau_{xy} = 3.16 \text{ N/mm}^2$ .

The shear force at **P** is equal to  $Q$  so that the shear stress at **P** is given by

$$\tau_{xy} = 3.16 = \frac{3Q}{2 \times 150 \times 300}$$

from which

$$Q = 94800 \text{ N} = 94.8 \text{ kN}.$$

## Solutions to Chapter 2 Problems

### S.2.1