Exercise 1.34. Many flying and swimming animals – as well as human-engineered vehicles – rely on some type of repetitive motion for propulsion through air or water. For this problem, assume the average travel speed U, depends on the repetition frequency f, the characteristic length scale of the animal or vehicle L, the acceleration of gravity g, the density of the animal or vehicle ρ_o, the density of the fluid ρ, and the viscosity of the fluid μ.
a) Formulate a dimensionless scaling law for U involving all the other parameters.
b) Simplify your answer for a) for turbulent flow where μ is no longer a parameter.
c) Fish and animals that swim at or near a water surface generate waves that move and propagate because of gravity, so g clearly plays a role in determining U. However, if fluctuations in the propulsive thrust are small, then f may not be important. Thus, eliminate f from your answer for b) while retaining L, and determine how U depends on L. Are successful competitive human swimmers likely to be shorter or taller than the average person?
d) When the propulsive fluctuations of a surface swimmer are large, the characteristic length scale may be U/f instead of L. Therefore, drop L from your answer for b). In this case, will higher speeds be achieved at lower or higher frequencies?
e) While traveling submerged, fish, marine mammals, and submarines are usually neutrally buoyant ($\rho_o \approx \rho$) or very nearly so. Thus, simplify your answer for b) so that g drops out. For this situation, how does the speed U depend on the repetition frequency f?
f) Although fully submerged, aircraft and birds are far from neutrally buoyant in air, so their travel speed is predominately set by balancing lift and weight. Ignoring frequency and viscosity, use the remaining parameters to construct dimensionally accurate surrogates for lift and weight to determine how U depends on ρ_o/ρ, L, and g.

Solution 1.34. a) Construct the parameter & units matrix

	U	f	L	g	ρ_o	ρ	μ
M	0	0	0	0	1	1	1
L	1	0	1	1	-3	-3	-1
T	-1	-1	0	-2	0	0	-1

The rank of this matrix is three. There are 7 parameters and 3 independent units, so there will be 4 dimensionless groups. First try to assemble traditional dimensionless groups, but its best to use the solution parameter U only once. Here U is used in the Froude number, so its dimensional counter part, $\sqrt{gL}$, is used in place of U in the Reynolds number.

$$\Pi_1 = \frac{U}{\sqrt{gL}} = \text{Froude number},\quad \Pi_2 = \frac{\rho\sqrt{gL^3}}{\mu} = \text{a Reynolds number}$$

The next two groups can be found by inspection:

$\Pi_3 = \dfrac{\rho_o}{\rho}$ = a density ratio , and the final group must include f: $\Pi_4 = \dfrac{f}{\sqrt{g/L}}$, and is a frequency ratio between f and that of simple pendulum with length L. Putting these together produces:

$$\frac{U}{\sqrt{gL}} = \psi_1\left(\frac{\rho\sqrt{gL^3}}{\mu}, \frac{\rho_o}{\rho}, \frac{f}{\sqrt{g/L}}\right)$$ where, throughout this problem solution, ψ_i, $i = 1, 2, 3, \ldots$ are unknown functions.

b) When μ is no longer a parameter, the Reynolds number drops out: $\frac{U}{\sqrt{gL}} = \psi_2\left(\frac{\rho_o}{\rho}, \frac{f}{\sqrt{g/L}}\right)$.

c) When f is no longer a parameter, then $U = \sqrt{gL} \cdot \psi_3(\rho_o/\rho)$, so that U is proportional to $\sqrt{L}$. This scaling suggests that taller swimmers have an advantage over shorter ones. [Human swimmers best approach the necessary conditions for this part of this problem while doing freestyle (crawl) or backstroke where the arms (and legs) are used for propulsion in an alternating (instead of simultaneous) fashion. Interestingly, this length advantage also applies to ships and sailboats. Aircraft carriers are the longest and fastest (non-planing) ships in any Navy, and historically the longer boat typically won the America's Cup races under the 12-meter rule. Thus, if you bet on a swimming or sailing race where the competitors aren't known to you but appear to be evenly matched, choose the taller swimmer or the longer boat.]

d) Dropping L from the answer for b) requires the creation of a new dimensionless group from f, g, and U to replace Π_1 and Π_4. The new group can be obtained via a product of original dimensionless groups: $\Pi_1\Pi_4 = \frac{U}{\sqrt{gL}}\frac{f}{\sqrt{g/L}} = \frac{Uf}{g}$. Thus, $\frac{Uf}{g} = \psi_4\left(\frac{\rho_o}{\rho}\right)$, or $U = \frac{g}{f}\psi_4\left(\frac{\rho_o}{\rho}\right)$. Here, U is inversely proportional to f which suggests that higher speeds should be obtained at lower frequencies. [Human swimmers of butterfly (and breaststroke to a lesser degree) approach the conditions required for this part of this problem. Fewer longer strokes are typically preferred over many short ones. Of course, the trick for reaching top speed is to properly lengthen each stroke without losing propulsive force].

e) When g is no longer a parameter, a new dimensionless group that lacks g must be made to replace Π_1 and Π_5. This new dimensionless group is $\frac{\Pi_1}{\Pi_5} = \frac{U/\sqrt{gL}}{f/\sqrt{g/L}} = \frac{U}{fL}$, so the overall scaling law must be: $U = fL \cdot \psi_5\left(\frac{\rho_o}{\rho}\right)$. Thus, U will be directly proportional to f. Simple observations of swimming fish, dolphins, whales, etc. verify that their tail oscillation frequency increases at higher swimming speeds, as does the rotation speed of a submarine or torpedo's propeller.

f) Dimensionally-accurate surrogates for weight and lift are: $\rho_o L^3 g$ and $\rho U^2 L^2$, respectively. Set these proportional to each other, $\rho_o L^3 g \propto \rho U^2 L^2$, to find $U \propto \sqrt{\rho_o gL/\rho}$, which implies that larger denser flying objects must fly faster. This result is certainly reasonable when comparing similarly shaped aircraft (or birds) of different sizes.

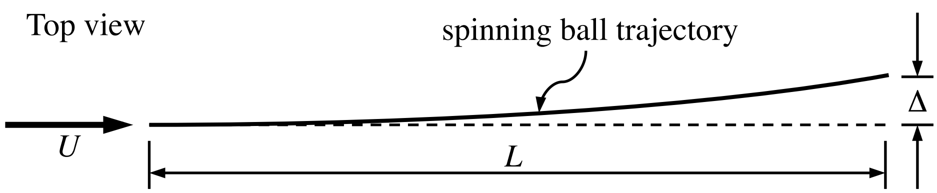
Exercise 1.35. The acoustic power W generated by a large industrial blower depends on its volume flow rate Q, the pressure rise ΔP it works against, the air density ρ, and the speed of sound c. If hired as an acoustic consultant to quiet this blower by changing its operating conditions, what is your first suggestion?

Solution 1.35. The boundary condition and material parameters are: Q, ρ, ΔP, and c. The solution parameter is W. Create the parameter matrix:

	W	Q	ΔP	ρ	c
Mass:	1	0	1	1	0
Length:	2	3	-1	-3	1
Time:	-3	-1	-2	0	-1

This rank of this matrix is three. Next, determine the number of dimensionless groups: 5 parameters - 3 dimensions = 2 groups. Construct the dimensionless groups: $\Pi_1 = W/Q\Delta P$, $\Pi_2 = \Delta P/\rho c^2$. Now write the dimensionless law: $W = Q\Delta P\Phi(\Delta P/\rho c^2)$, where Φ is an unknown function. Since the sound power W must be proportional to volume flow rate Q, you can immediately suggest a decrease in Q as means of lowering W. At this point you do not know if Q must be maintained at high level, so this solution may be viable even though it may oppose many of the usual reasons for using a blower. Note that since Φ is unknown the dependence of W on ΔP cannot be determined from dimensional analysis alone.

Exercise 1.36. The horizontal displacement Δ of the trajectory of a spinning ball depends on the mass m and diameter d of the ball, the air density ρ and viscosity μ, the ball's rotation rate ω, the ball's speed U, and the distance L traveled.
a) Use dimensional analysis to predict how Δ can depend on the other parameters.
b) Simplify your result from part a) for negligible viscous forces.
c) It is experimentally observed that Δ for a spinning sphere becomes essentially independent of the rotation rate once the surface rotation speed, $\omega d/2$, exceeds twice U. Simplify your result from part b) for this high-spin regime.
d) Based on the result in part c), how does Δ depend on U?



Solution 1.36. a) Create the parameter matrix using the solution parameter is Δ, and the boundary condition and material parameters are: Q, ρ, ΔP, and c.

	Δ	m	d	ρ	μ	ω	U	L
Mass:	0	1	0	1	1	0	0	0
Length:	1	0	1	-3	-1	0	1	1
Time:	0	0	0	0	-1	-1	-1	0

This rank of this matrix is three. Next, determine the number of dimensionless groups: 8 parameters - 3 dimensions = 5 groups. Construct the dimensionless groups: $\Pi_1 = \Delta/d$, $\Pi_2 = m/\rho d^3$, $\Pi_3 = \rho U d/\mu$, $\Pi_4 = \omega d/U$, and $\Pi_5 = L/d$. Thus, the dimensionless law for Δ is:

$$\frac{\Delta}{d} = \Phi\left(\frac{m}{\rho d^3}, \frac{\rho U d}{\mu}, \frac{\omega d}{U}, \frac{L}{d}\right),$$

where Φ is an undetermined function.
b) When the viscosity is no longer a parameter, then the third dimensionless group (the Reynolds number) must drop out, so the part a) result simplifies to:

$$\frac{\Delta}{d} = \Psi\left(\frac{m}{\rho d^3}, \frac{\omega d}{U}, \frac{L}{d}\right),$$

where Ψ is another undetermined function.
c) When the rotation rate is no longer a parameter, the fourth dimensionless group from part a) (the Strouhal number) must drop out, so the part b) result simplifies to:

$$\frac{\Delta}{d} = \Theta\left(\frac{m}{\rho d^3}, \frac{L}{d}\right),$$

where Θ is another undetermined function.
d) Interestingly, the part c) result suggests that Δ does not depend on U at all!

Exercise 1.37. A machine that fills peanut-butter jars must be reset to accommodate larger jars. The new jars are twice as large as the old ones but they must be filled in the same amount of time by the same machine. Fortunately, the viscosity of peanut butter decreases with increasing temperature, and this property of peanut butter can be exploited to achieve the desired results since the existing machine allows for temperature control.

a) Write a dimensionless law for the jar-filling time t_f based on: the density of peanut butter ρ, the jar volume V, the viscosity of peanut butter μ, the driving pressure that forces peanut butter out of the machine P, and the diameter of the peanut butter-delivery tube d.

b) Assuming that the peanut butter flow is dominated by viscous forces, modify the relationship you have written for part a) to eliminate the effects of fluid inertia.

c) Make a reasonable assumption concerning the relationship between t_f and V when the other variables are fixed so that you can determine the viscosity ratio μ_{new}/μ_{old} necessary for proper operation of the old machine with the new jars.

d) Unfortunately, the auger mechanism that pumps the liquid peanut butter develops driving pressure through viscous forces so that P is proportional to μ. Therefore, to meet the new jar-filling requirement, what part of the machine should be changed and how much larger should it be?

Solution 1.37. a) First create the parameter matrix. The solution parameter is t_f. The boundary condition and material parameters are: V, ρ, P, μ, and d.

	t_f	V	P	ρ	d	μ
Mass:	0	0	1	1	0	1
Length:	0	3	-1	-3	1	-1
Time:	1	0	-2	0	0	-1

This rank of this matrix is three. Next, determine the number of dimensionless groups: 6 parameters - 3 dimensions = 3 groups. Construct the dimensionless groups: $\Pi_1 = Pt_f/\mu$, $\Pi_2 = \mu^2/\rho d^2 P$, $\Pi_3 = V/d^3$, and write a dimensionless law: $t_f = (\mu/P)\Phi(\mu^2/\rho d^2 P, V/d^3)$, where Φ is an unknown function.

b) When fluid inertia is not important the fluid's density is not a parameter. Therefore, drop Π_2 from the dimensional analysis formula: $t_f = (\mu/P)\Psi(V/d^3)$, where Ψ is yet another unknown function.

c) One might reasonably expect that $t_f \propto V$ (these are the two extensive variables). Therefore, we end up with $t_f =$ const$\cdot\mu V/Pd^3$. Now form a ratio between the old and new conditions and cancel common terms:

$$\frac{(t_f)_{new}}{(t_f)_{old}} = 1 = \frac{(\mu V/Pd^3)_{new}}{(\mu V/Pd^3)_{old}} = \frac{(\mu V)_{new}}{(\mu V)_{old}}, \text{ so } \frac{V_{new}}{V_{old}} = 2 \rightarrow \frac{\mu_{new}}{\mu_{old}} = \frac{1}{2}$$

d) If P is proportional to μ, then to achieve the same filling time for twice the volume using the part c) result for t_f implies,

$$\frac{V_{old}}{(d_{old})^3} = \frac{2V_{old}}{(d_{new})^3}$$

Thus, the machine's nozzle diameter must be increased so that $d_{new} = \sqrt[3]{2}d_{old}$.

Exercise 1.38. As an idealization of fuel injection in a Diesel engine, consider a stream of high-speed fluid (called a *jet*) that emerges into a quiescent air reservoir at $t = 0$ from a small hole in an infinite plate to form a *plume* where the fuel and air mix.
a) Develop a scaling law via dimensional analysis for the penetration distance D of the plume as a function of: Δp the pressure difference across the orifice that drives the jet, d_o the diameter of the jet orifice, ρ_o the density of the fuel, μ_∞ and ρ_∞ the viscosity and density of the air, and t the time since the jet was turned on.
b) Simplify this scaling law for turbulent flow where air viscosity is no longer a parameter.
c) For turbulent flow and $D << d_o$, d_o and ρ_∞ are not parameters. Recreate the dimensionless law for D.
d) For turbulent flow and $D >> d_o$, only the momentum flux of the jet matters, so Δp and d_o are replaced by the single parameter J_o = jet momentum flux (J_o has the units of force and is approximately equal to $\Delta p d_o^2$). Recreate the dimensionless law for D using the new parameter J_o.

Solution 1.38. a) The parameters are: D, t, Δp, ρ_o, ρ_∞, μ_∞, and d_o. With D as the solution parameter, create the parameter matrix:

	D	t	Δp	ρ_o	ρ_∞	μ_∞	d_o
Mass:	0	0	1	1	1	1	0
Length:	1	0	-1	-3	-3	-1	1
Time:	0	1	-2	0	0	-1	0

Next, determine the number of dimensionless groups. This rank of this matrix is three so 7 parameters - 3 dimensions = 4 groups, and construct the groups: $\Pi_1 = D/d_o$, $\Pi_2 = \rho_o/\rho_\infty$, $\Pi_3 = \Delta p t^2/\rho_\infty d_o^2$, and $\Pi_4 = \rho_\infty \Delta p d_o^2/\mu_\infty^2$. Thus, the dimensionless law is:

$$\frac{D}{d_o} = f\left(\frac{\rho_o}{\rho_\infty}, \frac{\Delta p t^2}{\rho_\infty d_o^2}, \frac{\rho_\infty \Delta p d_o^2}{\mu_\infty^2}\right),$$ where f is an unknown function.

b) For high Reynolds number turbulent flow when the reservoir viscosity is no longer a parameter, the above result becomes:

$$\frac{D}{d_o} = g\left(\frac{\rho_o}{\rho_\infty}, \frac{\Delta p t^2}{\rho_\infty d_o^2}\right),$$

where g is an unknown function.
c) When d_o and ρ_∞ are not parameters, there is only one dimensionless group: $\Delta p t^2/\rho_o D^2$, so the dimensionless law becomes: $D = const \cdot t\sqrt{\Delta p/\rho_o}$.
d) When Δp and d_o are replaced by the single parameter J_o = jet momentum flux, there are two dimensionless parameters: $J_o t^2/\rho_\infty D^4$, and ρ_o/ρ_∞, so the dimensionless law becomes:

$$D = \left(J_o t^2/\rho_\infty\right)^{1/4} F(\rho_o/\rho_\infty),$$

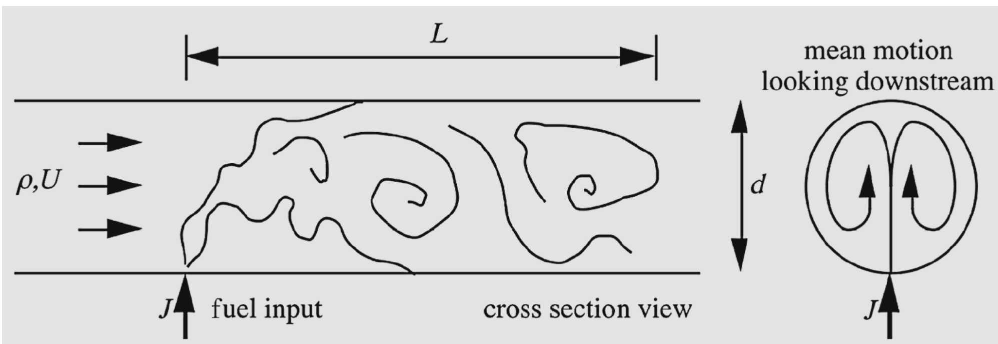
where F is an unknown function.
[The results presented here are the fuel-plume penetration scaling laws for fuel injection in Diesel engines where more than half of the world's petroleum ends up being burned.]

Exercise 1.39. One of the simplest types of gasoline carburetors is a tube with small port for transverse injection of fuel. It is desirable to have the fuel uniformly mixed in the passing air stream as quickly as possible. A prediction of the mixing length L is sought. The parameters of this problem are: ρ = density of the flowing air, d = diameter of the tube, μ = viscosity of the flowing air, U = mean axial velocity of the flowing air, and J = momentum flux of the fuel stream.

a) Write a dimensionless law for L.

b) Simplify your result from part a) for turbulent flow where μ must drop out of your dimensional analysis.

c) When this flow is turbulent, it is observed that mixing is essentially complete after one rotation of the counter rotating vortices driven by the injected-fuel momentum (see downstream-view of the drawing for this problem), and that the vortex rotation rate is directly proportional to J. Based on this information, assume that $L \propto$ (rotation time)(U) to eliminate the arbitrary function in the result of part b). The final formula for L should contain an undetermined dimensionless constant.



Solution 1.39. a) The parameters are: L, J, d, μ, ρ, and U. Use these to create the parameter matrix with L as the solution parameter:

	L	J	d	μ	ρ	U
Mass:	0	1	0	1	1	0
Length:	1	1	1	-1	-3	1
Time:	0	-2	0	-1	0	-1

Next, determine the number of dimensionless groups. This rank of this matrix is three so 6 parameters - 3 dimensions = 3 groups, and construct them: $\Pi_1 = L/d$, $\Pi_2 = \rho Ud/\mu$, $\Pi_3 = \rho U^2 d^2/J$. And, finally write a dimensionless law: $L = d\,\Phi(\rho Ud/\mu, \rho U^2 d^2/J)$, where Φ is an unknown function.

b) At high Reynolds numbers, μ must not be a parameter. Therefore: $L = d\Psi(\rho U^2 d^2/J)$ where Ψ is an unknown function.

c) Let Ω = vortex rotation rate. The units of Ω are 1/time and Ω must be proportional to J. Putting this statement in dimensionless terms based on the boundary condition and material parameters of this problem means: $\Omega = \text{const}\,\dfrac{J}{\rho U d^3} = (\text{rotation time})^{-1}$

Therefore: $L = \text{const}\,(\Omega^{-1})U = \text{const}\,\dfrac{\rho U^2 d^3}{J}$, or $\dfrac{L}{d} = \text{const}\,\dfrac{\rho U^2 d^2}{J}$. Thus, for transverse injection, more rapid mixing occurs (L decreases) when the injection momentum increases.

Exercise 1.40. Consider dune formation in a large horizontal desert of deep sand.
a) Develop a scaling relationship that describes how the height h of the dunes depends on the average wind speed U, the length of time the wind has been blowing Δt, the average weight and diameter of a sand grain w and d, and the air's density ρ and kinematic viscosity ν.
b) Simplify the result of part a) when the sand-air interface is fully rough and ν is no longer a parameter.
c) If the sand dune height is determined to be proportional to the density of the air, how do you expect it to depend on the weight of a sand grain?

Solution 1.40. a) The solution parameter is h. The boundary condition and material parameters are: U, Δt, w, d, ρ, and ν. First create the parameter matrix:

	h	U	Δt	w	d	ρ	ν
Mass:	0	0	0	1	0	1	0
Length:	1	1	0	1	1	-3	2
Time:	0	-1	1	-2	0	0	-1

Next determine the number of dimensionless groups. This rank of this matrix is three so 7 parameters - 3 dimensions = 4 groups. Construct the dimensionless groups: $\Pi_1 = h/d$, $\Pi_2 = Ud/\nu$, $\Pi_3 = w/\rho U^2 d^2$, and $\Pi_4 = U\Delta t/d$. Thus, the dimensionless law is

$$\frac{h}{d} = \Phi\left(\frac{Ud}{\nu}, \frac{w}{\rho U^2 d^2}, \frac{U\Delta t}{d}\right),$$

where Φ is an unknown function.
b) When ν is no longer a parameter, Π_2 drops out:

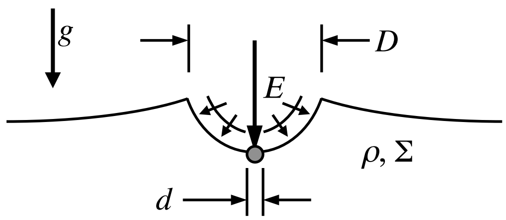
$$\frac{h}{d} = \Psi\left(\frac{w}{\rho U^2 d^2}, \frac{U\Delta t}{d}\right),$$

where Ψ is another unknown function.
c) When h is proportional to ρ, then

$$\frac{h}{d} = \frac{\rho U^2 d^2}{w}\Theta\left(\frac{U\Delta t}{d}\right),$$

where Θ is another unknown function. Under this condition, dune height will be inversely proportional to w the sand grain weight.

Exercise 1.41. The rim-to-rim diameter D of the impact crater produced by a vertically-falling object depends on d = average diameter of the object, E = kinetic energy of the object lost on impact, ρ = density of the ground at the impact site, Σ = yield stress of the ground at the impact site, and g = acceleration of gravity.
a) Using dimensional analysis, determine a scaling law for D.
b) Simplify the result of part a) when $D >> d$, and d is no longer a parameter.
c) Further simplify the result of part b) when the ground plastically deforms to absorb the impact energy and ρ is irrelevant. In this case, does gravity influence D? And, if E is doubled how much bigger is D?
d) Alternatively, further simplify the result of part b) when the ground at the impact site is an unconsolidated material like sand where Σ is irrelevant. In this case, does gravity influence D? And, if E is doubled how much bigger is D?
e) Assume the relevant constant is unity and invert the algebraic relationship found in part d) to estimate the impact energy that formed the 1.2-km-diameter Barringer Meteor Crater in Arizona using the density of Coconino sandstone, 2.3 g/cm^3, at the impact site. The impact energy that formed this crater is likely between 10^{16} and 10^{17} J. How close to this range is your dimensional analysis estimate?



Solution 1.41. The solution parameter is D. The boundary condition and material parameters are: d, E, θ, ρ, Σ, and g. First create the parameter matrix:

	D	d	E	ρ	Σ	g
M	0	0	1	1	1	0
L	1	1	2	–3	–1	1
T	0	0	–2	0	–2	–2

The rank of this matrix is 3, so there are 6 – 3 = 4 dimensionless groups. These groups are: $\Pi_1 = D/d$, $\Pi_2 = E/\rho g d^4$, and $\Pi_3 = \Sigma/\rho g d$. Thus, the scaling law is:

$D/d = fn(E/\rho g d^4, \Sigma/\rho g d)$,

where fn is an undetermined function.
b) Use the second dimensionless group to remove d from the other two:

$$\frac{D}{d}\left(\frac{E}{\rho g d^4}\right)^{-1/4} = \Phi\left(\frac{\Sigma}{\rho g d}\left(\frac{E}{\rho g d^4}\right)^{-1/4}\right), \text{ or } \frac{D}{(E/\rho g)^{1/4}} = \Phi\left(\frac{\Sigma}{(\rho^3 g^3 E)^{1/4}}\right).$$

where Φ is an undetermined function.
c) In this case, the two dimensionless groups in the part b) must be combined to eliminate ρ,

$$\frac{D}{(E/\rho g)^{1/4}}\frac{\Sigma^{1/3}}{(\rho^3 g^3 E)^{1/12}} = \frac{D}{(E/\Sigma)^{1/3}} = const.,$$

and this lone dimensionless group must be constant. In this case, gravity does not influence the crater diameter, and a doubling of the energy E increases D by a factor of $2^{1/3} \cong 1.26$.

d) When S is irrelevant, then the part b) result reduces to:

$$\frac{D}{(E/\rho g)^{1/4}} = const.$$

In this case, gravity does influence the crater diameter, and a doubling of the energy E increases D a factor of $2^{1/4} \cong 1.19$.

e) If the part d) constant is unity, then that result implies: $E = \rho g D^4$. For the Barringer crater, this energy is:

$$E = (2300 \text{ kgm}^{-3})(9.81 \text{ ms}^{-2})(1200\text{m})^4 \approx 4.7 \times 10^{16} \text{ J},$$

an estimate that falls in the correct range, a remarkable result given the simplicity of the analysis.

Exercise 1.42. An isolated nominally spherical bubble with radius R undergoes shape oscillations at frequency f. It is filled with air having density ρ_a and resides in water with density ρ_w and surface tension σ. What frequency ratio should be expected between two isolated bubbles with 2 cm and 4 cm diameters undergoing geometrically similar shape oscillations? If a soluble surfactant is added to the water that lowers σ by a factor of two, by what factor should air bubble oscillation frequencies increase or decrease?

Solution 1.42. The boundary condition and material parameters are: R, ρ_a, ρ_w, and σ. The solution parameter is f. First create the parameter matrix:

	f	R	ρ_a	ρ_w	σ
Mass:	0	0	1	1	1
Length:	0	1	-3	-3	0
Time:	-1	0	0	0	-2

Next determine the number of dimensionless groups. This rank of this matrix is three, so 5 parameters - 3 dimensions = 2 groups. Construct the dimensionless groups: $\Pi_1 = f\sqrt{\rho_w R^3/\sigma}$, and $\Pi_2 = \rho_w/\rho_a$. Thus, the dimensionless law is

$$f = \sqrt{\frac{\sigma}{\rho_w R^3}}\Phi\left(\frac{\rho_w}{\rho_a}\right),$$

where Φ is an unknown function. For a fixed density ratio, $\Phi(\rho_w/\rho_a)$ will be constant so f is proportional to $R^{-3/2}$ and to $\sigma^{1/2}$. Thus, the required frequency ratio between different sizes bubbles is:

$$\frac{(f)_{2cm}}{(f)_{4cm}} = \left(\frac{2cm}{4cm}\right)^{-3/2} = 2\sqrt{2} \cong 2.83.$$

Similarly, if the surface tension is decreased by a factor of two, then

$$\frac{(f)_{\sigma/2}}{(f)_{\sigma}} = \left(\frac{1/2}{1}\right)^{-1/2} = \frac{1}{\sqrt{2}} \cong 0.707.$$

Exercise 1.43. In general, boundary layer skin friction, τ_w, depends on the fluid velocity U above the boundary layer, the fluid density ρ, the fluid viscosity μ, the nominal boundary layer thickness δ, and the surface roughness length scale ε.
a) Generate a dimensionless scaling law for boundary layer skin friction.
b) For laminar boundary layers, the skin friction is proportional to μ. When this is true, how must τ_w depend on U and ρ?
c) For turbulent boundary layers, the dominant mechanisms for momentum exchange within the flow do not directly involve the viscosity μ. Reformulate your dimensional analysis without it. How must τ_w depend on U and ρ when μ is not a parameter?
d) For turbulent boundary layers on smooth surfaces, the skin friction on a solid wall occurs in a viscous sublayer that is very thin compared to δ. In fact, because the boundary layer provides a buffer between the outer flow and this viscous sub-layer, the viscous sublayer thickness l_v does not depend directly on U or δ. Determine how l_v depends on the remaining parameters.
e) Now consider nontrivial roughness. When ε is larger than l_v a surface can no longer be considered fluid-dynamically smooth. Thus, based on the results from parts a) through d) and anything you may know about the relative friction levels in laminar and turbulent boundary layers, are high- or low-speed boundary layer flows more likely to be influenced by surface roughness?

Solution 1.43. a) Construct the parameter & units matrix and recognizing that τ_w is a stress and has units of pressure.

	τ_w	U	ρ	μ	δ	ε
M	1	0	1	1	0	0
L	-1	1	-3	-1	1	1
T	-2	-1	0	1	0	0

The rank of this matrix is three. There are 6 parameters and 3 independent units, thus there will be 6 – 3 = 3 dimensionless groups. By inspection these groups are: a skin-friction coefficient = $\Pi_1 = \frac{\tau_w}{\rho U^2}$, a Reynolds number = $\Pi_2 = \frac{\rho U\delta}{\mu}$, and the relative roughness = $\Pi_3 = \frac{\varepsilon}{\delta}$. Thus the dimensionless law is: $\frac{\tau_w}{\rho U^2} = f\left(\frac{\rho U\delta}{\mu}, \frac{\varepsilon}{\delta}\right)$ where f is an undetermined function.

b) Use the result of part a) and set $\tau_w \propto \mu$. This involves requiring Π_1 to be proportional to $1/\Pi_2$ so the revised form of the dimensionless law in part a) is: $\frac{\tau_w}{\rho U^2} = \frac{\mu}{\rho U\delta} g\left(\frac{\varepsilon}{\delta}\right)$, where g is an undetermined function. Simplify this relationship to find: $\tau_w = \frac{\mu U}{\delta} g\left(\frac{\varepsilon}{\delta}\right)$. Thus, in laminar boundary layers, τ_w is proportional to U and independent of ρ.

c) When μ is not a parameter the second dimensionless group from part a) must be dropped. Thus, the dimensionless law becomes: $\frac{\tau_w}{\rho U^2} = h\left(\frac{\varepsilon}{\delta}\right)$ where h is an undetermined function. Here we see that $\tau_w \propto \rho U^2$. Thus, in turbulent boundary layers, τ_w is linearly proportional to ρ and quadratically proportional to U. In reality, completely dropping μ from the dimensional analysis is not quite right, and the skin-friction coefficient (Π_1 in the this problem) maintains a weak dependence on the Reynolds number when $\varepsilon/\delta << 1$.

d) For this part of this problem, it is necessary to redo the dimensional analysis with the new length scale l_ν and the three remaining parameters: τ_w, ρ, and μ. Here there are four parameters and three units, so there is only one dimensionless group: $\Pi = \frac{l_\nu\sqrt{\rho\tau_w}}{\mu}$. This means that:

$$l_\nu \propto \mu\Big/\sqrt{\rho\tau_w} = \nu\Big/\sqrt{\tau_w/\rho} = \nu/u_* .$$

In the study of wall bounded turbulent flows, the length scale l_ν is commonly known as the viscous wall unit and u_* is known as the friction or shear velocity.

e) The results of part b) and part c) both suggest that τ_w will be larger at high flow speeds than at lower flow speeds. This means that l_ν will be smaller at high flow speeds for both laminar and turbulent boundary layers. Thus, boundary layers in high-speed flows are more likely to be influenced by constant-size surface roughness.

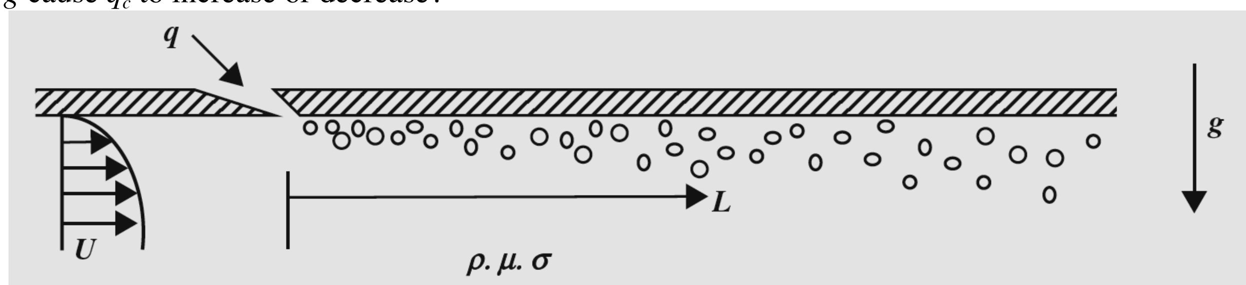
Exercise 1.44. Turbulent boundary layer skin friction is one of the fluid phenomena that limit the travel speed of aircraft and ships. One means for reducing the skin friction of liquid boundary layers is to inject a gas (typically air) from the surface on which the boundary layer forms. The shear stress, τ_w, that is felt a distance L downstream of such an air injector depends on: the volumetric gas flux per unit span q (in m^2/s), the free stream flow speed U, the liquid density ρ, the liquid viscosity μ, the surface tension σ, and gravitational acceleration g.

a) Formulate a dimensionless law for τ_w in terms of the other parameters.

b) Experimental studies of air injection into liquid turbulent boundary layers on flat plates has found that the bubbles may coalesce to form an air film that provides near perfect lubrication, $\tau_w \to 0$ for $L > 0$, when q is high enough and gravity tends to push the injected gas toward the plate surface. Reformulate your answer to part a) by dropping τ_w and L to determine a dimensionless law for the minimum air injection rate, q_c, necessary to form an air layer.

c) Simplify the result of part c) when surface tension can be neglected.

d) Experimental studies (Elbing et al. 2008) find that q_c is proportional to U^2. Using this information, determine a scaling law for q_c involving the other parameters. Would an increase in g cause q_c to increase or decrease?



Solution 1.44. a) Construct the parameter & units matrix and recognizing that τ_w is a stress and has units of pressure.

	τ_w	L	q	U	ρ	μ	σ	g
M	1	0	0	0	1	1	1	0
L	-1	1	2	1	-3	-1	0	1
T	-2	0	-1	-1	0	-1	-2	-2

This rank of this matrix is three. There are 8 parameters and 3 independent units, thus there will be 8 – 3 = 5 dimensionless groups. By inspection these groups are: a skin-friction coefficient = $\Pi_1 = \frac{\tau_w}{\rho U^2}$, a Reynolds number = $\Pi_2 = \frac{\rho UL}{\mu}$, a Froude number = $\Pi_3 = \frac{U}{\sqrt{gL}}$, a capillary number = $\Pi_4 = \frac{\mu U}{\sigma}$, and flux ratio = $\Pi_5 = \frac{\rho q}{\mu}$. Thus the dimensionless law is:

$$\frac{\tau_w}{\rho U^2} = f\left(\frac{\rho UL}{\mu}, \frac{U}{\sqrt{gL}}, \frac{\mu U}{\sigma}, \frac{\rho q}{\mu}\right)$$ where f is an undetermined function.

b) Dropping τ_w means dropping Π_1. Dropping L means combining Π_2 and Π_3 to form a new dimensionless group: $\Pi_2\Pi_3^2 = \frac{\rho UL}{\mu}\frac{U^3}{gL} = \frac{\rho U^3}{\mu g}$. Thus, with Π_5 as the solution parameter, the

scaling law for the minimum air injection rate, q_c, necessary to form an air layer is:
$\rho q_c/\mu = \phi\left(\rho U^3/\mu g, \mu U/\sigma\right)$ where ϕ is an undetermined function.

c) When σ is not a parameter, Π_4 can be dropped leaving: $\rho q_c/\mu = \varphi\left(\rho U^3/\mu g\right)$ where φ is an undetermined function.

d) When q_c is proportional to U^2, then dimensional analysis requires:

$$q_c = (\mu/\rho) const.\left(\rho U^3/\mu g\right)^{2/3} = const. U^2\left(\mu/\rho g^2\right)^{1/3}.$$

So, an increase in g would cause q_c to decrease.

Exercise 1.45. An industrial cooling system is in the design stage. The pumping requirements are known and the drive motors have been selected. For maximum efficiency the pumps will be directly driven (no gear boxes). The number N_p and type of water pumps are to be determined based on pump efficiency η (dimensionless), the total required volume flow rate Q, the required pressure rise ΔP, the motor rotation rate Ω, and the power delivered by one motor W. Use dimensional analysis and simple physical reasoning for the following items.
a) Determine a formula for the number of pumps.
b) Using Q, N_p, ΔP, Ω, and the density (ρ) and viscosity (μ) of water, create the appropriate number of dimensionless groups using ΔP as the dependent parameter.
c) Simplify the result of part b) by requiring the two extensive variables to appear as a ratio.
d) Simplify the result of part c) for high Reynolds number pumping where μ is no longer a parameter.
e) Manipulate the remaining dimensionless group until Ω appears to the first power in the numerator. This dimensionless group is known as the specific speed, and its value allows the most efficient type of pump to be chosen (see Sabersky et al. 1999).

Solution 1.45. a) The total power that must be delivered to the fluid is $Q\Delta P$. The power that one pump delivers to the fluid will be ηW. Thus, N_p will be the next integer larger than $Q\Delta P/\eta W$.
b) Construct the parameter & units matrix using ΔP as the solution parameter

	ΔP	Q	N_p	Ω	ρ	μ
M	1	0	0	0	1	1
L	-1	3	0	0	-3	-1
T	-2	-1	0	-1	-0	-1

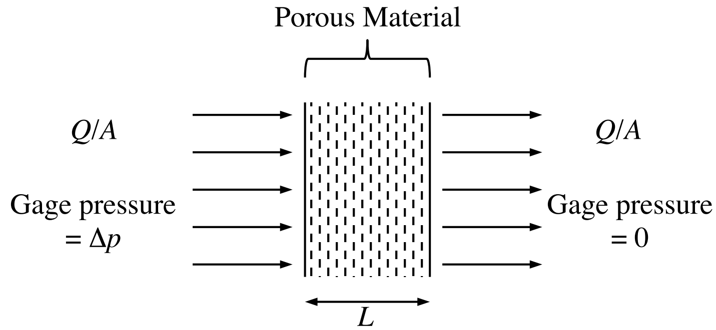
The rank of this matrix is three. There are 6 parameters and 3 independent units, thus there will be 6 – 3 = 2 dimensionless groups. By inspection these groups are: a pressure coefficient = $\Pi_1 = \dfrac{\Delta P}{\rho(\Omega^2 Q)^{2/3}}$, the number of pumps = $\Pi_2 = N_p$, and a Reynolds number = $\Pi_3 = \dfrac{\rho Q^{2/3}\Omega^{1/3}}{\mu}$.

c) The two extensive parameters (Q & N_p) must for a ratio, so defining $q = Q/N_p$, the two dimensionless groups are: $\dfrac{\Delta P}{\rho(\Omega^2 q)^{2/3}}$, and $\dfrac{\rho q^{2/3}\Omega^{1/3}}{\mu}$.

d) At high Reynolds number, the second dimensionless group will not matter. Thus, $\dfrac{\Delta P}{\rho(\Omega^2 q)^{2/3}}$ alone will characterize the pump, and this group will be a constant.

e) The specific speed = $\dfrac{\Omega q^{1/2}}{(\Delta P/\rho)^{3/4}}$. Low values of the specific speed (below ~1/2 or so) correspond centrifugal pumps that move relatively small amounts of liquid against relatively-high pressure differences (a water pump for circulating a coolant through narrow passageways). High values of the specific speed (above ~2 or so) correspond to propeller pumps that move relatively high volumes of fluid against relatively-low pressure differences (a ventilation fan).

Exercise 1.46. Nearly all types of fluid filtration involve pressure driven flow through a porous material.
a) For a given volume flow rate per unit area = Q/A, predict how the pressure difference across the porous material = Δp, depends on the thickness of the filter material = L, the surface area per unit volume of the filter material = Ψ, and other relevant parameters using dimensional analysis.
b) Often the Reynolds number of the flow in the filter pores is very much less than unity so fluid inertia becomes unimportant. Redo the dimensional analysis for this situation.
c) To minimize pressure losses in heating, ventilating, and air-conditioning (HVAC) ductwork, should hot or cold air be filtered?
d) If the filter material is changed and Ψ is lowered to one half its previous value, estimate the change in Δp if all other parameters are constant. (Hint: make a reasonable assumption about the dependence of Δp on L; they are both *extensive* variables in this situation).



Solution 1.46. This question can be answered with dimensional analysis. The parameters are drawn from the problem statement and the two fluid properties ρ = density and μ = viscosity. The solution parameter is Δp, and the unit matrix is:

	Δp	Q/A	L	Ψ	ρ	μ
M	1	0	0	0	1	1
L	–1	1	1	–1	–3	–1
T	–2	–1	0	0	0	–1

There will be: 6 – 3 = 3 dimensionless groups. These groups are: a pressure coefficient = $\Pi_1 = \Delta p / \rho (Q/A)^2$, a dimensionless thickness = $\Pi_2 = L\Psi$, and a thickness-based Reynolds number $\Pi_3 = \rho (Q/A) L/\mu$. The dimensionless relationship must take the form:

$$\frac{\Delta p}{\rho (Q/A)^2} = fn\left(L\Psi, \frac{\rho (Q/A)L}{\mu} \right).$$

b) Dropping the density reduces the number of dimensionless groups. The product of the first and third group is independent of the density, thus the revised dimensional analysis result is:

$$\frac{\Delta p}{\rho (Q/A)^2} \cdot \frac{\rho (Q/A)L}{\mu} = \frac{\Delta p L}{(Q/A)\mu} = fn(L\Psi).$$

c) The viscosity of gases increases with increasing temperature, thus to keep Δp low for a given filter element ($L\Psi$) the viscosity should be low. So, filter cold air.

d) A reasonable assumption is that Δp will be proportional to the thickness of the porous material, and this implies:

$$\Delta p = \frac{(Q/A)\mu}{L} \cdot \text{const.}\left(L\Psi\right)^2 = \text{const.}(Q/A)\mu L \Psi^2 .$$

So, if Ψ is lowered by a factor of ½, then Δp will be lowered to ¼ of is previous value.

Exercise 1.47. A new industrial process requires a volume V of hot air with initial density ρ to be moved quickly from a spherical reaction chamber to a larger evacuated chamber using a single pipe of length L and interior diameter of d. The vacuum chamber is also spherical and has a volume of V_f. If the hot air cannot be transferred fast enough, the process fails. Thus, a prediction of the transfer time t is needed based on these parameters, the air's ratio of specific heats γ, and initial values of the air's speed of sound c and viscosity μ.
a) Formulate a dimensionless scaling law for t, involving six dimensionless groups.
b) Inexpensive small-scale tests of the air-transfer process are untaken before construction of the commercial-scale reaction facility. Can all these dimensionless groups be matched if the target size for the pipe diameter in the small-scale tests is $d' = d/10$? Would lowering or raising the initial air temperature in the small-scale experiments help match the dimensionless numbers?

Solution 1.47. a) This question can be answered with dimensional analysis. The parameters are drawn from the problem statement. The solution parameter is t, and the unit matrix is:

	t	V	V_f	L	d	γ	c	ρ	μ
M	0	0	0	0	0	0	0	1	1
L	0	3	3	1	1	0	1	-3	-1
T	1	0	0	0	0	0	-1	0	-1

The rank of this matrix is 3, so there will be: 9 – 3 = 6 dimensionless groups. These groups are: a dimensionless time: $\Pi_1 = ct/d$; a volume ratio: $\Pi_2 = V/V_f$; another volume ratio: $\Pi_2 = d^3/V_f$; an aspect ratio: $\Pi_4 = L/d$; the ratio of specific heats: $\Pi_5 = \gamma$; and a sonic Reynolds number: $\Pi_6 = \rho cd/\mu$. Thus, the scaling law is:

$$\frac{ct}{d} = \phi\left(\frac{V}{V_f}, \frac{d^3}{V_f}, \frac{L}{d}, \gamma, \frac{\rho cd}{\mu}\right).$$

b) The first dimensionless group will be matched if the other five are matched. So, let primes denote the small-scale test parameter. Matching the five independent dimensionless groups means:

$$\frac{V}{V_f} = \frac{V'}{V_f'}\ ,\ \frac{d^3}{V_f} = \frac{d'^3}{V_f'}\ ,\ \frac{L}{d} = \frac{L'}{d'},\ \gamma = \gamma',\ \text{and}\ \frac{\rho cd}{\mu} = \frac{\rho' c' d'}{\mu'}. \qquad (1, 2, 3, 4, 5)$$

Starting from the target pipe size ratio, $d' = d/10$, (2) implies:

$$\frac{d^3}{d'^3} = \frac{V_f}{V_f'} = 10^3 \quad \text{or} \quad V_f' = V_f/10^3,\ \text{and}\ V' = V/10^3.$$

where the finding for V' follows from (1). Similarly, from (3), the results for the length of the pipe are:

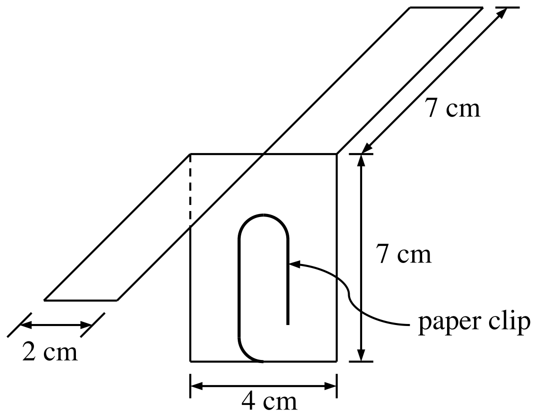
$$\frac{d}{d'} = \frac{L}{L'} = 10\ ,\ \text{so}\quad L' = L/10.$$

The next independent dimensionless group, γ, can be matched by using air in the scale model tests since it will be used in the full-scale device. The final independent dimensionless group is likely to be the most difficult to match. Using (5) and $d' = d/10$, implies:

$$\frac{\rho c d}{\mu} = \frac{\rho' c' d}{10\mu'} \text{ or } \frac{\rho' c'}{\mu'} = 10\frac{\rho c}{\mu},$$

and this relationship only involves material properties. The initial temperature (and pressure P) of the small scale experiments can be varied, ρ is proportional to T^{-1} (and P^{+1}), while both c and μ are proportional to $T^{1/2}$ (and nearly independent of P). Thus, small scale testing at reduced temperature (and elevated pressure) might make it possible to match values of this dimensionless group between the large- and small-scale experiments.

Exercise 1.48. Create a small passive helicopter from ordinary photocopy-machine paper (as shown) and drop it from a height of 2 m or so. Note the helicopter's rotation and decent rates once it's rotating steadily. Repeat this simple experiment with different sizes of paper clips to change the helicopter's weight, and observe changes in the rotation and decent rates.
a) Using the helicopter's weight W, blade length l, and blade width (chord) c, and the air's density ρ and viscosity μ as independent parameters, formulate two independent dimensionless scaling laws for the helicopter's rotation rate Ω, and decent rate dz/dt.
b) Simplify both scaling laws for the situation where μ is no longer a parameter.
c) Do the dimensionless scaling laws correctly predict the experimental trends?
d) If a new paper helicopter is made with all dimensions smaller by a factor of two. Use the scaling laws found in part b) to predict changes in the rotation and decent rates. Make the new smaller paper helicopter and see if the predictions are correct.



Solution 1.48. The experiments clearly show that Ω and dz/dt increase with increasing W.
a) This question can be answered with dimensional analysis. The parameters are drawn from the problem statement. The first solution parameter is Ω (the rotation rate) and the units matrix is:

	Ω	W	l	c	ρ	μ
M	0	1	0	0	1	1
L	0	1	1	1	-3	-1
T	-1	-2	0	0	0	-1

The rank of this matrix is 3, so there will be: 6 – 3 = 3 dimensionless groups. These groups are: $\Pi_1 = \Omega\rho^{1/2}l^2/W^{1/2}$, $\Pi_2 = l/c$, and $\Pi_3 = \rho W/\mu^2$. Thus, the scaling law is:

$$\Omega\sqrt{\frac{\rho l^4}{W}} = \phi\left(\frac{l}{c},\frac{\rho W}{\mu^2}\right),$$

where ϕ is an undetermined function. The second solution parameter is dz/dt (the descent rate), and it has the same units as Ωl, so it's scaling law is:

$$\left(\frac{dz}{dt}\right)\sqrt{\frac{\rho l^2}{W}} = \psi\left(\frac{l}{c},\frac{\rho W}{\mu^2}\right).$$

where ψ is an undetermined function.

b) When μ is no longer a parameter, both scaling laws simplify

$$\Omega\sqrt{\frac{\rho l^4}{W}} = \Phi\left(\frac{l}{c}\right) \text{ and } \left(\frac{dz}{dt}\right)\sqrt{\frac{\rho l^2}{W}} = \Psi\left(\frac{l}{c}\right),$$

where Φ and Ψ are different undetermined functions. These laws imply

$$\Omega = \frac{1}{l^2}\sqrt{\frac{W}{\rho}}\Phi\left(\frac{l}{c}\right) \text{ and } \frac{dz}{dt} = \frac{1}{l}\sqrt{\frac{W}{\rho}}\Psi\left(\frac{l}{c}\right),$$

c) These scaling laws are consistent with experimental results; the rotation and decent rates both increase when W increases.

d) If the aspect ratio, l/c, is fixed, then decreasing l by a factor of two should increase Ω by a factor of 4, and dz/dt by a factor of two. When the smaller helicopter is made, the experiments do appear to confirm these predictions; both Ω and dz/dt do increase, and the increase for Ω is larger.

Exercise 2.1. For three spatial dimensions, rewrite the following expressions in index notation and evaluate or simplify them using the values or parameters given, and the definitions of δ_{ij} and ε_{ijk} wherever possible. In b) through e), **x** is the position vector, with components x_i.

a) $\mathbf{b}\cdot\mathbf{c}$ where $\mathbf{b} = (1, 4, 17)$ and $\mathbf{c} = (-4, -3, 1)$

b) $(\mathbf{u}\cdot\nabla)\mathbf{x}$ where **u** a vector with components u_i.

c) $\nabla\phi$, where $\phi = \mathbf{h}\cdot\mathbf{x}$ and **h** is a constant vector with components h_i.

d) $\nabla\times\mathbf{u}$, where $\mathbf{u} = \mathbf{\Omega}\times\mathbf{x}$ and $\mathbf{\Omega}$ is a constant vector with components Ω_i.

e) $\mathbf{C}\cdot\mathbf{x}$, where $\mathbf{C} = \begin{Bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{Bmatrix}$

Solution 2.1. a) $\mathbf{b}\cdot\mathbf{c} = b_i c_i = 1(-4) + 4(-3) + 17(1) = -4 - 12 + 17 = +1$

b) $$(\mathbf{u}\cdot\nabla)\mathbf{x} = u_j \frac{\partial}{\partial x_j} x_i = \left[u_1\left(\frac{\partial}{\partial x_1}\right) + u_2\left(\frac{\partial}{\partial x_2}\right) + u_3\left(\frac{\partial}{\partial x_3}\right)\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1\left(\frac{\partial x_1}{\partial x_1}\right) + u_2\left(\frac{\partial x_1}{\partial x_2}\right) + u_3\left(\frac{\partial x_1}{\partial x_3}\right) \\ u_1\left(\frac{\partial x_2}{\partial x_1}\right) + u_2\left(\frac{\partial x_2}{\partial x_2}\right) + u_3\left(\frac{\partial x_2}{\partial x_3}\right) \\ u_1\left(\frac{\partial x_3}{\partial x_1}\right) + u_2\left(\frac{\partial x_3}{\partial x_2}\right) + u_3\left(\frac{\partial x_3}{\partial x_3}\right) \end{bmatrix} = \begin{bmatrix} u_1\cdot 1 + u_2\cdot 0 + u_3\cdot 0 \\ u_1\cdot 0 + u_2\cdot 1 + u_3\cdot 0 \\ u_1\cdot 0 + u_2\cdot 0 + u_3\cdot 1 \end{bmatrix} = u_j\delta_{ij} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_i$$

c) $$\nabla\phi = \frac{\partial\phi}{\partial x_j} = \frac{\partial}{\partial x_j}(h_i x_i) = h_i\frac{\partial x_i}{\partial x_j} = h_i\delta_{ij} = h_j = \mathbf{h}$$

d) $$\nabla\times\mathbf{u} = \nabla\times(\mathbf{\Omega}\times\mathbf{x}) = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}\Omega_l x_m) = \varepsilon_{ijk}\varepsilon_{klm}\Omega_l\delta_{jm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\Omega_l\delta_{jm} = (\delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl})\Omega_l$$

$$= (3\delta_{il} - \delta_{il})\Omega_l = 2\delta_{il}\Omega_l = 2\Omega_l = 2\mathbf{\Omega}$$

Here, the following identities have been used: $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, $\delta_{ij}\delta_{jk} = \delta_{ik}$, $\delta_{jj} = 3$, and $\delta_{ij}\Omega_j = \Omega_i$

e) $$\mathbf{C}\cdot\mathbf{x} = C_{ij}x_j = \begin{Bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{Bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 + 2x_3 \\ x_3 \end{Bmatrix}$$

Exercise 2.2. Starting from (2.1) and (2.3), prove (2.7).

Solution 2.2. The two representations for the position vector are:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \text{ or } \mathbf{x} = x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3'.$$

Develop the dot product of $\mathbf{x}$ with $\mathbf{e}_1$ from each representation,

$$\mathbf{e}_1\cdot\mathbf{x} = \mathbf{e}_1\cdot\left(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3\right) = x_1\mathbf{e}_1\cdot\mathbf{e}_1 + x_2\mathbf{e}_1\cdot\mathbf{e}_2 + x_3\mathbf{e}_1\cdot\mathbf{e}_3 = x_1\cdot 1 + x_2\cdot 0 + x_3\cdot 0 = x_1,$$

$$\text{and } \mathbf{e}_1\cdot\mathbf{x} = \mathbf{e}_1\cdot\left(x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3'\right) = x_1'\mathbf{e}_1\cdot\mathbf{e}_1' + x_2'\mathbf{e}_1\cdot\mathbf{e}_2' + x_3'\mathbf{e}_1\cdot\mathbf{e}_3' = x_i'C_{1i},$$

set these equal to find:

$$x_1 = x_i'C_{1i},$$

where $C_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j'$ is a 3 × 3 matrix of direction cosines. In an entirely parallel fashion, forming the dot product of $\mathbf{x}$ with $\mathbf{e}_2$, and $\mathbf{x}$ with $\mathbf{e}_2$ produces:

$$x_2 = x_i'C_{2i} \text{ and } x_3 = x_i'C_{3i}.$$

Thus, for any component x_j, where j = 1, 2, or 3, we have:

$$x_j = x_i'C_{ji},$$

which is (2.7).

Exercise 2.3. For two three-dimensional vectors with Cartesian components a_i and b_i, prove the Cauchy-Schwartz inequality: $(a_ib_i)^2 \leq (a_i)^2(b_i)^2$.

Solution 2.3. Expand the left side term,

$$(a_ib_i)^2 = (a_1b_1 + a_2b_2 + a_3b_3)^2 = a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + 2a_2b_2a_3b_3,$$

then expand the right side term,

$$(a_i)^2(b_i)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$
$$= a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + (a_1^2b_2^2 + a_2^2b_1^2) + (a_1^2b_3^2 + a_3^2b_1^2) + (a_3^2b_2^2 + a_2^2b_3^2).$$

Subtract the left side term from the right side term to find:

$$(a_i)^2(b_i)^2 - (a_ib_i)^2$$
$$= (a_1^2b_2^2 - 2a_1b_1a_2b_2 + a_2^2b_1^2) + (a_1^2b_3^2 - 2a_1b_1a_3b_3 + a_3^2b_1^2) + (a_3^2b_2^2 - 2a_2b_2a_3b_3 + a_2^2b_3^2)$$
$$= (a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_3b_2 - a_2b_3)^2 = |\mathbf{a} \times \mathbf{b}|^2.$$

Thus, the difference $(a_i)^2(b_i)^2 - (a_ib_i)^2$ is greater than zero unless **a** = (const.)**b** then the difference is zero.

Exercise 2.4. For two three-dimensional vectors with Cartesian components a_i and b_i, prove the triangle inequality: $|\mathbf{a}|+|\mathbf{b}| \geq |\mathbf{a}+\mathbf{b}|$.

Solution 2.4. To avoid square roots, square both side of the equation; this operation does not change the equation's meaning. The left side becomes:

$$\left(|\mathbf{a}|+|\mathbf{b}|\right)^2 = |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 \ ,$$

and the right side becomes:

$$|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} = |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \ .$$

So,

$$\left(|\mathbf{a}|+|\mathbf{b}|\right)^2 - |\mathbf{a}+\mathbf{b}|^2 = 2|\mathbf{a}||\mathbf{b}| - 2\mathbf{a}\cdot\mathbf{b} \ .$$

Thus, to prove the triangle equality, the right side of this last equation must be greater than or equal to zero. This requires:

$$|\mathbf{a}||\mathbf{b}| \geq \mathbf{a}\cdot\mathbf{b} \quad \text{or using index notation:} \quad \sqrt{a_i^2 b_i^2} \geq a_i b_i \ ,$$

which can be squared to find:

$$a_i^2 b_i^2 \geq (a_i b_i)^2 \ ,$$

and this is the Cauchy-Schwartz inequality proved in Exercise 2.3. Thus, the triangle equality is proved.

Exercise 2.5. Using Cartesian coordinates where the position vector is $\mathbf{x} = (x_1, x_2, x_3)$ and the fluid velocity is $\mathbf{u} = (u_1, u_2, u_3)$, write out the three components of the vector: $(\mathbf{u}\cdot\nabla)\mathbf{u} = u_i(\partial u_j/\partial x_i)$.

Solution 2.5.

a) $$(\mathbf{u}\cdot\nabla)\mathbf{u} = u_i\left(\frac{\partial u_j}{\partial x_i}\right) = u_1\left(\frac{\partial u_j}{\partial x_1}\right) + u_2\left(\frac{\partial u_j}{\partial x_2}\right) + u_3\left(\frac{\partial u_j}{\partial x_3}\right) = \begin{Bmatrix} u_1\left(\frac{\partial u_1}{\partial x_1}\right) + u_2\left(\frac{\partial u_1}{\partial x_2}\right) + u_3\left(\frac{\partial u_1}{\partial x_3}\right) \\ u_1\left(\frac{\partial u_2}{\partial x_1}\right) + u_2\left(\frac{\partial u_2}{\partial x_2}\right) + u_3\left(\frac{\partial u_2}{\partial x_3}\right) \\ u_1\left(\frac{\partial u_3}{\partial x_1}\right) + u_2\left(\frac{\partial u_3}{\partial x_2}\right) + u_3\left(\frac{\partial u_3}{\partial x_3}\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} u\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial u}{\partial y}\right) + w\left(\frac{\partial u}{\partial z}\right) \\ u\left(\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial v}{\partial y}\right) + w\left(\frac{\partial v}{\partial z}\right) \\ u\left(\frac{\partial w}{\partial x}\right) + v\left(\frac{\partial w}{\partial y}\right) + w\left(\frac{\partial w}{\partial z}\right) \end{Bmatrix}$$

The vector in this exercise, $(\mathbf{u}\cdot\nabla)\mathbf{u} = u_i(\partial u_j/\partial x_i)$, is an important one in fluid mechanics. As described in Ch. 3, it is the nonlinear advective acceleration.

Exercise 2.6. Convert $\nabla\times\nabla\rho$ to indicial notation and show that it is zero in Cartesian coordinates for any twice-differentiable scalar function ρ.

Solution 2.6. Start with the definitions of the cross product and the gradient.

$$\nabla\times(\nabla\rho)=\varepsilon_{ijk}\frac{\partial}{\partial x_j}(\nabla\rho)_k=\varepsilon_{ijk}\frac{\partial^2\rho}{\partial x_j\partial x_k}$$

Write out the vector component by component recalling that $\varepsilon_{ijk}=0$ if any two indices are equal. Here the "i" index is the free index.

$$\varepsilon_{ijk}\frac{\partial^2\rho}{\partial x_j\partial x_k}=\begin{Bmatrix}\varepsilon_{123}\dfrac{\partial^2\rho}{\partial x_2\partial x_3}+\varepsilon_{132}\dfrac{\partial^2\rho}{\partial x_3\partial x_2}\\ \varepsilon_{213}\dfrac{\partial^2\rho}{\partial x_1\partial x_3}+\varepsilon_{231}\dfrac{\partial^2\rho}{\partial x_3\partial x_1}\\ \varepsilon_{312}\dfrac{\partial^2\rho}{\partial x_1\partial x_2}+\varepsilon_{321}\dfrac{\partial^2\rho}{\partial x_2\partial x_1}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial^2\rho}{\partial x_2\partial x_3}-\dfrac{\partial^2\rho}{\partial x_3\partial x_2}\\ -\dfrac{\partial^2\rho}{\partial x_1\partial x_3}+\dfrac{\partial^2\rho}{\partial x_3\partial x_1}\\ \dfrac{\partial^2\rho}{\partial x_1\partial x_2}-\dfrac{\partial^2\rho}{\partial x_2\partial x_1}\end{Bmatrix}=0,$$

where the middle equality follows from the definition of ε_{ijk} (2.18), and the final equality follows when ρ is twice differentiable so that $\dfrac{\partial^2\rho}{\partial x_j\partial x_k}=\dfrac{\partial^2\rho}{\partial x_k\partial x_j}$.

Exercise 2.7. Using indicial notation, show that **a** × (**b** × **c**) = (**a** · **c**)**b** – (**a** · **b**)**c**. [*Hint*: Call **d** ≡ **b** × **c**. Then $(\mathbf{a}\times\mathbf{d})_m = \varepsilon_{pqm}a_p d_q = \varepsilon_{pqm}a_p\varepsilon_{ijq}b_i c_j$. Using (2.19), show that $(\mathbf{a}\times\mathbf{d})_m = (\mathbf{a}\cdot\mathbf{c})\boldsymbol{b_m} - (\mathbf{a}\cdot\mathbf{b})\boldsymbol{c_m}$.]

Solution 2.7. Using the hint and the definition of ε_{ijk} produces:

$$(\mathbf{a}\times\mathbf{d})_m = \varepsilon_{pqm}a_p d_q = \varepsilon_{pqm}a_p\varepsilon_{ijq}b_i c_j = \varepsilon_{pqm}\varepsilon_{ijq}\, b_i c_j a_p = -\varepsilon_{ijq}\varepsilon_{qpm}\, b_i c_j a_p.$$

Now use the identity (2.19) for the product of epsilons:

$$(\mathbf{a}\times\mathbf{d})_m = -(\delta_{ip}\delta_{jm} - \delta_{im}\delta_{pj})\, b_i c_j a_p = -b_p c_m a_p + b_m c_p a_p.$$

Each term in the final expression involves a sum over "p", and this is a dot product; therefore

$$(\mathbf{a}\times\mathbf{d})_m = -(\mathbf{a}\cdot\mathbf{b})c_m + b_m(\mathbf{a}\cdot\mathbf{c}).$$

Thus, for any component m = 1, 2, or 3,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = -(\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{a}\cdot\mathbf{c})\mathbf{b} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

Exercise 2.8. Show that the condition for the vectors **a**, **b**, and **c** to be coplanar is $\varepsilon_{ijk}a_i b_j c_k = 0$.

Solution 2.8. The vector **b** × **c** is perpendicular to **b** and **c**. Thus, **a** will be coplanar with **b** and **c** if it too is perpendicular to **b** × **c**. The condition for a to be perpendicular with **b** × **c** is:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0.$$

In index notation, this is $a_i \varepsilon_{ijk} b_j c_k = 0 = \varepsilon_{ijk} a_i b_j c_k$.

Exercise 2.9. Prove the following relationships: $\delta_{ij}\delta_{ij} = 3$, $\varepsilon_{pqr}\varepsilon_{pqr} = 6$, and $\varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij}$.

Solution 2.9. (*i*) $\delta_{ij}\delta_{ij} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$.
For the second two, the identity (2.19) is useful.
(*ii*) $\varepsilon_{pqr}\varepsilon_{pqr} = \varepsilon_{pqr}\varepsilon_{rpq} = \delta_{pp}\delta_{qq} - \delta_{pq}\delta_{pq} = 3(3) - \delta_{pp} = 9 - 3 = 6$.
(*iii*) $\varepsilon_{pqi}\varepsilon_{pqj} = \varepsilon_{ipq}\varepsilon_{pqj} = -\varepsilon_{ipq}\varepsilon_{qpj} = -(\delta_{ip}\delta_{pj} - \delta_{ij}\delta_{pp}) = -\delta_{ij} + 3\delta_{ij} = 2\delta_{ij}$.

Exercise 2.10. Show that $\mathbf{C}\cdot\mathbf{C}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{C} = \boldsymbol{\delta}$, where **C** is the direction cosine matrix and $\boldsymbol{\delta}$ is the matrix of the Kronecker delta. Any matrix obeying such a relationship is called an *orthogonal matrix* because it represents transformation of one set of orthogonal axes into another.

Solution 2.10. To show that $\mathbf{C}\cdot\mathbf{C}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{C} = \boldsymbol{\delta}$, where **C** is the direction cosine matrix and $\boldsymbol{\delta}$ is the matrix of the Kronecker delta. Start from (2.5) and (2.7), which are

$$x'_j = x_i C_{ij} \quad \text{and} \quad x_j = x'_i C_{ji},$$

respectively, and change the index "i" into "m" on (2.5): $x'_j = x_m C_{mj}$. Substitute this into (2.7) to find:

$$x_j = x'_i C_{ji} = (x_m C_{mi}) C_{ji} = C_{mi} C_{ji} x_m.$$

However, we also have $x_j = \delta_{jm} x_m$, so

$$\delta_{jm} x_m = C_{mi} C_{ji} x_m \quad \rightarrow \quad \delta_{jm} = C_{mi} C_{ji},$$

which can be written:

$$\delta_{jm} = C_{mi} C^{\mathrm{T}}_{ij} = \mathbf{C}\cdot\mathbf{C}^{\mathrm{T}},$$

and taking the transpose of the this produces:

$$(\delta_{jm})^{\mathrm{T}} = \delta_{mj} = (C_{mi} C^{\mathrm{T}}_{ij})^{\mathrm{T}} = C^{\mathrm{T}}_{mi} C_{ij} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{C}.$$

Exercise 2.11. Show that for a second-order tensor **A**, the following quantities are invariant under the rotation of axes:

$$I_1 = A_{ii}\,,\quad I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix},\ \text{and } I_3 = \det(A_{ij}).$$

[*Hint*: Use the result of Exercise 2.8 and the transformation rule (2.12) to show that $I'_1 = A'_{ii} = A_{ii} = I_1$. Then show that $A_{ij}A_{ji}$ and $A_{ij}A_{jk}A_{ki}$ are also invariants. In fact, *all* contracted scalars of the form $A_{ij}A_{jk} \cdots A_{mi}$ are invariants. Finally, verify that $I_2 = \frac{1}{2}\left[I_1^2 - A_{ij}A_{ji}\right]$, $I_3 = \frac{1}{3}\left[A_{ij}A_{jk}A_{ki} - I_1 A_{ij}A_{ji} + I_2 A_{ii}\right]$. Because the right-hand sides are invariant, so are I_2 and I_3.]

Solution 2.11. First prove I_1 is invariant by using the second order tensor transformation rule (2.12):

$$A'_{mn} = C_{im}C_{jn}A_{ij}.$$

Replace C_{jn} by C_{nj}^{T} and set $n = m$,

$$A'_{mn} = C_{im}C_{nj}^{\mathrm{T}}A_{ij} \to A'_{mm} = C_{im}C_{mj}^{\mathrm{T}}A_{ij}.$$

Use the result of Exercise 2.8, $\delta_{ij} = C_{im}C_{mj}^{\mathrm{T}} =$, to find:

$$I_1 = A'_{mm} = \delta_{ij}A_{ij} = A_{ii}.$$

Thus, the first invariant is does not depend on a rotation of the coordinate axes.

Now consider whether or not $A_{mn}A_{nm}$ is invariant under a rotation of the coordinate axes. Start with a double application of (2.12):

$$A'_{mn}A'_{nm} = \left(C_{im}C_{jn}A_{ij}\right)\left(C_{pn}C_{qm}A_{pq}\right) = \left(C_{jn}C_{np}^{\mathrm{T}}\right)\left(C_{im}C_{mq}^{\mathrm{T}}\right)A_{ij}A_{pq}.$$

From the result of Exercise 2.8, the factors in parentheses in the last equality are Kronecker delta functions, so

$$A'_{mn}A'_{nm} = \delta_{jp}\delta_{iq}A_{ij}A_{pq} = A_{ij}A_{ji}.$$

Thus, the matrix contraction $A_{mn}A_{nm}$ does not depend on a rotation of the coordinate axes.

The manipulations for $A_{mn}A_{np}A_{pm}$ are a straightforward extension of the prior efforts for A_{ii} and $A_{ij}A_{ji}$.

$$A'_{mn}A'_{np}A'_{pm} = \left(C_{im}C_{jn}A_{ij}\right)\left(C_{qn}C_{rp}A_{qr}\right)\left(C_{sp}C_{tm}A_{st}\right) = \left(C_{jn}C_{nq}^{\mathrm{T}}\right)\left(C_{rp}C_{ps}^{\mathrm{T}}\right)\left(C_{im}C_{mt}^{\mathrm{T}}\right)A_{ij}A_{qr}A_{st}.$$

Again, the factors in parentheses are Kronecker delta functions, so

$$A'_{mn}A'_{np}A'_{pm} = \delta_{jq}\delta_{rs}\delta_{it}A_{ij}A_{qr}A_{st} = A_{iq}A_{qs}A_{si},$$

which implies that the matrix contraction $A_{ij}A_{jk}A_{ki}$ does not depend on a rotation of the coordinate axes.

Now, for the second invariant, verify the given identity, starting from the given definition for I_2.

$$\begin{aligned}
I_2 &= \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} \\
&= A_{11}A_{22} - A_{12}A_{21} + A_{22}A_{33} - A_{23}A_{32} + A_{11}A_{33} - A_{13}A_{31} \\
&= A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33} - \left(A_{12}A_{21} + A_{23}A_{32} + A_{13}A_{31}\right) \\
&= \tfrac{1}{2}A_{11}^2 + \tfrac{1}{2}A_{22}^2 + \tfrac{1}{2}A_{33}^2 + A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33} - \left(A_{12}A_{21} + A_{23}A_{32} + A_{13}A_{31} + \tfrac{1}{2}A_{11}^2 + \tfrac{1}{2}A_{22}^2 + \tfrac{1}{2}A_{33}^2\right) \\
&= \tfrac{1}{2}\left[A_{11} + A_{22} + A_{33}\right]^2 - \tfrac{1}{2}\left(2A_{12}A_{21} + 2A_{23}A_{32} + 2A_{13}A_{31} + A_{11}^2 + A_{22}^2 + A_{33}^2\right)
\end{aligned}$$

$$= \tfrac{1}{2}I_1^2 - \tfrac{1}{2}\left(A_{11}A_{11} + A_{12}A_{21} + A_{13}A_{31} + A_{12}A_{21} + A_{22}A_{22} + A_{23}A_{32} + A_{13}A_{31} + A_{23}A_{32} + A_{33}A_{33}\right)$$
$$= \tfrac{1}{2}I_1^2 - \tfrac{1}{2}\left(A_{ij}A_{ji}\right) = \tfrac{1}{2}\left(I_1^2 - A_{ij}A_{ji}\right)$$

Thus, since I_2 only depends on I_1 and $A_{ij}A_{ji}$, it is invariant under a rotation of the coordinate axes because I_1 and $A_{ij}A_{ji}$ are invariant under a rotation of the coordinate axes.

The manipulations for the third invariant are a tedious but not remarkable. Start from the given definition for I_3, and group like terms.

$$I_3 = \det\left(A_{ij}\right) = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$
$$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21} - \left(A_{11}A_{23}A_{32} + A_{22}A_{13}A_{31} + A_{33}A_{12}A_{21}\right) \qquad \text{(a)}$$

Now work from the given identity. The triple matrix product $A_{ij}A_{jk}A_{ki}$ has twenty-seven terms:

$$A_{11}^3 + A_{11}A_{12}A_{21} + A_{11}A_{13}A_{31} + A_{12}A_{21}A_{11} + A_{12}A_{22}A_{21} + A_{12}A_{23}A_{31} + A_{13}A_{31}A_{11} + A_{13}A_{32}A_{21} + A_{13}A_{33}A_{31} +$$
$$A_{21}A_{11}A_{12} + A_{21}A_{12}A_{22} + A_{21}A_{13}A_{32} + A_{22}A_{21}A_{12} + A_{22}^3 + A_{22}A_{23}A_{32} + A_{23}A_{31}A_{12} + A_{23}A_{32}A_{22} + A_{23}A_{33}A_{32} +$$
$$A_{31}A_{11}A_{13} + A_{31}A_{12}A_{23} + A_{31}A_{13}A_{33} + A_{32}A_{21}A_{13} + A_{32}A_{22}A_{23} + A_{32}A_{23}A_{33} + A_{33}A_{31}A_{13} + A_{33}A_{32}A_{23} + A_{33}^3$$

These can be grouped as follows:

$$A_{ij}A_{jk}A_{ki} = 3(A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21}) + A_{11}(A_{11}^2 + 3A_{12}A_{21} + 3A_{13}A_{31}) +$$
$$A_{22}(3A_{21}A_{12} + A_{22}^2 + 3A_{23}A_{32}) + A_{33}(3A_{31}A_{13} + 3A_{32}A_{23} + A_{33}^2) \qquad \text{(b)}$$

The remaining terms of the given identity are:

$$-I_1A_{ij}A_{ji} + I_2A_{ii} = I_1(I_2 - A_{ij}A_{ji}) = I_1(I_2 + 2I_2 - I_1^2) = 3I_1I_2 - I_1^3,$$

where the result for I_2 has been used. Evaluating the first of these two terms leads to:

$$3I_1I_2 = 3(A_{11} + A_{22} + A_{33})(A_{11}A_{22} - A_{12}A_{21} + A_{22}A_{33} - A_{23}A_{32} + A_{11}A_{33} - A_{13}A_{31})$$
$$= 3(A_{11} + A_{22} + A_{33})(A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33}) - 3(A_{11} + A_{22} + A_{33})(A_{12}A_{21} + A_{23}A_{32} + A_{13}A_{31}).$$

Adding this to (b) produces:

$$A_{ij}A_{jk}A_{ki} + 3I_1I_2 = 3(A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21}) + 3(A_{11} + A_{22} + A_{33})(A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33}) +$$
$$A_{11}(A_{11}^2 - 3A_{23}A_{32}) + A_{22}(A_{22}^2 - 3A_{13}A_{31}) + A_{33}(A_{33}^2 - 3A_{12}A_{21})$$
$$= 3(A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21} - A_{11}A_{23}A_{32} - A_{22}A_{13}A_{31} - A_{33}A_{12}A_{21}) +$$
$$3(A_{11} + A_{22} + A_{33})(A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33}) + A_{11}^3 + A_{22}^3 + A_{33}^3 \qquad \text{(c)}$$

The last term of the given identity is:

$$I_1^3 = A_{11}^3 + A_{22}^3 + A_{33}^3 + 3(A_{11}^2A_{22} + A_{11}^2A_{33} + A_{22}^2A_{11} + A_{22}^2A_{33} + A_{33}^2A_{11} + A_{33}^2A_{22}) + 6A_{11}A_{22}A_{33}$$
$$= A_{11}^3 + A_{22}^3 + A_{33}^3 + 3(A_{11} + A_{22} + A_{33})(A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33}) - 3A_{11}A_{22}A_{33}$$

Subtracting this from (c) produces:

$$A_{ij}A_{jk}A_{ki} + 3I_1I_2 - I_1^3 = 3(A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21} - A_{11}A_{23}A_{32} - A_{22}A_{13}A_{31} - A_{33}A_{12}A_{21} + A_{11}A_{22}A_{33})$$
$$= 3I_3.$$

This verifies that the given identity for I_3 is correct. Thus, since I_3 only depends on I_1, I_2, and $A_{ij}A_{jk}A_{ki}$, it is invariant under a rotation of the coordinate axes because these quantities are invariant under a rotation of the coordinate axes as shown above.

Exercise 2.12. If **u** and **v** are vectors, show that the products $u_i v_j$ obey the transformation rule (2.12), and therefore represent a second-order tensor.

Solution 2.12. Start by applying the vector transformation rule (2.5 or 2.6) to the components of **u** and **v** separately,

$$u'_m = C_{im} u_i \text{ , and } v'_n = C_{jn} v_j .$$

The product of these two equations produces:

$$u'_m v'_n = C_{im} C_{jn} u_i v_j ,$$

which is the same as (2.12) for second order tensors.

Exercise 2.13. Show that δ_{ij} is an isotropic tensor. That is, show that $\delta'_{ij} = \delta_{ij}$ under rotation of the coordinate system. [*Hint*: Use the transformation rule (2.12) and the results of Exercise 2.10.]

Solution 2.13. Apply (2.12) to δ_{ij},

$$\delta'_{mn} = C_{im}C_{jn}\delta_{ij} = C_{im}C_{in} = C^{\mathrm{T}}_{mi}C_{in} = \delta_{mn}.$$

where the final equality follows from the result of Exercise 2.10. Thus, the Kronecker delta is invariant under coordinate rotations.

Exercise 2.14. If **u** and **v** are arbitrary vectors resolved in three-dimensional Cartesian coordinates, use the definition of vector magnitude, $|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}$, and the Pythagorean theorem to show that $\mathbf{u}\cdot\mathbf{v} = 0$ when **u** and **v** are perpendicular.

Solution 2.14. Consider the magnitude of the sum **u** + **v**,

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (u_1+v_1)^2 + (u_2+v_2)^2 + (u_3+v_3)^2 \\ &= u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 + 2u_1v_1 + 2u_2v_2 + 2u_3v_3 \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v},\end{aligned}$$

which can be rewritten:

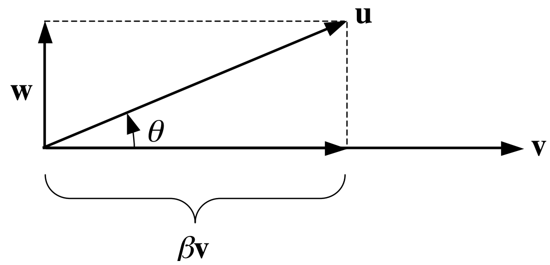
$$\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 2\mathbf{u}\cdot\mathbf{v}.$$

When **u** and **v** are perpendicular, the Pythagorean theorem requires the left side to be zero. Thus,

$$\mathbf{u}\cdot\mathbf{v} = 0.$$

Exercise 2.15. If **u** and **v** are vectors with magnitudes u and υ, use the finding of Exercise 2.14 to show that $\mathbf{u}\cdot\mathbf{v} = u\upsilon\cos\theta$ where θ is the angle between **u** and **v**.

Solution 2.15. Start with two arbitrary vectors (**u** and **v**), and view them so that the plane they define is coincident with the page and **v** is horizontal. Consider two additional vectors, $\beta\mathbf{v}$ and **w**, that are perpendicular ($\mathbf{v}\cdot\mathbf{w} = 0$) and can be summed together to produce **u**: $\mathbf{w} + \beta\mathbf{v} = \mathbf{u}$.



Compute the dot-product of **u** and **v**:

$$\mathbf{u}\cdot\mathbf{v} = (\mathbf{w} + \beta\mathbf{v})\cdot\mathbf{v} = \mathbf{w}\cdot\mathbf{v} + \beta\mathbf{v}\cdot\mathbf{v} = \beta\upsilon^2.$$

where the final equality holds because $\mathbf{v}\cdot\mathbf{w} = 0$. From the geometry of the figure:

$$\cos\theta \equiv \frac{\|\beta\mathbf{v}\|}{\|\mathbf{u}\|} = \frac{\beta\upsilon}{u}, \text{ or } \beta = \frac{u}{\upsilon}\cos\theta.$$

Insert this into the final equality for $\mathbf{u}\cdot\mathbf{v}$ to find:

$$\mathbf{u}\cdot\mathbf{v} = \left(\frac{u}{\upsilon}\cos\theta\right)\upsilon^2 = u\upsilon\cos\theta.$$

Exercise 2.16. Determine the components of the vector **w** in three-dimensional Cartesian coordinates when **w** is defined by: **u·w** = 0, **v·w** = 0, and **w·w** = $u^2\upsilon^2\sin^2\theta$, where **u** and **v** are known vectors with components u_i and υ_i and magnitudes u and υ, respectively, and θ is the angle between **u** and **v**. Choose the sign(s) of the components of **w** so that **w** = $\mathbf{e}_3$ when **u** = $\mathbf{e}_1$ and **v** = $\mathbf{e}_2$.

Solution 2.16. The effort here is primarily algebraic. Write the three constraints in component form:

$$\mathbf{u}\cdot\mathbf{w} = 0, \text{ or } u_1w_1 + u_2w_2 + u_3w_3 = 0, \quad (1)$$

$$\mathbf{v}\cdot\mathbf{w} = 0, \text{ or } \upsilon_1w_1 + \upsilon_2w_2 + \upsilon_3w_3 = 0, \text{ and} \quad (2)$$

The third one requires a little more effort since the angle needs to be eliminated via a dot product:

$$\mathbf{w}\cdot\mathbf{w} = u^2\upsilon^2\sin^2\theta = u^2\upsilon^2(1-\cos^2\theta) = u^2\upsilon^2 - (\mathbf{u}\cdot\mathbf{w})^2 \text{ or}$$

$$w_1^2 + w_2^2 + w_3^2 = (u_1^2+u_2^2+u_3^2)(\upsilon_1^2+\upsilon_2^2+\upsilon_3^2) - (u_1\upsilon_1 + u_2\upsilon_2 + u_3\upsilon_3)^2, \text{ which leads to}$$

$$w_1^2 + w_2^2 + w_3^2 = (u_1\upsilon_2 - u_2\upsilon_1)^2 + (u_1\upsilon_3 - u_3\upsilon_1)^2 + (u_2\upsilon_3 - u_3\upsilon_2)^2. \quad (3)$$

Equation (1) implies:

$$w_1 = -(w_2u_2 + w_3u_3)/u_1 \quad (4)$$

Combine (2) and (4) to eliminate w_1, and solve the resulting equation for w_2:

$$-\upsilon_1(w_2u_2 + w_3u_3)/u_1 + \upsilon_2w_2 + \upsilon_3w_3 = 0, \text{ or } \left(-\frac{\upsilon_1}{u_1}u_2 + \upsilon_2\right)w_2 + \left(-\frac{\upsilon_1}{u_1}u_3 + \upsilon_3\right)w_3 = 0.$$

Thus:

$$w_2 = +w_3\left(\frac{\upsilon_1}{u_1}u_3 - \upsilon_3\right)\bigg/\left(-\frac{\upsilon_1}{u_1}u_2 + \upsilon_2\right) = w_3\left(\frac{u_3\upsilon_1 - u_1\upsilon_3}{u_1\upsilon_2 - u_2\upsilon_1}\right). \quad (5)$$

Combine (4) and (5) to find:

$$w_1 = -\frac{w_3}{u_1}\left(\left(\frac{\upsilon_1u_3 - \upsilon_3u_1}{\upsilon_2u_1 - \upsilon_1u_2}\right)u_2 + u_3\right) = -\frac{w_3}{u_1}\left(\frac{\upsilon_1u_3u_2 - \upsilon_3u_1u_2 + \upsilon_2u_1u_3 - \upsilon_1u_2u_3}{\upsilon_2u_1 - \upsilon_1u_2} + \right)$$

$$= -\frac{w_3}{u_1}\left(\frac{-\upsilon_3u_1u_2 + \upsilon_2u_1u_3}{\upsilon_2u_1 - \upsilon_1u_2}\right) = w_3\left(\frac{u_2\upsilon_3 - u_3\upsilon_2}{u_1\upsilon_2 - u_2\upsilon_1}\right). \quad (6)$$

Put (5) and (6) into (3) and factor out w_3 on the left side, then divide out the extensive common factor that (luckily) appears on the right and as the numerator inside the big parentheses.

$$w_3^2\left(\frac{(u_2\upsilon_3 - u_3\upsilon_2)^2 + (u_3\upsilon_1 - u_1\upsilon_3)^2 + (u_1\upsilon_2 - u_2\upsilon_1)^2}{(u_1\upsilon_2 - u_2\upsilon_1)^2}\right) = (u_1\upsilon_2 - u_2\upsilon_1)^2 + (u_1\upsilon_3 - u_3\upsilon_1)^2 + (u_2\upsilon_3 - u_3\upsilon_2)^2$$

$$w_3^2\left(\frac{1}{(u_1\upsilon_2 - u_2\upsilon_1)^2}\right) = 1, \text{ so } w_3 = \pm(u_1\upsilon_2 - u_2\upsilon_1).$$

If **u** = (1,0,0), and **v** = (0,1,0), then using the plus sign produces $w_3 = +1$, so $w_3 = +(u_1\upsilon_2 - u_2\upsilon_1)$.
Cyclic permutation of the indices allows the other components of w to be determined:

$$w_1 = u_2\upsilon_3 - u_3\upsilon_2,$$

$$w_2 = u_3\upsilon_1 - u_1\upsilon_3,$$

$$w_3 = u_1\upsilon_2 - u_2\upsilon_1.$$