1.1 Substitute (1.2-2) into (1.2-1):

$u_1 = \beta_1$ hence $\beta_1 = u_1$

$u_2 = \beta_1 + \beta_2 a = u_1 + \beta_2 a$ hence $\beta_2 = \dfrac{u_2 - u_1}{a}$

$u_3 = \beta_1 + \beta_3 b = u_1 + \beta_3 b$ hence $\beta_3 = \dfrac{u_3 - u_1}{b}$

(1.2-1a) becomes $u = u_1 + \dfrac{u_2 - u_1}{a}x + \dfrac{u_3 - u_1}{b}y$

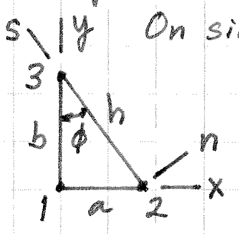
or $u = \left(1 - \dfrac{x}{a} - \dfrac{y}{b}\right)u_1 + \dfrac{x}{a}u_2 + \dfrac{y}{b}u_3$

Similarly for v

1.2 On side 1-2, $y = 0$; subs. into (1.2-3b):

$$v = \left(1 - \frac{x}{a}\right)v_1 + \frac{x}{a}v_2$$

Depends only on v_1 and v_2 and is a linear function of x, ∴ 1-2 remains straight.



On side 2-3:

$x = a - s\sin\phi = a - \dfrac{a}{h}s$

$y = s\cos\phi = \dfrac{b}{h}s$

Substitute into (1.2-3a):

$u = \left(1 - 1 + \dfrac{s}{h} - \dfrac{s}{h}\right)u_1 + \left(1 - \dfrac{s}{h}\right)u_2 + \dfrac{s}{h}u_3$

$u = \left(1 - \dfrac{s}{h}\right)u_2 + \dfrac{s}{h}u_3$, similarly for v

Now u & v along 2-3 depend only on u_2, v_2, u_3, v_3 & vary linearly with s, ∴ edge remains straight.

1.3 In all parts, substitute coords. into (1.2-1a); v equation is similar.

(a) $u_1 = \beta_1 + \beta_2 a$ (A)
$u_2 = \beta_1 + \beta_2 a + \beta_3 b$ $\Big\}$ $\beta_3 = \dfrac{u_2 - u_1}{b}$ (B)
$u_3 = \beta_1 + \beta_3 b$ (C)

(B) & (C) yield $\beta_1 = u_1 - u_2 + u_3$ (D)

(A) & (D) yield $\beta_2 = \dfrac{u_2 - u_3}{a}$

(1.2-1a) becomes

$u = (u_1 - u_2 + u_3) + \dfrac{u_2 - u_3}{a}x + \dfrac{u_2 - u_1}{b}y$

$u = \left(1 - \dfrac{y}{b}\right)u_1 + \left(-1 + \dfrac{x}{a} + \dfrac{y}{b}\right)u_2 + \left(1 - \dfrac{x}{a}\right)u_3$

(b) $u_1 = \beta_1 - \beta_2 a$
$u_2 = \beta_1 + \beta_2 a$ $\Big\}$ $\beta_1 = \dfrac{u_1 + u_2}{2}$
$u_3 = \beta_1 + \beta_3 b$ $\quad \beta_2 = \dfrac{u_2 - u_1}{2a}$

$\beta_3 = \dfrac{u_3 - \beta_1}{b} = \dfrac{2u_3 - u_1 - u_2}{2b}$

$u = \dfrac{u_1 + u_2}{2} + \dfrac{u_2 - u_1}{2a}x + \dfrac{2u_3 - u_1 - u_2}{2b}y$

$u = \left(\dfrac{1}{2} - \dfrac{x}{2a} - \dfrac{y}{2b}\right)u_1 + \left(\dfrac{1}{2} + \dfrac{x}{2a} - \dfrac{y}{2b}\right)u_2 + \dfrac{y}{b}u_3$

(c) $u_1 = \beta_1$ $\quad \beta_1 = u_1$
$u_2 = \beta_1 + \beta_2 a + \beta_3 b$ $\Big\}$ $\beta_2 = \dfrac{u_2 - u_3}{2a}$
$u_3 = \beta_1 - \beta_2 a + \beta_3 b$

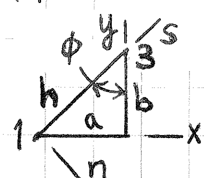
Add: $u_2 + u_3 = 2u_1 + 2\beta_3 b$, $\beta_3 = \dfrac{u_2 + u_3 - 2u_1}{2b}$

$u = u_1 + \dfrac{u_2 - u_3}{2a}x + \dfrac{u_2 + u_3 - 2u_1}{2b}y$

$u = \left(1 - \dfrac{y}{b}\right)u_1 + \left(\dfrac{x}{2a} + \dfrac{y}{2b}\right)u_2 + \left(-\dfrac{x}{2a} + \dfrac{y}{2b}\right)u_3$

1.4 (a) Same as side 2-3 in Prob. 1.2.

(b) Use ans. to Prob. 1.3b



$x = -a + s\sin\phi = -a + \dfrac{a}{h}s$

$y = s\cos\phi = \dfrac{b}{h}s$

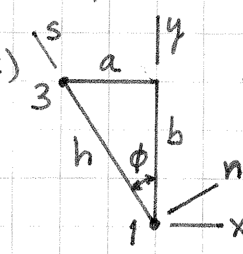
$u = \left(\dfrac{1}{2} + \dfrac{1}{2} - \dfrac{s}{2h} - \dfrac{s}{2h}\right)u_1 + \left(\dfrac{1}{2} - \dfrac{1}{2} + \dfrac{s}{2h} - \dfrac{s}{2h}\right)u_2 + \dfrac{s}{h}u_3$

$u = \left(1 - \dfrac{s}{h}\right)u_1 + \dfrac{s}{h}u_3$

Likewise $v = \left(1 - \dfrac{s}{h}\right)v_1 + \dfrac{s}{h}v_3$

Now u & v along 1-3 depend only on u_1, v_1, u_3, v_3 & vary linearly with s, ∴ edge remains straight.

(c)



$x = -s\sin\phi = -\dfrac{a}{h}s$

$y = s\cos\phi = \dfrac{b}{h}s$

Use ans. to Prob. 1.3c

$u = \left(1 - \dfrac{s}{h}\right)u_1 + \left(-\dfrac{s}{2h} + \dfrac{s}{2h}\right)u_2 + \left(\dfrac{s}{2h} + \dfrac{s}{2h}\right)u_3$

$u = \left(1 - \dfrac{s}{h}\right)u_1 + \dfrac{s}{h}u_3$

Likewise $v = \left(1 - \dfrac{s}{h}\right)v_1 + \dfrac{s}{h}v_3$

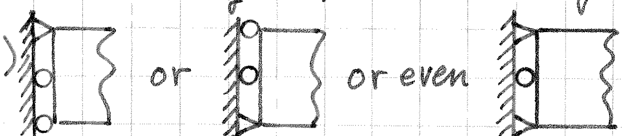
Conclusion: as for part (b).

1.5 (a) $\epsilon_x = \frac{\partial u}{\partial x} = \beta_2 + 2\beta_4 x + \beta_5 y$

$$\epsilon_y = \frac{\partial v}{\partial y} = \beta_9 + \beta_{11} x + 2\beta_{12} y$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \beta_3 + \beta_8 + (\beta_5 + 2\beta_{10})x + (2\beta_6 + \beta_{11})y$$

(b) To permit y-direction displacement associated with the Poisson effect. If all supports at $x=0$ were pins, we would not obtain $\sigma_y = 0$, as beam theory says.

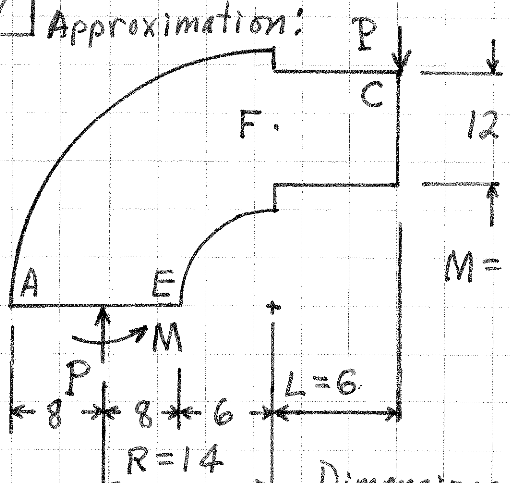
(c) 

(there is no relative y displacement between top and bottom surfaces)

(d) According to beam theory, $\epsilon_x = cy$, $\epsilon_y = \nu\epsilon_x$, and $\gamma_{xy} = 0$, where c = const.
Therefore $\beta_2 = \beta_4 = \beta_9 = \beta_{11} = 0$, and from the γ_{xy} expression $\beta_3 + \beta_8 = 0$.
Also, from γ_{xy}, $\beta_6 = 0$ and $\beta_5 + 2\beta_{10} = 0$.

1.6 Should approach, with mesh refinement:
$\sigma_x = 0$ along AB, CD, and at E
$\sigma_y = 0$ along BC and the straight part of ED
$\tau_{xy} = 0$ along all of the boundary except the arc of radius r
σ_y and τ_{xy} very large & negative at C
σ_y large and negative at E

1.7 Approximation:



Dimensions in mm

(a) $I = \frac{t}{12}16^3 = 341t$, $A = 16t$

$$\frac{Mc}{I} = \frac{20P(8)}{341t} = 0.469\frac{P}{t}$$

$$\frac{P}{A} = \frac{P}{16t} = 0.0625\frac{P}{t} \quad \text{(compressive)}$$

$$\sigma_{yA} = 0.469\frac{P}{t} - 0.0625\frac{P}{t} = 0.406\frac{P}{t}$$

$$\sigma_{yE} = -0.469\frac{P}{t} - 0.0625\frac{P}{t} = -0.531\frac{P}{t}$$

(b) Use theory and formulas of Ref. 2.2

$$\int\frac{dA}{r} = t\ln\frac{22}{6} = 1.2993t$$

$$r_n = \frac{16t}{1.2993t} = 12.315 \quad \text{radius of neutral ax.}$$

$$e = R - r_n = 1.685$$

At E, $\frac{My}{Aer} = \frac{-20P(12.315-6)}{16t(1.685)6} = -0.780\frac{P}{t}$

At A, $\frac{My}{Aer} = \frac{-20P(12.315-22)}{16t(1.685)22} = 0.326\frac{P}{t}$

$$\sigma_{yA} = 0.326\frac{P}{t} - 0.0625\frac{P}{t} = 0.264\frac{P}{t}$$

$$\sigma_{yE} = -0.780\frac{P}{t} - 0.0625\frac{P}{t} = -0.843\frac{P}{t}$$

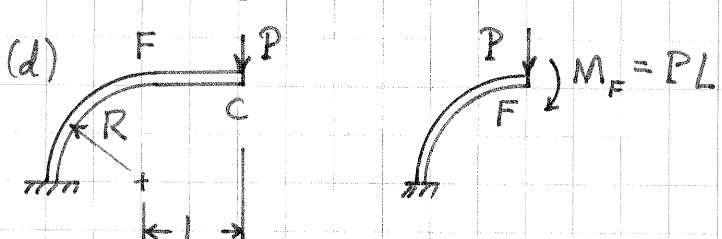
(c) At $r = r_n$, $\int\frac{dA}{r} = t\ln\frac{12.315}{6} = 0.719t$

Largest σ_r is almost at the neutral axis. Let's evaluate at the neutral axis.

$$\sigma_r = \frac{M}{Aetr_n}\left(r_n\int_b^{r_n}\frac{dA}{r} - A_r\right)$$

$$\sigma_r = \frac{-20P}{16t(1.685)t(12.315)}\left(12.315(0.719t) - 6.315t\right)$$

$$\sigma_r = -0.153\frac{P}{t}$$

(d) 

Using theory & formulas of Ref. 2.1,

$$\theta_F = \frac{PR^2}{EI} + \frac{\pi}{2}\frac{M_F R}{EI} \qquad I = \frac{t}{12}16^3$$

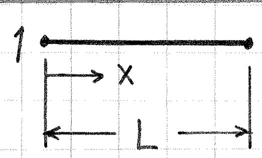
$$v_F = \frac{\pi}{4}\frac{PR^3}{EI} + \frac{M_F R^2}{EI}$$

$V_D = V_C + \theta_C L + \frac{PL^3}{3EI_s}$, $I_s = \frac{t\,6^3}{12}$

$$V_D = \frac{PR^2}{EI}\left(\frac{\pi R}{4} + L + L + \frac{\pi L^2}{2R}\right) + \frac{PL^3}{3EI_s}$$

$$V_D = \frac{P(14)^2}{341Et}\left(\frac{\pi}{4}14 + 12 + \frac{\pi}{2}\frac{6^2}{14}\right) + \frac{P(6)^3}{3E(144t)} = \frac{16.0\,P}{Et} \quad \text{(down)}$$

1.8



$u = \beta_1 + \beta_2 x$ (a)
(form of Eq. 1.2-1)

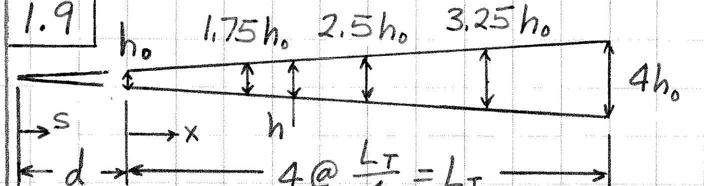
To obtain form of Eq. 1.2-3, evaluate (a) at $x=0$ and at $x=L$.

$$\left.\begin{aligned} x=0: &\quad u_1 = \beta_1 \\ x=L: &\quad u_2 = \beta_1 + \beta_2 L \end{aligned}\right\} \therefore \quad \beta_1 = u_1, \quad \beta_2 = \frac{u_2 - u_1}{L}$$

Eq. (a) becomes $u = u_1 + \frac{u_2 - u_1}{L}x$

or $u = \left(1 - \frac{x}{L}\right)u_1 + \frac{x}{L}u_2$

1.9



(a) $\frac{4h_0}{L_T + d} = \frac{h_0}{d}$, $d = \frac{L_T}{3}$

$\frac{h}{s} = \frac{h_0}{d}$, $h = \frac{h_0}{d}s = \frac{3h_0}{L_T}s$

$$\Delta_{ex} = \int \epsilon\, ds = \int_d^{d+L_T} \frac{P}{Eth}\, ds = \int_{L_T/3}^{4L_T/3} \frac{P\, ds}{\frac{3th_0}{L_T}sE}$$

$$\Delta_{ex} = \frac{PL_T}{3Eth_0}\ln 4 = 0.4621\frac{PL_T}{Eth_0}$$

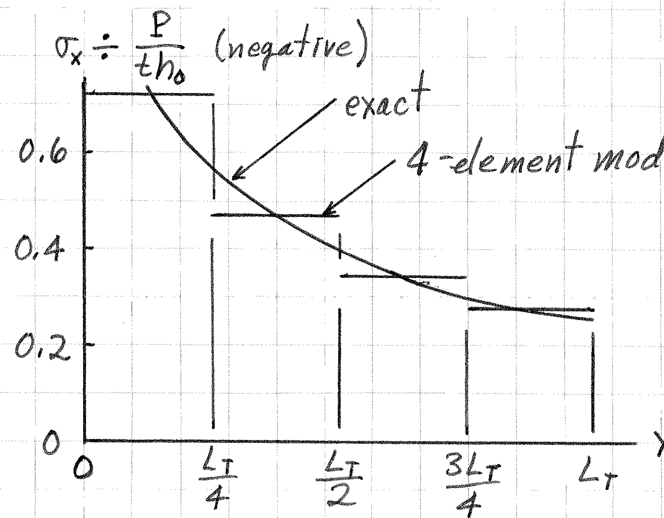
(b) $\Delta_1 = \frac{PL_T}{(2.5th_0)E} = 0.4\frac{PL_T}{th_0E}$ $\quad -13.4\%$

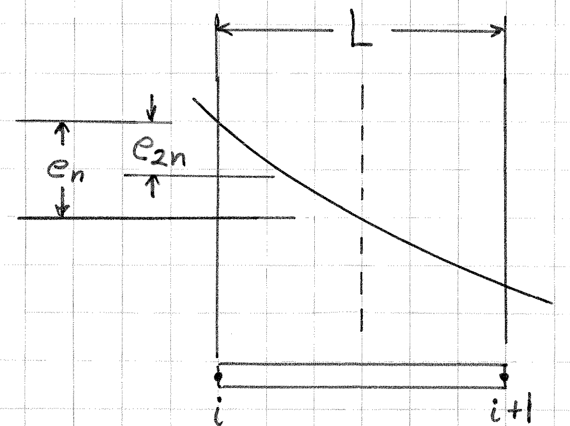
$\Delta_2 = \frac{P(L_T/2)}{(1.75th_0)E} + \frac{P(L_T/2)}{(3.25th_0)E} = 0.4396\frac{PL_T}{Eth_0}$ $\quad -4.9\%$

$$\Delta_3 = \frac{P(L_T/4)}{(1.375th_0)E} + \frac{P(L_T/4)}{(2.125th_0)E} + \frac{P(L_T/4)}{(2.875th_0)E} + \frac{P(L_T/4)}{(3.625th_0)E}$$

$\Delta_3 = 0.4554\frac{PL_T}{Eth_0}$ $\quad -1.5\%$

(c) $(\sigma_x)_{ex} = -\frac{P}{A} = \frac{-P}{t\left(h_0 + \frac{3h_0}{L_T}x\right)} = \frac{-P}{th_0\left(1 + \frac{3x}{L_T}\right)}$





e_n = el. nodal σ error for n elements
e_{2n} = " " " " " 2n "

We see that when elements are small enough for the curve to be almost a straight line over L, stress error is halved when no. of els. is doubled.