

1.1

$$f = f(x, t) = f(x(x', t'), t(t'))$$

$$\frac{\partial f}{\partial t'} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t'} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial t'}$$

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial x'}$$

$$\frac{Df'}{Dt'} \equiv \frac{\partial f'}{\partial t'} + v' \frac{\partial f'}{\partial x'} = \frac{\partial f}{\partial t'} + v' \frac{\partial f}{\partial x'}$$

$$= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t'} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial t'} + v' \left[\frac{\partial f}{\partial x} \frac{\partial x}{\partial x'} \right]$$

$$= V \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + v' \frac{\partial f}{\partial x} = (V + v') \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} \quad \text{Same form as before}$$

$$\frac{Dv'}{Dt'} \equiv \frac{\partial v'}{\partial t'} + v' \frac{\partial v'}{\partial x'} = \frac{\partial}{\partial t'} (v - V) + (v - V) \frac{\partial}{\partial x'} (v - V)$$

$$= \frac{\partial v}{\partial t'} + (v - V) \frac{\partial v}{\partial x'}$$

$$= \frac{\partial v}{\partial x} \frac{\partial x}{\partial t'} + \frac{\partial v}{\partial t} \frac{\partial t}{\partial t'} + (v - V) \left[\frac{\partial v}{\partial x} \frac{\partial x}{\partial x'} \right]$$

$$= V \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - V \frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \quad \text{Same form as before}$$

1.2 Arbitrary Region. Vapor crosses the surface and the volume decreases.

1.3 The velocity of a material region is the velocity of the interface plus the circulation velocity of the liquid (and gas) along the interface.

1.4 Yes If $\underline{w}$ is the normal velocity of the fluid at the control surface, $\underline{w} \equiv (\underline{n} \cdot \underline{u}) \underline{n}$, then fluid does not cross the surface. It can, however, slide along the surface

1.5
$$\underline{v}_i' \equiv \underline{v}_i - \underline{u}$$

$$m_i \underline{v}_i' = m_i \underline{v}_i - m_i \underline{u}$$

$$\sum m_i \underline{v}_i' = \sum m_i \underline{v}_i - \sum m_i \underline{u}$$

$$\sum m_i \underline{v}_i' = \sum m_i \underline{v}_i - \underline{u} \sum m_i$$

$$\frac{\sum m_i \underline{v}_i'}{\sum m_i} = \frac{\sum m_i \underline{v}_i}{\sum m_i} - \underline{u}$$

$$\frac{\sum m_i \underline{v}_i'}{\sum m_i} = \underline{u} - \underline{u} = 0$$

1.6

$$\underline{v}_i = \underline{v}_i' + \underline{v}$$

$$\underline{v}_i \cdot \underline{v}_i = \underline{v}_i' \cdot \underline{v}_i' + 2 \underline{v} \cdot \underline{v}_i' + \underline{v} \cdot \underline{v}$$

$$\sum m_i \frac{1}{2} \underline{v}_i \cdot \underline{v}_i = \sum m_i \frac{1}{2} \underline{v}_i' \cdot \underline{v}_i' + \underbrace{2 \underline{v} \cdot \sum m_i \underline{v}_i'}_{=0 \text{ from 1.5}} + \sum m_i \frac{1}{2} \underline{v} \cdot \underline{v}$$

1.8

$$\frac{\text{momentum}}{\text{unit vol}} = \underline{P} = \lim \frac{\sum m_i \underline{v}_i}{V} = \lim \frac{\sum m_i}{V} \frac{\sum m_i \underline{v}_i}{\sum m_i}$$

$$= \lim \frac{\sum m_i}{V} \lim \frac{\sum m_i \underline{v}_i}{\sum m_i}$$

$$= \rho \underline{v}$$

1.7 Yes. Unsteady processes must maintain equilibrium among the particles. A time could be formed using the typical particle velocity v_p and the continuum length say $l_{\text{mean free path}}$

$$t_{\text{collision}} \sim \frac{l_{\text{mfp}}}{v_p}$$

1.8

1.8

$$\rho = \lim_{L \rightarrow 0} \frac{\sum m_i}{V}$$

$$\underline{v} = \lim_{L \rightarrow 0} \frac{\sum m_i \underline{v}_i}{\sum m_i}$$

$$\rho \underline{v} = \lim_{L \rightarrow 0} \frac{\sum m_i}{V} \frac{\sum m_i \underline{v}_i}{\sum m_i}$$

$$= \lim_{L \rightarrow 0} \frac{\sum m_i \underline{v}_i}{V} = \lim_{L \rightarrow 0} \frac{\sum \underline{P}_i}{V}$$

- 2.1 (a) no, shear forces present
 (b) no, system not homogeneous

3 (d) Could be, neglect contraction & assume normal force only.
 No, if true stress state with compression, tension and shear stresses are accounted for

2.2 Yes, S/E is invariant $S_2 = \lambda S_1$, $E_2 = \lambda E_1$

$$\frac{S_2}{E_2} = \frac{S_1}{E_1}$$

2.3
$$S = R_0 N \left(\frac{E}{E_0} \right)^{1/2} \left(\frac{V}{V_0} \right) \left(\frac{N}{N_0} \right)^{-3/2}$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_{V,N} = R_0 N \frac{1}{2} \left(\frac{1}{E E_0} \right)^{1/2} \left(\frac{V}{V_0} \right) \left(\frac{N}{N_0} \right)^{-3/2}$$

$$\frac{P}{T} = \left(\frac{\partial S}{\partial V} \right)_{E,N} = \frac{R_0 N}{V_0} \left(\frac{E}{E_0} \right)^{1/2} \left(\frac{N}{N_0} \right)^{-3/2}$$

$$\frac{\mu}{T} = \left(\frac{\partial S}{\partial N} \right)_{E,V} = -\frac{1}{2} \frac{R_0 N^{-3/2}}{E_0} \left(\frac{E}{E_0} \right)^{1/2} N_0^{3/2}$$

Euler's Eq
$$S = \frac{1}{T} E + \frac{P}{T} V + \frac{\mu}{T} N$$

$$S = \frac{1}{2} R_0 N \left(\frac{E}{E_0} \right)^{1/2} \left(\frac{V}{V_0} \right) \left(\frac{N}{N_0} \right)^{-3/2} + R_0 N \frac{V}{V_0} \left(\frac{E}{E_0} \right)^{1/2} \left(\frac{N}{N_0} \right)^{-3/2} - \frac{1}{2} R_0 N \left(\frac{E}{E_0} \right)^{1/2} \left(\frac{N}{N_0} \right)^{3/2}$$

$$S = R_0 N \frac{V}{V_0} \left(\frac{E}{E_0} \right)^{1/2} \left(\frac{N}{N_0} \right)^{-3/2} \text{ which check the original equation}$$

2.4

$$dE = TdS - pdV - \mu dN$$

so

$$dS = \frac{1}{T} dE + \frac{p}{T} dV + \frac{\mu}{T} dN$$

$$\frac{1}{T} \equiv \left. \frac{\partial S}{\partial E} \right|_{V,N} \quad \text{or} \quad T = \frac{1}{\left. \frac{\partial S}{\partial E} \right|_{V,N}}$$

$$\frac{p}{T} \equiv \left. \frac{\partial S}{\partial V} \right|_{E,N} \quad \text{or} \quad p = \frac{\left. \partial S / \partial V \right|_{E,N}}{\left. \partial S / \partial E \right|_{V,N}}$$

$$\frac{\mu}{T} \equiv \left. \frac{\partial S}{\partial N} \right|_{E,V} \quad \text{or} \quad \mu = \frac{\left. \partial S / \partial N \right|_{E,V}}{\left. \partial S / \partial E \right|_{V,N}}$$

2.5 Divide 2.7.26 by m to get the fundamental equation in a unit mass basis

$$de = Tds - pdv$$

From the definition of enthalpy 2.9.4 and $v = \frac{1}{\rho}$

$$h = e + pv$$

$$dh = de + pdv + vdp \quad \text{OR} \\ = Tds + pdv$$

Thus since $dh = \left. \frac{\partial h}{\partial s} \right|_v ds + \left. \frac{\partial h}{\partial v} \right|_s dv$

$$T = \left. \frac{\partial h}{\partial s} \right|_v \quad \& \quad v = \left. \frac{\partial h}{\partial v} \right|_s$$

At $p = 30 \text{ psia}$ $h = 1050 \text{ Btu/Lbm} - 650 \text{ Btu/Lbm}$
 $\rho = 1.8 \text{ Btu/(ft}^3 \text{ } ^\circ\text{R)}$
 $T = 1050 \text{ } ^\circ\text{R}$

$$v = \left. \frac{\partial h}{\partial v} \right|_s \approx \left. \frac{\Delta h}{\Delta v} \right|_s = \frac{-630 - (-650)}{(35 - 30) \cdot 144}$$

$$= 0.275 \frac{\text{Btu}}{\text{Lbm}} \frac{\text{ft}^3}{\text{ft}^3} \cdot 778 \frac{\text{ft-Lbm}}{\text{Btu}}$$

$$= 21.40 \frac{\text{ft}^3}{\text{Lbm}}$$

$$\rho = \frac{1}{v} = 0.0463 \frac{\text{Lbm}}{\text{ft}^3}$$

$$h = 1050 \text{ Btu/Lbm} - 650 = 1240 \frac{\text{Btu}}{\text{Lbm}}$$

$$e = h - pv = 1240 - \frac{30 \cdot 144 \cdot 21.4}{778}$$

$$= 1240 - 118.8$$

$$= 1121 \text{ Btu/Lbm}$$

2.6

$$G \equiv H - TS = E + PV - TS$$

but from 2.4.4

$$dE = Tds - p dV - \mu dN$$

so

$$\begin{aligned} dG &= dE + P dV - V dP - T ds - s dT \\ &= -\mu dN - V dP - s dT \end{aligned}$$

This is the fundamental equation of thermodynamics for G and the independent variables are:

$$N, T, P$$

or

$$G = G(T, P, N)$$

gives all thermodynamic information about a substance,

2.7

$$p = \rho R T$$

$$\rho = \frac{p}{RT}$$

$$\alpha = \left(\frac{1}{\rho} \frac{\partial \rho}{\partial p} \right)_T$$

$$= \frac{1}{\rho} \frac{1}{RT} = \frac{1}{p}$$

$$\beta = - \left(\frac{1}{\rho} \frac{\partial \rho}{\partial T} \right)_p$$

$$= - \frac{1}{\rho} \frac{p}{R} \left(- \frac{1}{T^2} \right) = \frac{p}{\rho R T^2} = \frac{1}{T}$$

2.8

$$T ds = de + p d\rho^{-1} \quad (2.7.2 \div M)$$

$$(2.9.3) \quad de = C_v dT + \rho^{-2} \left[p - T \left(\frac{\partial p}{\partial T} \right)_\rho \right] d\rho$$

$$T ds = C_v dT + (-p \rho^{-2}) d\rho + \rho^{-2} \left[p - T \left(\frac{\partial p}{\partial T} \right)_\rho \right] d\rho$$

$$= C_v dT - \left(\frac{1}{\rho^2} \frac{\partial p}{\partial T} \right)_\rho d\rho$$

$$ds = C_v \frac{dT}{T} - \frac{1}{\rho^2} \left(\frac{\partial p}{\partial T} \right)_\rho d\rho$$

2.9

$$p = \rho R T$$

$$\left(\frac{\partial p}{\partial T} \right)_\rho = \rho R$$

$$ds = \int_{T_0, \rho_0}^{T, \rho} \frac{C_v(T, \rho_{const})}{T} dT - R \int_{T_0, \rho_0}^{T, \rho} \frac{d\rho}{\rho}$$

$$s - s_0 = C_v \ln \frac{T}{T_0} - R \ln \frac{\rho}{\rho_0}$$

2.4

$$TdS = dE + pdV + \mu dN$$

or

$$dE = TdS - pdV - \mu dN$$

1st ed

equations of state are

$$T = \left(\frac{\partial E}{\partial S} \right)_{V,N}$$

$$-p = \left(\frac{\partial E}{\partial V} \right)_{S,N} \quad \mu = \left(\frac{\partial E}{\partial N} \right)_{S,N}$$

2.5 see page 2-3

2.6 ~~skip~~ see next page

~~2.6~~ Consider fixed - adiabatic - impermeable is changed to
 old fixed - diathermal - permeable



$\Delta E, \Delta N$

$S = S_A + S_B$ from state ① to state ② $\Delta E, \Delta N$ change

$$\Delta S = \Delta S_A + \Delta S_B = \frac{1}{T_A} \Delta E_A + \frac{1}{T_B} \Delta E_B + \frac{\mu_A}{T_A} \Delta N_A + \frac{\mu_B}{T_B} \Delta N_B$$

for the manifold we find the max state by $\Delta S = 0$

$$\text{Now } \Delta E_A = -\Delta E_B \quad \Delta N_A = -\Delta N_B$$

$$\Delta S = 0 = \left(\frac{1}{T_A} - \frac{1}{T_B} \right) \Delta E_A + \left(\frac{\mu_A}{T_A} - \frac{\mu_B}{T_B} \right) \Delta N_A$$

for arbitrary small changes in ΔE_A & ΔN_A . Hence

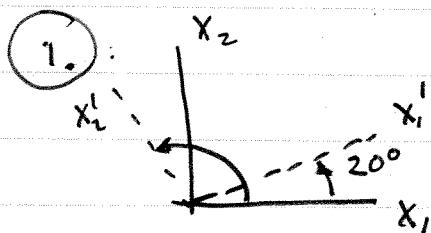
$$T_A = T_B$$

and

$$\mu_A = \mu_B$$

for the member of the manifold with maximum entropy.

Chapter 3 Solutions



In new system coordinates of P are x'_1, x'_2, x'_3 . They are

$$x'_j = C_{ij} x_i$$

Now

$$\begin{aligned} C_{11} &= \cos 20^\circ = .940 & C_{12} &= \cos 110^\circ = -.342 & C_{13} &= \cos 90^\circ = 0 \\ C_{21} &= \cos 70^\circ = .342 & C_{22} &= \cos 20^\circ = .940 & C_{23} &= 0 \\ C_{31} &= \cos 90^\circ = 0 & C_{32} &= 0 & C_{33} &= \cos 0^\circ = 1 \end{aligned}$$

$$\begin{aligned} x'_1 &= C_{i1} x_i = C_{11} x_1 + C_{21} x_2 + C_{31} x_3 \\ &= .94(5) + .342(4) + 0(0) \\ &= 6.07 \end{aligned}$$

$$\begin{aligned} x'_2 &= C_{i2} x_i = C_{12} x_1 + C_{22} x_2 + C_{32} x_3 \\ &= -.342(5) + .940(4) + 0(0) \\ &= 2.05 \end{aligned}$$

$$\begin{aligned} x'_3 &= C_{i3} x_i = C_{13} x_1 + C_{23} x_2 + C_{33} x_3 \\ &= 0 + 0 + 1 \cdot 0 \\ &= 0 \end{aligned}$$

2.

$$a = b_i c_{ij} d_j \quad \text{OK}$$

$$a = b_i c_i + d_j \quad \text{No free index } j \text{ occurs in only one term}$$

$$a_i = \delta_{ij} b_j + c_i \quad \text{OK} \quad \text{" " " " " "}$$

$$a_k = b_i c_{ki} \quad \text{OK}$$

$$a_k = b_k c + d_i e_{ik} \quad \text{OK}$$

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$$a_i = b_i + c_{ij} d_{ji} \quad \text{No } i \text{ appears 3 times in last term}$$

$$a_l = \sum_{j,k} b_j c_k \quad \text{No free index on left is } l \text{ but on right is } i$$

$$a_{ij} = b_{ji} \quad \text{OK}$$

$$a_{ij} = b_i c_j + c_{jk} \quad \text{No free index } k \text{ in last term is not the same as } i \text{ in other terms}$$

$$a_{kl} = b_i c_{ki} d_l + c_{kl} \quad \text{No free index } i \text{ in last term is not same as } l \text{ in others}$$

3. (a) if $u \perp v$ then $u \cdot v = 0$

$$u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 = 3 \cdot 4 + 2 \cdot 1 + (-7) \cdot 2 = 12 + 2 - 14 = 0 \Rightarrow \perp$$

(b)

$$v = \sqrt{4^2 + 1^2 + 2^2} = 4.583$$

$$w = \sqrt{6^2 + 4^2 + (-5)^2} = 8.775$$

(c)

$$\begin{aligned} \underline{v \cdot w} &= v w \cos \theta = v_1 w_1 + v_2 w_2 + v_3 w_3 \\ &= 4 \cdot 6 + 1 \cdot 4 + 2 \cdot (-5) \\ &= 18 \end{aligned}$$

$$\cos \theta = \frac{18}{4.583 \cdot 8.775} = .488$$

$$\underline{\theta = 63.4^\circ}$$

(d) $\alpha_i = \frac{w_i}{w} \quad \alpha_1 = \frac{6}{8.775} = .684$

$$\alpha_2 = \frac{4}{8.775} = .456 \quad \alpha_3 = \frac{-5}{8.775} = -.570$$

(e) $\alpha \cdot u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = .684(3) + .456(2) - .570(-7) = 6.95$

Chapter 3

4. No; the cosines are items which cannot be measured with a single coordinate system

5. δ_{ij} is defined as $\delta_{ii} = 1$, $\delta_{i1} = 0$, $\delta_{i2} = 0$ etc. In a new coordinate system, it would be

$$\delta_{ij}^{''} = C_{ki}^{''} C_{kj}^{''} \delta_{kl}$$

By definition as the substitution tensor we set $l=k$

$$\delta_{ij}^{''} = C_{ki}^{''} C_{kj}^{''}$$

But the RHS by the law of cosines 3.1.16, is δ_{ij} hence

$$\delta_{ij}^{''} = \delta_{ij}$$

The components are the same in any coordinate system

6. $(a \times b) \cdot c = a \cdot (b \times c) = (c \times a) \cdot b$

$$\epsilon_{ijk} a_j b_k c_i = \epsilon_{jki} a_j b_k c_i \quad \text{since } \epsilon_{jki} = \epsilon_{ijk}$$

$$= a \cdot (b \times c)$$

$$\epsilon_{ijk} a_j b_k c_i = \epsilon_{kij} c_i a_j b_k \quad \text{since } \epsilon_{kij} = \epsilon_{ijk}$$

$$= (c \cdot a) \times b$$

7. See next page for the other parts of problem 6

$$T_{(ij)} \equiv \frac{1}{2} T_{ij} + \frac{1}{2} T_{ji} \quad T_{[ij]} \equiv \frac{1}{2} T_{ij} - \frac{1}{2} T_{ji}$$

$$\begin{bmatrix} 6 & 3 & 1 \\ 7 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & \frac{4+3}{2} & \frac{1+1}{2} \\ \frac{7}{2} & 0 & \frac{5+3}{2} \\ 1 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3-4}{2} & \frac{1-1}{2} \\ +\frac{1}{2} & 0 & \frac{5-3}{2} \\ 0 & -1 & 0 \end{bmatrix}$$

7. continued

$$\begin{matrix} T_{ij} & & T_{ji} & & T_{Lij} \\ \begin{bmatrix} 6 & 3 & 1 \\ 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} & = & \begin{bmatrix} 6 & \frac{7}{2} & 1 \\ \frac{7}{2} & 0 & 4 \\ 1 & 4 & 2 \end{bmatrix} & + & \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ +\frac{1}{2} & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \end{matrix}$$

Dual vector

$$d_i = \varepsilon_{ijk} T_{jk}$$

$$d_1 = \varepsilon_{1jk} T_{jk} = \varepsilon_{123} T_{23} + \varepsilon_{132} T_{32}$$

$$= -3 + 5 = +2$$

$$d_2 = \varepsilon_{2jk} T_{jk} = \varepsilon_{231} T_{31} + \varepsilon_{213} T_{13}$$

$$= 1 - 1 = 0$$

$$d_3 = \varepsilon_{3jk} T_{jk} = \varepsilon_{312} T_{12} + \varepsilon_{321} T_{21}$$

$$= 3 - 4 = -1$$

$$\underline{d} = (-2, 0, -1)$$

continued

$$(6) \quad \underline{t} \times (\underline{u} \times \underline{v}) = \underline{u} (\underline{t} \cdot \underline{v}) - \underline{v} (\underline{t} \cdot \underline{u})$$

$$\varepsilon_{pqi} \underline{t}_i \varepsilon_{ijk} u_j v_k = \varepsilon_{ipq} \varepsilon_{ijk} \underline{t}_i u_j v_k$$

$$= (\delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj}) \underline{t}_i u_j v_k$$

$$= v_k \underline{t}_i u_p - v_p \underline{t}_i u_k$$

which is the left side of the equation above

$$\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

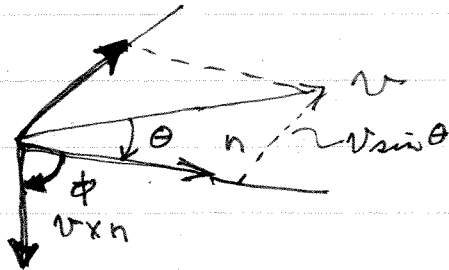
$$\varepsilon_{ijk} u_j v_k = -\varepsilon_{ikj} v_k u_j$$

8. $S_{ij} T_{ji}$ since $S_{ij} = S_{ji}$ & $T_{ji} = -T_{ij}$

$$S_{ij} T_{ji} = S_{ji} (-T_{ij}) = -S_{ji} T_{ij} = -S_{ij} T_{ji}$$

a quantity equal to its negative must be zero

9.



$$v \times n = |v| \sin \theta$$

$$n \times (v \times n) = 1 \cdot v \sin \theta \cdot \sin \phi$$

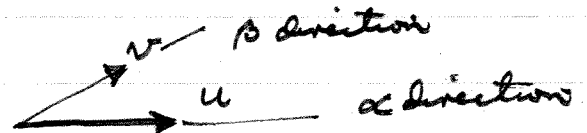
$$\sin \phi = 1$$

$$n \times (v \times n) = v \sin \theta$$

10.

$$w_i = \epsilon_{ijk} u_j v_k$$

$$w = uv \sin(\alpha, \beta)$$



choose x_i system so u_i : $u_1 = u, u_2 = 0, u_3 = 0$

choose x_i^* system so v_i : $v_1 = v, v_2 = 0, v_3 = 0$

$$\begin{aligned} \text{Now } w^2 &= w_i w_i = \epsilon_{ijk} u_j v_k \epsilon_{ipq} u_p v_q = \epsilon_{ijk} \epsilon_{ipq} u_j v_k u_p v_q \\ &= (\delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}) u_j v_k u_p v_q \\ &= u_j u_j v_k v_k - u_j v_j u_k v_k \\ &= u^2 v^2 - u_j v_j u_k v_k \end{aligned}$$

$$\text{Next since } v_k = C_{ik} v_i, u_k v_k = u_k C_{ik} v_i = u_i C_{ii} v_i$$

$$u_k v_k = uv \cos(\alpha, \beta)$$

$\therefore$

$$\begin{aligned} w^2 &= u^2 v^2 - u^2 v^2 \cos^2(\alpha, \beta) = u^2 v^2 (1 - \cos^2(\alpha, \beta)) \\ &= u^2 v^2 \sin^2(\alpha, \beta) \end{aligned}$$

11.

$$\partial_i \delta_j = \partial_i x_j$$

$$\partial_1 x_2 = 0, \partial_1 x_1 = 1 \text{ etc therefore}$$

$$\partial_i x_j = \delta_{ij}$$

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$$\text{div}(\phi \underline{v}) = \phi \text{div} \underline{v} + \underline{v} \cdot \text{grad} \phi$$

$$\nabla \cdot (\phi \underline{v}) = \phi \nabla \cdot \underline{v} + \underline{v} \cdot \nabla \phi$$

$$\partial_i (\phi v_i) = \phi \partial_i v_i + v_i \partial_i \phi$$

$$\text{div}(\underline{u} \times \underline{v}) = \underline{v} \cdot \text{curl} \underline{u} - \underline{u} \cdot \text{curl} \underline{v}$$

$$\nabla \cdot (\underline{u} \times \underline{v}) = \underline{v} \cdot (\nabla \times \underline{u}) - \underline{u} \cdot (\nabla \times \underline{v})$$

$$\partial_i (\varepsilon_{ijk} u_j v_k) = \varepsilon_{ijk} \partial_i u_j v_k = \varepsilon_{ijk} u_j \partial_i v_k + \varepsilon_{ijk} v_k \partial_i u_j$$

$$= u_j (\varepsilon_{jik} \partial_i v_k) + v_k \varepsilon_{kij} \partial_i u_j$$

$$= -\underline{u} \cdot (\nabla \times \underline{v}) + \underline{v} \cdot (\nabla \times \underline{u})$$

$$\text{curl}(\underline{u} \times \underline{v}) = \underline{v} \cdot \text{grad} \underline{u} - \underline{u} \cdot \text{grad} \underline{v} + \underline{u} \text{div} \underline{v} - \underline{v} \text{div} \underline{u}$$

$$\nabla \times (\underline{u} \times \underline{v}) = (\underline{v} \cdot \nabla) \underline{u} - (\underline{u} \cdot \nabla) \underline{v} + \underline{u} \nabla \cdot \underline{v} - \underline{v} \nabla \cdot \underline{u}$$

$$\varepsilon_{lmi} \partial_m (\varepsilon_{ijk} u_j v_k) = \varepsilon_{ilem} \varepsilon_{ijk} \partial_m (u_j v_k)$$

$$= (\delta_{lj} \delta_{mk} - \delta_{lk} \delta_{mj}) \partial_m (u_j v_k)$$

$$= \partial_k (u_l v_k) - \partial_j (u_j v_l)$$

$$= u_l \partial_k v_k + v_k \partial_k u_l - u_j \partial_j v_l - v_l \partial_j u_j$$

which is

$$\underline{u} \nabla \cdot \underline{v} + (\underline{v} \cdot \nabla) \underline{u} - (\underline{u} \cdot \nabla) \underline{v} - (\underline{v} \nabla \cdot \underline{u})$$

Chapter 3

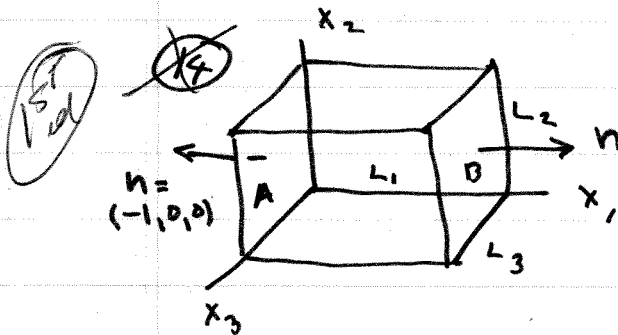
(13) $\partial_i \partial_j (\cdot)$ is symmetric because the order of two differentiation processes may be interchanged.

$$\text{curl grad } \phi, \quad \epsilon_{kji} \partial_j \partial_i \phi = 0$$

This is always zero because ϵ_{kji} is antisymmetric and $\partial_j \partial_i$ is symmetric so the product is zero as in problem 3.8

$$\text{div curl } v, \quad \partial_i (\epsilon_{ijk} \partial_j v_k)$$

$$= \epsilon_{ijk} \partial_i \partial_j v_k = 0$$



$$\int_R \partial_i (T_{jk}) dV = \int_S n_i T_{jk} dS$$

$$T_{jk} = \phi = x_1$$

$$\int_R \partial_i (x_1) dV = \int_S n_i x_1 dS$$

for $i = 1, \partial_i x_1 = 1$

$$\int_R dV = Vol = L_1 \cdot L_2 \cdot L_3$$

$$\begin{aligned} \int_S n_i x_1 dS &= \int_B 1 \cdot L_1 dS + \int_A (-1) \cdot (0) dS \quad n_i = 0 \text{ on all other sides} \\ &= L_1 \int_B dS = L_1 \cdot L_2 \cdot L_3 \end{aligned}$$

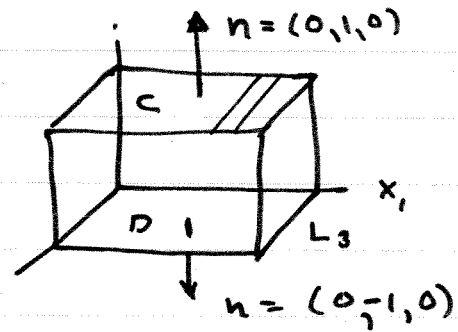
Chapter 3

14 continued 1st ed

for $i=2$ $\partial_2 x_i = 0$

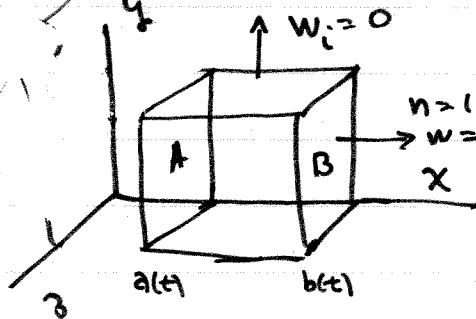
$$\int_R \partial_2 V = 0$$

$$\begin{aligned} \int_S n_2 x_i dS &= \int_C 1 x_i dS + \int_D (-1) x_i dS \\ &= \int x_i L_3 dx_1 - \int x_i L_3 dx_1 \\ &= 0 \end{aligned}$$



for $i=3$ $\partial_3 x_i = 0$ and both integrals are again zero

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der
Graaf



$$\frac{d}{dt} \int_R f(x,t) dV = \int_R \frac{\partial f}{\partial t} dV + \int_S n_i w_i f dS$$

Take $T_{ij}(x_i, t) = f(x, t)$. Let R be $a(t), b(t)$ on x axis and one unit on y & z axes

$$\begin{aligned} \frac{d}{dt} \int_0^1 \int_0^1 \int_a^b f(x,t) dx dy dz &= \frac{d}{dt} \int_a^b f dx \\ \int_0^1 \int_0^1 \int_a^b \frac{\partial f}{\partial t} dV &= \int_a^b \frac{\partial f}{\partial t} dx \end{aligned}$$

$$\begin{aligned} \int_S n_i w_i f dS &= \int_B (1) \frac{db}{dt} f(x,t) dy dz + \int_A (-1) \frac{da}{dt} f(x,t) dy dz \\ &= \frac{db}{dt} f(b,t) - \frac{da}{dt} f(a,t) \end{aligned}$$

$$T_{ij} = -\frac{2}{3}\mu \delta_{ij} \partial_k v_k + 2\mu \partial_i v_j$$

$$\begin{aligned} \text{1st ed } \partial_i T_{ij} &= -\frac{2}{3}\mu \delta_{ij} \partial_i \underbrace{\partial_k v_k}_{=0} + 2\mu \partial_i \left[\frac{1}{2} \partial_i v_j + \frac{1}{2} \partial_j v_i \right] \\ &= \mu \partial_i \partial_i v_j + \mu \partial_i \partial_j v_i \end{aligned}$$

$$= \mu \partial_i v_i v_j + \mu \partial_j \underbrace{\partial_i v_i}_{=0}$$

2nd & 3rd edition

(15)

$$v_j \partial_j v_i = \partial_i \left(\frac{1}{2} v^2 \right) - \varepsilon_{ijk} v_j \omega_k$$

$$\omega_k = \varepsilon_{klm} \partial_l v_m$$

$$\begin{aligned} v_j \partial_j v_i &\stackrel{?}{=} \partial_i \left(\frac{1}{2} v^2 \right) - \varepsilon_{ijk} \varepsilon_{klm} v_j \partial_l v_m \\ &\stackrel{?}{=} \partial_i \left(\frac{1}{2} v^2 \right) - \varepsilon_{kij} \varepsilon_{klm} v_j \partial_l v_m \\ &\stackrel{?}{=} \partial_i \left(\frac{1}{2} v^2 \right) - [\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}] v_j \partial_l v_m \\ &\stackrel{?}{=} \frac{1}{2} v_k \partial_i v_k + \frac{1}{2} v_k \partial_i v_k - v_j \partial_i v_j + v_j \partial_j v_i \\ &= v_j \partial_j v_i \quad \text{OK} \end{aligned}$$

Chapter 3

3.16 $\epsilon_{ijk} \epsilon_{ijl} = ? = 2\delta_{kl}$

$$\epsilon_{ijk} \epsilon_{ipl} = \delta_{jp} \delta_{kl} - \delta_{jl} \delta_{kp}$$

now set $p = j$

$$\epsilon_{ijk} \epsilon_{ijl} = \delta_{jj} \delta_{kl} - \delta_{jl} \delta_{kj}$$

$$\delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$\begin{aligned} \delta_{jl} \delta_{kj} &= \delta_{1l} \delta_{k1} + \delta_{2l} \delta_{k2} + \delta_{3l} \delta_{k3} \\ &= \delta_{kl} \end{aligned}$$

3.17 NEXT PAGE

→ $\epsilon_{ijk} \epsilon_{ijl} = 3\delta_{kl} - \delta_{kl} = 2\delta_{kl}$

3.18 $-\nabla \times \nabla \times \underline{v} = ? = \nabla^2 \underline{v} = \nabla \cdot (\nabla \underline{v})$

$$-\epsilon_{lmn} \partial_m \epsilon_{ijk} \partial_j v_k = \partial_j \partial_j v_l$$

$$-\epsilon_{ilm} \epsilon_{ijk} \partial_m \partial_j v_k =$$

$$-[\delta_{lj} \delta_{mk} - \delta_{lk} \delta_{mj}] \partial_m \partial_j v_k =$$

$$-\{\partial_k \partial_l v_k - \partial_j \partial_j v_l\} =$$

$$\partial_j \partial_j v_l - \underbrace{\partial_k \partial_l v_k}_{=0} = \partial_j \partial_j v_l$$

Chapter 3

3.17

$$\nabla v : \nabla v = S : S - \frac{1}{2} \omega^2$$

$$S_{ij} = \partial_i v_j$$

$$\omega_i = \epsilon_{ijk} \partial_j v_k$$

$$\nabla v \Rightarrow \partial_i v_j = \partial_{ij} v_j + \frac{1}{2} \epsilon_{ijk} d_k \quad (3.6.13)$$

$$d_k = \epsilon_{ijk} \partial_j v_k \text{ or } d_k = \epsilon_{klm} \partial_l v_m \quad (3.6.9)$$

$$\partial_i v_j = \partial_{ij} v_j + \frac{1}{2} \epsilon_{ijk} \underbrace{(\epsilon_{klm} \partial_l v_m)}_{= \omega_k}$$

$$\partial_i v_j \partial_j v_i = \left[\partial_{ij} v_j + \frac{1}{2} \epsilon_{ijk} \omega_k \right] \left[\partial_{ji} v_i + \frac{1}{2} \epsilon_{jik} \omega_k \right]$$

$$= \partial_{ij} v_j \partial_{ji} v_i + \frac{1}{2} \underbrace{\epsilon_{ijk} \omega_k}_{\text{ANTI} \times \text{SYM} = 0} \partial_{ji} v_i$$

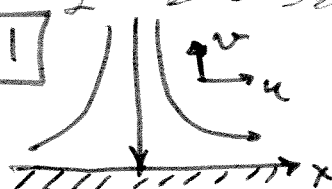
$$+ \frac{1}{4} \epsilon_{ijk} \epsilon_{jik} \omega_k \omega_l + \frac{1}{2} \underbrace{\epsilon_{jik} \omega_k}_{\text{ANTI} \times \text{SYM} = 0} \partial_{ji} v_i$$

$$= \partial_{ij} v_j \partial_{ji} v_i + \frac{1}{4} \underbrace{\epsilon_{ijk} \epsilon_{ijl}}_{= 2 \delta_{kl}} \omega_k \omega_l$$

$$= 2 \delta_{kl} \text{ prob 3.16}$$

$$= \partial_{ij} v_j \partial_{ji} v_i - \frac{1}{2} \omega_k \omega_k$$

4.1



along the stagnation streamline

$$v = -cy$$

since

$$v = \frac{dy}{dt}$$

where y is the particle position

$$t = \int dt = \int \frac{dy}{v} = \int_{y_1}^0 \frac{dy}{-cy} = -\frac{1}{c} \ln \frac{0}{y_1} = \infty$$

4.2

A. round tube

$$v_z = f(r) = v_0 \left[1 - \left(\frac{r}{R} \right)^2 \right]$$

$$\underline{w} = \nabla \times \underline{v}$$

$$w_r = 0, w_z = 0$$

$$w_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = +2v_0 \frac{r}{R^2}$$



B. ideal vortex

$$v_\theta = \frac{\Gamma}{2\pi r}$$

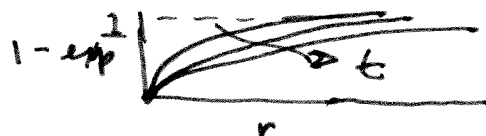
$$w_r = w_\theta = 0$$

$$w_z = \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\Gamma}{2\pi r} \right) = 0$$

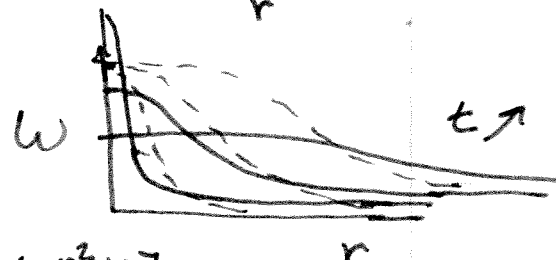
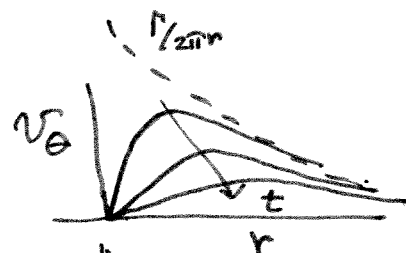


C. viscous vortex

$$v_\theta = \frac{\Gamma}{2\pi r} \left[1 - \exp\left(-\frac{r^2}{4\nu t}\right) \right]$$



$$w_r = w_\theta = 0$$



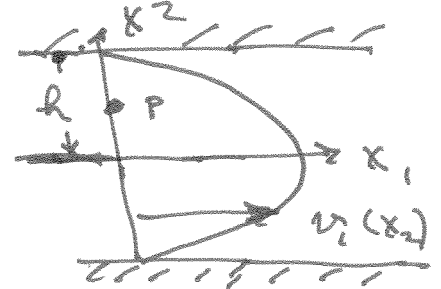
$$w_z = \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) = \frac{1}{r} \frac{\partial}{\partial r} \left[\frac{\Gamma}{2\pi} \left(1 - \exp\left(-\frac{r^2}{4\nu t}\right) \right) \right]$$

$$= \frac{\Gamma}{2\pi r} \frac{2r}{4\nu t} \exp\left(-\frac{r^2}{4\nu t}\right) = \frac{\Gamma}{4\pi \nu t} \exp\left(-\frac{r^2}{4\nu t}\right)$$

Gaussian curve that spreads with $\sqrt{t}$
 Amp decays $\frac{1}{t}$

4.3 (Nondimensional form for answer)

$$v_1 = \frac{3}{2} v_{ave} \left[1 - \left(\frac{x_2}{h} \right)^2 \right]$$



let $v_1/v_{ave} \rightarrow v_1$

and $x_2/h \rightarrow x_2$

so that in nondimensional variables

$$v_1 = \frac{3}{2} [1 - x_2^2]$$

VORTICITY

$$\omega_i = \epsilon_{ijk} \partial_j v_k$$

$$\omega_1 = \omega_2 = 0 ; \quad \omega_3 = -\partial_2 v_1 = 3x_2$$

STRAIN RATE

$$S_{ij} = \frac{1}{2} \partial_i v_j + \frac{1}{2} \partial_j v_i$$

$$S_{11} = \partial_1 v_1 = 0 ; \quad S_{22} = \partial_2 v_2 = 0$$

$$S_{12} = \frac{1}{2} \partial_1 v_2 + \frac{1}{2} \partial_2 v_1 = \frac{1}{2} (-3x_2)$$

$$S_{21} = S_{12}$$

At P where $x_2 = 1/2$

$$\omega_3 = 3/2 ; \quad S_{12} = -3/4$$

Solid Body Rotation $\frac{dv_j^{(r)}}{ds} = \varepsilon_{jki} \left(\frac{\omega_k}{2} \right) \alpha_i$

$$\frac{dv_j^{(r)}}{ds} = \varepsilon_{j3i} \frac{3}{2 \cdot 2} \alpha_i$$

$$\frac{dv_1^{(r)}}{ds} = \varepsilon_{132} \frac{3}{4} \alpha_2 = -\frac{3}{4} \alpha_2 \quad (A)$$

$$\frac{dv_2^{(r)}}{ds} = \varepsilon_{231} \frac{3}{4} \alpha_1 = \frac{3}{4} \alpha_1 \quad (B)$$

Strain (Deformation)

$$\frac{dv_j^{(s)}}{ds} = d_j = \alpha_i S_{ij}$$

$$\begin{aligned} \frac{dv_1^{(s)}}{ds} &= d_1 = \alpha_1 S_{11} + \alpha_2 S_{21} \\ &= \alpha_2 \left(-\frac{3}{4} \right) \end{aligned} \quad (C)$$

$$\begin{aligned} \frac{dv_2^{(s)}}{ds} &= d_2 = \alpha_1 S_{12} + \alpha_2 S_{22} \\ &= \alpha_1 \left(-\frac{3}{4} \right) \end{aligned} \quad (D)$$

Strain (deformation)

$$d_j = \sigma_i S_{ij} \\ = \sigma_1 S_{1j} + \sigma_2 S_{2j}$$

$$d_1 = \sigma_1 S_{11} + \sigma_2 S_{21} \\ = \sigma_2 \left(\frac{-v_0}{2h} \right) \quad (D)$$

$$d_2 = \sigma_1 S_{12} + \sigma_2 S_{22} \\ = \sigma_1 \left(\frac{-v_0}{2h} \right) \quad (E)$$

$$^{(a)} dv_j = d_j ds \\ = d_j (.1h) \quad (F)$$

$$^{(a)} dv_1 = d_1 (.1h) = -.05 v_0 \sigma_2 \quad (G)$$

$$^{(a)} dv_2 = d_2 (.1h) = -.05 v_0 \sigma_1 \quad (H)$$

Elongational Strain

$$dv_j^{(a)} = \sigma_i \sigma_k d_k = \sigma_j [\sigma_1 d_1 + \sigma_2 d_2] ds \quad (I)$$

$$= \sigma_j \left[\sigma_1 \sigma_2 \left(\frac{-v_0}{2h} \right) + \sigma_2 \sigma_1 \left(\frac{-v_0}{2h} \right) \right] ds$$

$$= \sigma_j \sigma_1 \sigma_2 \left(\frac{-v_0}{h} \right) ds$$

$$= \sigma_j \sigma_1 \sigma_2 (-.1 v_0)$$

$$dv_1^{(a)} = \sigma_1^2 \sigma_2 (-.1 v_0) \quad (J)$$

$$dv_2^{(a)} = \sigma_1 \sigma_2^2 (-.1 v_0) \quad (K)$$

Elongation of Strain (strain $\parallel \underline{\alpha}$)

$$\begin{aligned}\frac{dv_i^{(ee)}}{ds} &= \alpha_j (\alpha_k d_k) = \alpha_j (\alpha_1 d_1 + \alpha_2 d_2) \\ &= \alpha_j \left[\alpha_1 \left(\frac{3}{4} \alpha_2 \right) + \alpha_2 \left(\frac{3}{4} \alpha_1 \right) \right] \\ &= -\frac{3}{2} \alpha_1 \alpha_2 \alpha_j\end{aligned}$$

$$\frac{dv_1^{(ee)}}{ds} = -\frac{3}{2} \alpha_1^2 \alpha_2 \quad (E)$$

$$\frac{dv_2^{(ee)}}{ds} = -\frac{3}{2} \alpha_1 \alpha_2^2 \quad (F)$$

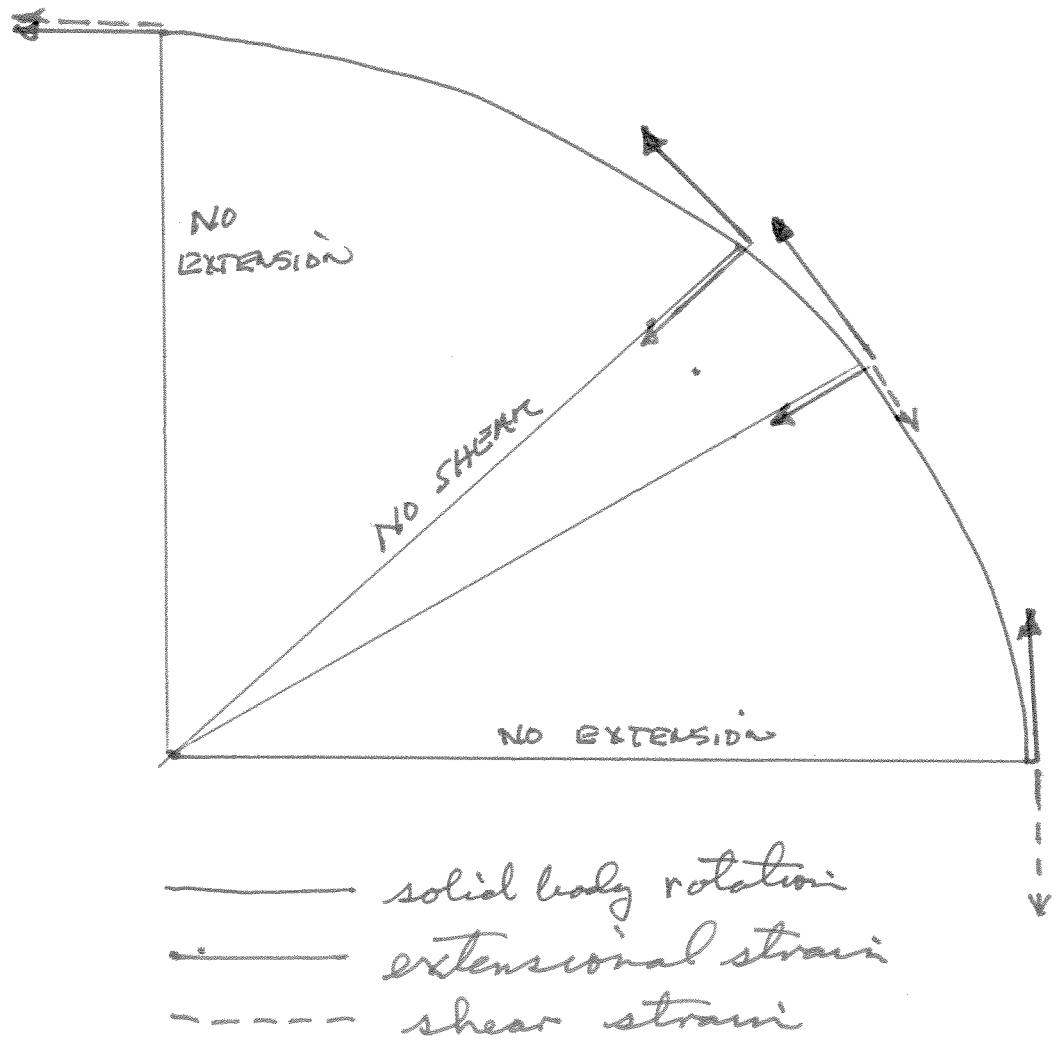
Shear Strain ($\perp \underline{\alpha}$)

$$dv_i^{(se)} = dv_i^{(e)} - dv_j^{(ee)}$$

$$\frac{dv_i^{(se)}}{ds} = d_j + 2\alpha_1 \alpha_2 \alpha_j$$

$$\frac{dv_1^{(se)}}{ds} = -\frac{3}{4} \alpha_2 + 2\alpha_1 \alpha_2^2 \quad (G)$$

$$\frac{dv_2^{(se)}}{ds} = -\frac{3}{4} \alpha_1 + 2\alpha_1 \alpha_2^2 \quad (H)$$



Solution to problem 4.3 using nondimensional velocity profile

$$v_1 = 3/2(1 - x_2^2)$$

v_1 is v_1/v_{ave} and x_2 is x_2/h where h is half width and coordinate is on the centerline.

P is a point at $x_2=0.5$. P' is at an angle θ and distance ds'

The vorticity is nondimensionalized $\Omega_3/v_{ave}/h = dv_1/dx_2$

$$\text{At P:} \quad \omega_1 := 0 \quad \omega_2 := 0 \quad \omega_3 := \frac{3}{2}$$

The strain rate tensor is nondimensionalized $S_{ij}/v_{ave}/h$

$$S_{ij} = 1/2(dv_j/dx_i + dv_i/dx_j)$$

$$\text{At P:} \quad S_{1,1} := 0 \quad S_{1,2} := -0.75 \quad S_{2,1} := -0.75 \quad S_{2,2} := 0$$

The first point at 0 degrees

$$\theta_{deg} := 0 \quad \theta := \theta_{deg} \cdot \frac{\pi}{180} \quad j := 1..2 \quad i := 1..2$$

$$\alpha_1 := \cos(\theta) \quad \alpha_2 := \sin(\theta) \quad \alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$d_i = \left[\sum_{j=1}^2 \alpha_j \cdot (S_{j,i}) \right] \quad d = \begin{bmatrix} 0 \\ 0.75 \end{bmatrix}$$

dv_r is the solid rotational velocity $dv(r)/ds$

$$dv_{r_i} = \sum_{j=1}^2 \epsilon(i, 3, j) \cdot \frac{\omega_3}{2} \cdot \alpha_j \quad dv_r = \begin{bmatrix} 0 \\ 0.75 \end{bmatrix}$$

Magnitude

$$dV_r = (dv_r \cdot dv_r)^{.5} \quad dV_r = 0.75$$

dv_s is the strain velocity $dv(s)/ds$

$$dv_{s_i} := d_i \quad dv_s = \begin{bmatrix} 0 \\ -0.75 \end{bmatrix}$$

$$\text{Magnitude} \quad dV_s := (dv_s \cdot dv_s)^{.5} \quad dV_s = 0.75$$

dv_{es} is the elongational strain velocity $dv(es)/ds$

$$dv_{es_i} := \left[\sum_{j=1}^2 (\alpha_j \cdot d_j) \right] \cdot \alpha_i \quad dv_{es} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Magnitude} \quad dV_{es} := (dv_{es} \cdot dv_{es})^{.5} \quad dV_{es} = 0$$

dv_{ss} is the shear strain velocity $dv(ss)/ds$

$$dv_{ss_i} = dv_{s_i} - dv_{es_i}$$

$$dv_{ss} = \begin{bmatrix} 0 \\ -0.75 \end{bmatrix}$$

Magnitude $dV_{ss} = (dv_{ss} \cdot dv_{ss})^{.5}$

$$dV_{ss} = 0.75$$

The second point at 30 degrees

$$\theta_{deg} = 30 \quad \theta = \theta_{deg} \cdot \frac{\pi}{180} \quad j := 1..2 \quad i := 1..2$$

$$\alpha_1 = \cos(\theta) \quad \alpha_2 = \sin(\theta) \quad \alpha = \begin{bmatrix} 0.866 \\ 0.5 \end{bmatrix}$$

$$d_i = \left[\sum_{j=1}^2 \alpha_j \cdot (S_{j,i}) \right] \quad d = \begin{bmatrix} -0.375 \\ -0.65 \end{bmatrix}$$

dv_r is the solid rotational velocity $dv(r)/ds$

$$dv_{r_i} = \sum_{j=1}^2 \epsilon(i, 3, j) \cdot \frac{\omega_3}{2} \cdot \alpha_j$$

$$dv_r = \begin{bmatrix} -0.375 \\ 0.65 \end{bmatrix}$$

Magnitude

$$dV_r = (dv_r \cdot dv_r)^{.5}$$

$$dV_r = 0.75$$

dv_s is the strain velocity $dv(s)/ds$

$$dv_{s_i} = d_i$$

$$dv_s = \begin{bmatrix} -0.375 \\ -0.65 \end{bmatrix}$$

Magnitude $dV_s = (dv_s \cdot dv_s)^{.5} \quad dV_s = 0.75$

dv_{es} is the elongational strain velocity $dv(es)/ds$

$$dv_{es_i} = \left[\sum_{j=1}^2 (\alpha_j \cdot d_j) \right] \cdot \alpha_i$$

$$dv_{es} = \begin{bmatrix} -0.563 \\ -0.325 \end{bmatrix}$$

Magnitude $dV_{es} = (dv_{es} \cdot dv_{es})^{.5}$

$$dV_{es} = 0.65$$

dv_{ss} is the shear strain velocity $dv(ss)/ds$

$$dv_{ss_i} = dv_{s_i} - dv_{es_i}$$

$$dv_{ss} = \begin{bmatrix} 0.188 \\ -0.325 \end{bmatrix}$$

$$\begin{bmatrix} -0.525 \\ 0 \end{bmatrix}$$

$$\text{Magnitude} \quad dV_{ss} := (dv_{ss} \cdot dv_{ss})^{.5}$$

$$dV_{ss} = 0.375$$

The first point at 45 degrees

$$\theta_{deg} = 45 \quad \theta := \theta_{deg} \cdot \frac{\pi}{180} \quad j := 1..2 \quad i := 1..2$$

$$\alpha_1 = \cos(\theta) \quad \alpha_2 = \sin(\theta) \quad \alpha = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

$$d_i = \left[\sum_{j=1}^2 \alpha_j \cdot (S_{j,i}) \right] \quad d = \begin{bmatrix} -0.53 \\ -0.53 \end{bmatrix}$$

dv_r is the solid rotational velocity $dv(r)/ds$

$$dv_{ri} = \sum_{j=1}^2 \epsilon(i,3,j) \cdot \frac{\omega_3}{2} \cdot \alpha_j$$

$$dv_r = \begin{bmatrix} -0.53 \\ 0.53 \end{bmatrix}$$

Magnitude

$$dV_r := (dv_r \cdot dv_r)^{.5}$$

$$dV_r = 0.75$$

dv_s is the strain velocity $dv(s)/ds$

$$dv_{si} = d_i$$

$$dv_s = \begin{bmatrix} -0.53 \\ -0.53 \end{bmatrix}$$

$$\text{Magnitude} \quad dV_s := (dv_s \cdot dv_s)^{.5}$$

$$dV_s = 0.75$$

dv_{es} is the elongational strain velocity $dv(es)/ds$

$$dv_{esi} = \left[\left[\sum_{j=1}^2 (\alpha_j \cdot d_j) \right] \right] \cdot \alpha_i$$

$$dv_{es} = \begin{bmatrix} -0.53 \\ -0.53 \end{bmatrix}$$

$$dV_{es} = 0.65$$

$$\text{Magnitude} \quad dV_{es} := (dv_{es} \cdot dv_{es})^{.5}$$

dv_{ss} is the shear strain velocity $dv(ss)/ds$

$$dv_{ssi} = dv_{si} - dv_{esi}$$

$$dv_{ss} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Magnitude} \quad dV_{ss} := (dv_{ss} \cdot dv_{ss})^{.5}$$

$$dV_{ss} = 0$$

The last point at 90 degrees

$$\theta_{deg} = 90 \quad \theta = \theta_{deg} \cdot \frac{\pi}{180} \quad j = 1..2 \quad i = 1..2$$

$$\alpha_1 = \cos(\theta) \quad \alpha_2 = \sin(\theta) \quad \alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$d_i = \left[\sum_{j=1}^2 \alpha_j \cdot (S_{j,i}) \right] \quad d = \begin{bmatrix} -0.75 \\ 0 \end{bmatrix}$$

dv_r is the solid rotational velocity dv(r)/ ds

$$dv_{r_i} = \sum_{j=1}^2 \varepsilon(i, 3, j) \cdot \frac{\omega_3}{2} \cdot \alpha_j \quad dv_r = \begin{bmatrix} -0.75 \\ 0 \end{bmatrix}$$

Magnitude

$$dV_r = (dv_r \cdot dv_r)^{.5} \quad dV_r = 0.75$$

dv_s is the strain velocity dv(s)/ ds

$$dv_{s_i} = d_i \quad dv_s = \begin{bmatrix} -0.75 \\ 0 \end{bmatrix}$$

$$\text{Magnitude} \quad dV_s = (dv_s \cdot dv_s)^{.5} \quad dV_s = 0.75$$

dv_es is the elongational strain velocity dv(es)/ ds

$$dv_{es_i} = \left[\sum_{j=1}^2 (\alpha_j \cdot d_j) \right] \cdot \alpha_i \quad dv_{es} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Magnitude} \quad dV_{es} = (dv_{es} \cdot dv_{es})^{.5} \quad dV_{es} = 0$$

dv_ss is the shear strain velocity dv(ss)/ ds

$$dv_{ss_i} = dv_{s_i} - dv_{es_i} \quad dv_{ss} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Magnitude} \quad dV_{ss} = (dv_{ss} \cdot dv_{ss})^{.5} \quad dV_{ss} = 0$$