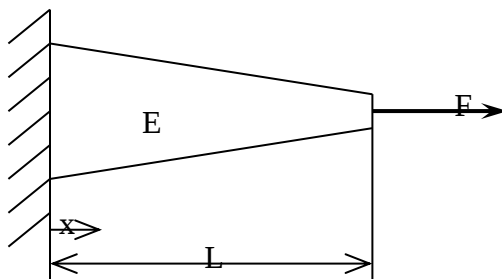


Problem 1.1

Using the principle of virtual work (PVW), find a quadratic displacement function $u(x)$ in $0 < x < L$ of a tapered slender rod of length L , fixed at the origin and loaded axially in tension at the free end. The cross-section area changes linearly and the areas are $A_1 > A_2$ at the fixed and free ends, respectively. The material is homogeneous and isotropic with modulus E .

Solution



The PVW says that:

$$\int_V (\sigma_{ij} \delta \epsilon_{ij} dV) - \int_S (t_i \delta u_i dS) - \int_V (f_i \delta u_i dV) = 0$$

Assuming $u(x)$ as: $u(x) = C_0 + C_1 x + C_2 x^2$

Using the boundary condition $u(x=0) = 0 \xrightarrow{\text{THEN}} C_0 = 0$

Because of it's a case of uniaxial load without body forces ($f_i = 0$), the PVW can be rewritten as:

$$\int_0^L (\sigma_x \delta \epsilon_x A_x dx) - F \delta u(x=L) = 0$$

So, we can calculate:

$$\delta u(x) = x \delta C_1 + x^2 \delta C_2$$

$$\epsilon_x = \frac{\partial u}{\partial x} = C_1 + 2C_2 x$$

$$\delta \epsilon_x = \delta C_1 + 2x \delta C_2$$

$$\sigma_x = E \epsilon_x = E(C_1 + 2C_2 x)$$

and

$$A_x = A_1 \frac{(L-x)}{L} + A_2 \frac{x}{L}$$

$$A_x = A_1 - \frac{(A_1 - A_2)}{L} x$$

Now,

$$E \int_0^L (C_1 + 2C_2 x) (\delta C_1 + 2x \delta C_2) \left(A_1 - \frac{(A_1 - A_2)}{L} x \right) dx - F(L \delta C_1 + L^2 \delta C_2) = 0$$

$$E \int_0^L (C_1 \delta C_1 + 2C_1 x \delta C_2 + 2C_2 x \delta C_1 + 4C_2 x^2 \delta C_2) \left(A_1 - \frac{(A_1 - A_2)}{L} x \right) dx - F(L \delta C_1 + L^2 \delta C_2) = 0$$

$$E \int_0^L (4C_2 x^2 \delta C_2 + 2(C_1 \delta C_2 + C_2 \delta C_1)x + C_1 \delta C_1) \left(A_1 - \frac{(A_1 - A_2)}{L} x \right) dx - F(L \delta C_1 + L^2 \delta C_2) = 0$$

$$EA_1 \int_0^L (4C_2 x^2 \delta C_2 + 2(C_1 \delta C_2 + C_2 \delta C_1)x + C_1 \delta C_1) dx$$

$$- E \frac{(A_1 - A_2)}{L} \int_0^L (4C_2 x^3 \delta C_2 + 2(C_1 \delta C_2 + C_2 \delta C_1)x^2 + C_1 \delta C_1 x) dx - F(L \delta C_1 + L^2 \delta C_2) = 0$$

$$EA_1 \left(\frac{4}{3} L^3 C_2 \delta C_2 + L^2 (C_1 \delta C_2 + C_2 \delta C_1) + L C_1 \delta C_1 \right)$$

$$- E \frac{(A_1 - A_2)}{L} \left(L^4 C_2 \delta C_2 + \frac{2}{3} L^3 (C_1 \delta C_2 + C_2 \delta C_1) + \frac{1}{2} L^2 C_1 \delta C_1 \right) - F(L \delta C_1 + L^2 \delta C_2) = 0$$

Reordering the equation to express it like a linear combination of the variations of C_i 's

$$\delta C_1 \left(EA_1 L^2 C_2 + EA_1 L C_1 - \frac{1}{2} E(A_1 - A_2) L C_1 - \frac{2}{3} E(A_1 - A_2) L^2 C_2 - FL \right) \\ + \delta C_2 \left(\frac{4}{3} EA_1 L^3 C_2 + EA_1 L^2 C_1 - E(A_1 - A_2) L^3 C_2 - \frac{2}{3} E(A_1 - A_2) L^2 C_1 - FL^2 \right) = 0$$

Since C_i 's are independent variations of C_i 's values there are two equations involved in the above one. The solution of those equations (2x2 system) will give the values of coefficients C_i 's.

$$\left(EA_1L - \frac{1}{2}E(A_1 - A_2)L\right)C_1 + \left(EA_1L^2 - \frac{2}{3}E(A_1 - A_2)L^2\right)C_2 = FL$$

$$\left(EA_1L^2 - \frac{2}{3}E(A_1 - A_2)L^2\right)C_1 + \left(\frac{4}{3}EA_1L^3 - E(A_1 - A_2)L^3\right)C_2 = FL^2$$

Calling

$$\alpha = EA_1 - \frac{1}{2}E(A_1 - A_2)$$

$$\beta = EA_1 - \frac{2}{3}E(A_1 - A_2)$$

$$\chi = \frac{4}{3}EA_1 - E(A_1 - A_2)$$

Then

$$\alpha C_1 + \beta LC_2 = F$$

$$\beta LC_1 + \chi LC_2 = F$$

The solution for this system is:

$$C_2 = \frac{(\beta - \alpha)F}{(\beta^2 - \alpha\chi)L}$$

$$C_1 = \frac{(\chi - \beta)F}{(\alpha\chi - \beta^2)}$$

Finally, the displacement solution is:

$$u(x) = \frac{(\chi - \beta)F}{(\alpha\chi - \beta^2)}x + \frac{(\beta - \alpha)F}{(\beta^2 - \alpha\chi)L}x^2$$

In terms of the known data

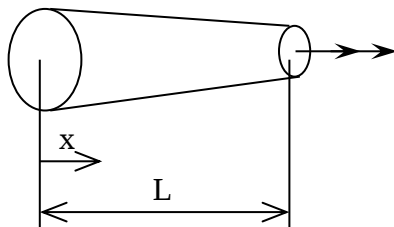
$$u(x) = \frac{6A_2}{E(A_1^2 + 4A_1A_2 + A_2^2)}Fx + \frac{3(A_1 - A_2)}{E(A_1^2 + 4A_1A_2 + A_2^2)}\frac{Fx}{L}$$

When $A_1 = A_2 = A$ the well-known result $u(x) = \frac{Fx}{EA}$ is obtained.

Problem 1.2

Using the principle of virtual work (PVW), find a quadratic rotation angle function $\theta(x)$ in $0 < x < L$ of a tapered slender shaft of circular cross section and length L , fixed at the origin and loaded by a torque T at the free end. The cross-section area changes lineally and the areas are $A_1 > A_2$ at the fixed and free ends, respectively. The material is homogeneous and isotropic with shear modulus G .

Solution



The PVW says that:

$$\int_V (\sigma_{ij} \delta \epsilon_{ij} dV) - \int_S (T_i \delta \theta_i dS) - \int_V (f_i \delta \theta_i dV) = 0$$

Assuming $\theta(x)$ as: $\theta(x) = C_0 + C_1 x + C_2 x^2$

Using the boundary condition $\theta(x=0) = 0 \xrightarrow{\text{THEN}} C_0 = 0$

Because of it's a case of pure torsion without body forces ($f_i = 0$), the PVW can be rewritten as:

$$\int_0^L (\tau_{xy} \delta (\frac{1}{2} \gamma_{xy}) A_x dx) - T \delta \theta(x=L) = 0$$

So, we can calculate:

$$\delta \theta(x) = x \delta C_1 + x^2 \delta C_2$$

$$\gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = r_x (C_1 + 2C_2 x)$$

$$u = 0 \xrightarrow{\frac{\partial}{\partial y}} \frac{\partial u}{\partial y} = 0$$

$$v = r_x \theta(x) \xrightarrow{\frac{\partial}{\partial x}} \frac{\partial v}{\partial x} = r_x (C_1 + 2C_2 x)$$

$$\delta (\frac{1}{2} \gamma_{xy}) = \frac{r_x}{2} (\delta C_1 + 2x \delta C_2)$$

$$\tau_{xy} = G \gamma_{xy} = G r_x (C_1 + 2C_2 x)$$

and

$$A_x = A_1 \frac{(L-x)}{L} + A_2 \frac{x}{L}$$

$$A_x = A_1 - \frac{(A_1 - A_2)}{L} x = \pi r_x^2$$

$$r_x^2 = r_1^2 - \frac{(r_1^2 - r_2^2)}{L} x$$

Now,

$$\int_0^L G r_x (C_1 + 2C_2 x)^{\frac{1}{2}} r_x (\delta C_1 + 2x \delta C_2) \pi r_x^2 dx - T(L \delta C_1 + L^2 \delta C_2) = 0$$

Solving the integral and organizing the result

$$\begin{aligned} & \delta C_1 \left(\frac{1}{12} \pi G \right) (2C_1 (r_1^4 + r_1^2 r_2^2 + r_2^4) + LC_2 (r_1^4 + 2r_1^2 r_2^2 + 3r_2^4)) \\ & + \delta C_2 \left(\frac{1}{60} \pi G \right) (5LC_1 (r_1^4 + 2r_1^2 r_2^2 + 3r_2^4) + 4L^2 C_2 (r_1^4 + 3r_1^2 r_2^2 + 6r_2^4)) \\ & = T \delta C_1 + LT \delta C_2 \end{aligned}$$

The coefficients of the terms δC_i 's must be equal on RHS and LHS, therefore a 2x2 system can be written.

$$2\alpha C_1 + \beta LC_2 = \delta$$

$$5\beta C_1 + 4\chi LC_2 = 5\delta$$

with

$$\alpha = r_1^4 + r_1^2 r_2^2 + r_2^4$$

$$\beta = r_1^4 + 2r_1^2 r_2^2 + 3r_2^4$$

$$\chi = r_1^4 + 3r_1^2 r_2^2 + 6r_2^4$$

$$\delta = \frac{12T}{\pi G}$$

Solving the system

$$C_1 = \left(\frac{5\beta - 4\chi}{5\beta^2 - 8\alpha\chi} \right) \delta$$

$$C_2 = \left(\frac{2\alpha - \beta}{5\beta^2 - 8\alpha\chi} \right) \frac{5\delta}{L}$$

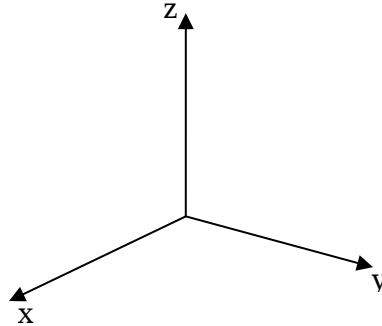
When $r_1 = r_2 = r$ the well-known result $\theta = \frac{2Tx}{G\pi r^4}$ is obtained.

Problem 1.3

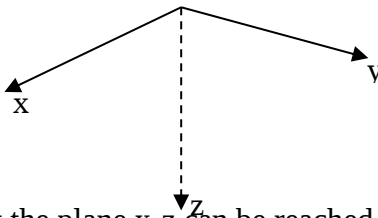
Construct a rotation matrix $[a]$ resulting from three consecutive reflections about (a) the x-y plane, (b) the x-z plane, (c) the y-z plane. The resulting system does not follow the right-hand rule.

Solution

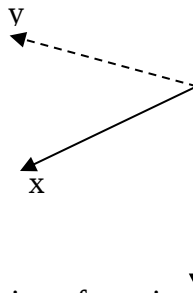
Initial system



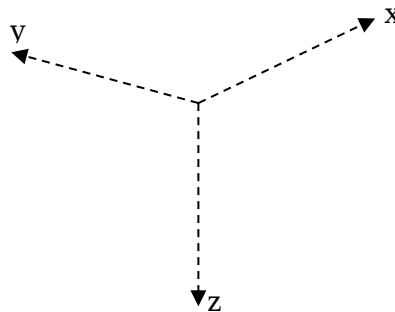
- (a) The reflection of z-axis about the plane x-y can be reached by one of the following four rotations: $\theta = (+\pi)$ or $(-\pi)$ about the x-axis or y-axis.



- (b) The reflection of y-axis about the plane x-z can be reached by one of the following four rotations: $\theta = (+\pi)$ or $(-\pi)$ about the x-axis or z-axis.



- (c) x' -axis represents the reflection of x-axis about the plane y-z. It can be reached by one of the following four rotations: $\theta = (+\pi)$ or $(-\pi)$ about the y-axis or z-axis.



Notice that the new system doesn't accomplish the right hand rule.

$$\text{Rotation matrix: } a_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Problem 1.4

Construct three rotation matrices $[a]$ for rotations $\theta = \pi$ about (a) the x-axis, (b) the y-axis, (c) the z-axis.

Solution

Rotation about the x-axis

$$a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \xrightarrow{\theta=\pi} a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Rotation about the y-axis

$$a_{ij} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \xrightarrow{\theta=\pi} a_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Rotation about the z-axis

$$a_{ij} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\theta=\pi} a_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 1.5

Using

$$\sigma = \begin{bmatrix} 10 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

and [a] of Ex. 1.2, verify that (1.29) yields the same result as (1.26).

Solution

Solved with Mathematica®:

```
T[({
  {l1_, m1_, n1_},
  {l2_, m2_, n2_},
  {l3_, m3_, n3_}
})] := ({
  {l1^2, m1^2, n1^2, 2 m1 n1, 2 l1 n1, 2 l1 m1},
  {l2^2, m2^2, n2^2, 2 m2 n2, 2 l2 n2, 2 l2 m2},
  {l3^2, m3^2, n3^2, 2 m3 n3, 2 l3 n3, 2 l3 m3},
  {l2 l3, m2 m3, n2 n3, m2 n3+n2 m3, l2 n3+n2 l3, l2 m3 +m2
13},
  {l1 l3, m1 m3, n1 n3, m1 n3+n1 m3, l1 n3+n1 l3, l1 m3 +m1
13},
  {l1 l2, m1 m2, n1 n2, m1 n2+n1 m2, l1 n2+n1 l2, l1 m2 +m1 l2}
});
Tbar[T_] := ({
  {1, 0, 0, 0, 0, 0},
  {0, 1, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0},
  {0, 0, 0, 2, 0, 0},
  {0, 0, 0, 0, 2, 0},
  {0, 0, 0, 0, 0, 2}
}).T.({
  {1, 0, 0, 0, 0, 0},
  {0, 1, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0},
  {0, 0, 0, 1/2, 0, 0},
  {0, 0, 0, 0, 1/2, 0},
  {0, 0, 0, 0, 0, 1/2}
});
Inverse[T[({
  {1, 0, 0},
  {0, Cos[π], Sin[π]},
  {0, -Sin[π], Cos[π]}
})]] == Tbar[T[({
  {1, 0, 0},
  {0, Cos[π], Sin[π]},
  {0, -Sin[π], Cos[π]}
})]]^T (* Eq1 .47 [T]^-1Overscript[=, ?][Overscript[T,
_] ]^T for (a) *)
```

```
Inverse[T[_{
    {Cos[π], 0, -Sin[π]},
    {0, 1, 0},
    {Sin[π], 0, Cos[π]}
}_]]==Tbar[T[_{
    {Cos[π], 0, -Sin[π]},
    {0, 1, 0},
    {Sin[π], 0, Cos[π]}
}_]]^T (* Eq1 .47 [T]^-1Overscript[=, ?][Overscript[T, _]]^T
for (b) *)
Inverse[T[_{
    {Cos[π], Sin[π], 0},
    {-Sin[π], Cos[π], 0},
    {0, 0, 1}
}_]]==Tbar[T[_{
    {Cos[π], Sin[π], 0},
    {-Sin[π], Cos[π], 0},
    {0, 0, 1}
}_]]^T (* Eq1 .47 [T]^-1Overscript[=, ?][Overscript[T,
_] ]^T for (c) *)
```

All three equations yield ``true''.