Solutions Manual to

Energy Principles and Variational Methods in in Applied Mechanics

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Preface

This solutions manual is prepared to help the instructor in discussing the solutions to problems at the end of each chapter of my book, *Energy Principles and Variational Methods in Applied Mechanics,* John Wiley & Sons, New York, 2002. Solutions to *most* problems of Chapters 2–10 (there is no problem set in Chapter 1) are included. Additional examples and problems on vectors, tensors, and variational principles and methods can be found in the following books of the author:

1. Reddy, J. N., and Rasmussen, M. L., *Advanced Engineering Analysis,* John Wiley, New York (1982); reprinted and marketed currently by Krieger Publishing Company, Melbourne, Florida (1990).
2. Reddy, J. N., *Energy and Variational Methods in Applied Mechanics,* John Wiley & Sons, New York (1984).
3. Reddy, J. N. *Applied Functional Analysis and Variational Methods in Engineering,* McGraw-Hill, New York, 1986; reprinted and marketed currently by Krieger Publishing Company, Melbourne, Florida (1991).
4. Reddy, J. N., *An Introduction to the Finite Element Method,* Second Edition, McGraw-Hill, New York (1993).
5. Reddy, J. N., *Mechanics of Laminated Composite Plates: Theory and Analysis,* CRC Press, Boca Raton, Florida (1997).
6. Reddy, J. N., *Theory and Analysis of Elastic Plates,* Taylor & Francis, Philadelphia (1999).

The instructor should make an effort to review the problems before assigning them. This allows the instructor to make comments and suggestions on the approach to be taken and nature of the answers expected. The instructor may wish to generate additional problems from those given in this book, especially when taught time and again from the same book.

As stated in the Preface to the book, this book is not free of errors. In the final printing of the book, the printer has used very light (instead of dark) printing and therefore there are many places where the arrow heads are seen but not the lines connecting the arrow heads (this will be corrected in the next printing of the book). They have also missed to make corrections in few places. I wish to thank in advance those colleagues who are willing to draw my attention to typos and errors, using the e-mail address: jnreddy@hotmail.com.

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Chapter 2

2.1 Let **C** be a vector along the line passing through the terminal point of vector **A** and parallel to vector **B**. The projection of vector **A** along vector **B** is

$$\mathbf{A}\cdot\hat{\mathbf{e}}_B, \quad \hat{\mathbf{e}}_B = \frac{\mathbf{B}}{|\mathbf{B}|}.$$

The vector along this projection is given by

$$\mathbf{A}_p \equiv (\mathbf{A}\cdot\hat{\mathbf{e}}_B)\,\hat{\mathbf{e}}_B.$$

Then $\mathbf{A}-\mathbf{A}_p$ is perpendicular to **B** and hence to **C**. Therefore,

$$\mathbf{C}\cdot[\mathbf{A}-(\mathbf{A}\cdot\hat{\mathbf{e}}_B)\,\hat{\mathbf{e}}_B]=0$$

is the equation of (or any multiple of it) the required line.

2.2 The vector connecting the terminal points of vectors **A** and **B** is $\mathbf{A}-\mathbf{B}$. Similarly, the vector connecting the terminal points of **B** and **C** is $\mathbf{B}-\mathbf{C}$. Finally, the vector connecting the terminal points of **A** and **C** is $\mathbf{A}-\mathbf{C}$. The three vectors $\mathbf{A}-\mathbf{B}$, $\mathbf{B}-\mathbf{C}$, and $\mathbf{A}-\mathbf{C}$ all lie in the plane connecting the terminal points of the three vectors **A**, **B**, and **C**. Then a necessary and sufficient condition for the three vectors to be coplanar is that

$$(\mathbf{A}-\mathbf{B})\times(\mathbf{B}-\mathbf{C})\cdot(\mathbf{A}-\mathbf{C})=0,$$

which provides an equation for the required plane.

2.3 We begin with the left side of the equality and arrive at the right side:

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= A_i\hat{\mathbf{e}}_i\times(\hat{\mathbf{e}}_\ell\varepsilon_{jk\ell}B_jC_k) = A_iB_jC_k\varepsilon_{jk\ell}\varepsilon_{pi\ell}\hat{\mathbf{e}}_p\\ &= (\delta_{jp}\delta_{ki}-\delta_{ji}\delta_{kp})A_iB_jC_k\hat{\mathbf{e}}_p = A_iB_jC_i\hat{\mathbf{e}}_j - A_iB_iC_k\hat{\mathbf{e}}_k\\ &= (\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}.\end{aligned}$$

2.4 This follows from Problem 2.3. We have

$$\mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{C}\cdot\mathbf{B})\mathbf{A}-(\mathbf{C}\cdot\mathbf{A})\mathbf{B}.$$

Let $\mathbf{B}=\mathbf{C}=\hat{\mathbf{e}}$ in the above identity and obtain the required result.

2.5

(a) $\delta_{ii}=\delta_{11}+\delta_{22}+\delta_{33}=1+1+1=3.$

(b) $F_{ij}\delta_{ij} = F_{i1}\delta_{i1}+F_{i2}\delta_{i2}+F_{i3}\delta_{i3}$. Clearly, the first term is zero unless $i=1$, the second is zero unless $i=2$, and the third is zero unless $i=3$. Thus, we have $F_{ij}\delta_{ij}=F_{11}+F_{22}+F_{33}=F_{ii}$. The given identity follows as a special case, $F_{ij}=\delta_{ij}$.

(c) $F_{ij}\delta_{jk} = F_{i1}\delta_{1k} + F_{i2}\delta_{2k} + F_{i3}\delta_{3k}$. Clearly, the first term is zero unless $k = 1$. If $k = 1$, the remaining two terms are zero giving F_{i1}. Similarly, the second term is zero unless $k = 2$, which makes the first and third terms zero so that the result is F_{i2}. Finally, the third term is zero unless $k = 3$, which makes the first two terms zero, giving F_{i3}. Thus, $F_{ij}\delta_{jk}$ is equal to F_{ik}. Alternatively, we have $F_{ij}\delta_{jk} = 0$ unless $j = k$ because $\delta_{jk} = 0$ unless $j = k$. (*Caution*: replace j by k only in the coefficient of δ_{jk} and then remove δ_{jk} from the expression; do not write $F_{ij}\delta_{jk} = F_{ik}\delta_{kk}$, which has no meaning as no subscript can appear more than twice!). The given identity follows as a special case, $F_{ij} = \delta_{ij}$.

(d) $\varepsilon_{mjk}\varepsilon_{njk} = \delta_{mn}\delta_{jj} - \delta_{mj}\delta_{jn} = 3\delta_{mn} - \delta_{mn} = 2\delta_{mn}$, where we have used the $\varepsilon - \delta$ identity (expanded about the common subscript k)and parts (a) and (c) of this problem in arriving at the result.

(e) Follows from Part (d).

(f) $F_{ij}\varepsilon_{ijk} = -F_{ij}\varepsilon_{ikj} = -F_{ji}\varepsilon_{jki}$. In the first step subscripts j and k are interchanged, which results in the change of sign. In the second step, the subscript i is renamed as j and subscript j as i (in the whole expression). Since $\varepsilon_{jki} = \varepsilon_{ijk}$, we have

$$F_{ij}\varepsilon_{ijk} = -F_{ji}\varepsilon_{jki} \quad \rightarrow \quad (F_{ij} + F_{ji})\,\varepsilon_{ijk} = 0.$$

Now replace F_{ij} with A_iA_j and find that

$$(A_iA_j + A_jA_i)\,\varepsilon_{ijk} = 2A_iA_j\varepsilon_{ijk} = 0,$$

which was to be proved. Thus, if F_{ij} is symmetric, it follows that $F_{ij}\varepsilon_{ijk} = 0$. Conversely, if $F_{ij}\varepsilon_{ijk} = 0$, then $F_{ij} = F_{ji}$, i.e., the second-order tensor $\overleftrightarrow{\mathbf{F}}$ is symmetric.

2.6 We have $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A})$

$$\begin{aligned}
&= (\varepsilon_{ijk}A_iB_j\hat{\mathbf{e}}_k) \cdot [(\varepsilon_{rst}B_rC_s\hat{\mathbf{e}}_t) \times (\varepsilon_{mnp}C_mA_n\hat{\mathbf{e}}_p)] \\
&= \varepsilon_{ijk}\varepsilon_{rst}\varepsilon_{mnp}A_iB_jB_rC_sC_mA_n\varepsilon_{tpq}\delta_{kq} \\
&= (\delta_{it}\delta_{jp} - \delta_{ip}\delta_{jt})\,\varepsilon_{rst}\varepsilon_{mnp}A_iB_jB_rC_sC_mA_n \\
&= (\varepsilon_{rsi}\varepsilon_{mnj} - \varepsilon_{rsj}\varepsilon_{mni})\,A_iB_jB_rC_sC_mA_n \\
&= (\mathbf{B} \times \mathbf{C} \cdot \mathbf{A})\,(\mathbf{C} \times \mathbf{A} \cdot \mathbf{B}) - (\mathbf{B} \times \mathbf{C} \cdot \mathbf{B})\,(\mathbf{C} \times \mathbf{A} \cdot \mathbf{A}) \\
&= (\mathbf{B} \times \mathbf{C} \cdot \mathbf{A})\,(\mathbf{C} \times \mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \times \mathbf{C} \cdot \mathbf{A})^2
\end{aligned}$$

Note that $\mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$.

2.7

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= (\varepsilon_{ijk}A_iB_j\hat{\mathbf{e}}_k) \times (\varepsilon_{mnp}C_mD_n\hat{\mathbf{e}}_p) \\
&= \varepsilon_{ijk}\varepsilon_{kpq}\varepsilon_{mnp}A_iB_jC_mD_n\hat{\mathbf{e}}_q \\
&= \varepsilon_{mnp}\,(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\,A_iB_jC_mD_n\hat{\mathbf{e}}_q \\
&= \varepsilon_{mni}A_iB_jC_mD_n\hat{\mathbf{e}}_j - \varepsilon_{mnj}A_iB_jC_mD_n\hat{\mathbf{e}}_i \\
&= (\mathbf{C} \times \mathbf{D} \cdot \mathbf{A})\,\mathbf{B} - (\mathbf{C} \times \mathbf{D} \cdot \mathbf{B})\,\mathbf{A}.
\end{aligned}$$

2.8 Let $A = (1,1,0,0)$, $B = (0,1,0,1), C = (0,0,1,1)$ and $D = (1,1,1,1)$. Then the relation

$$\alpha A + \beta B + \gamma C + \mu D = 0$$

implies

$$\alpha+\mu=0, \quad \alpha+\beta+\mu=0, \quad \gamma+\mu=0, \quad \beta+\gamma+\mu=0,$$

whose solution is $\alpha=\gamma=-\mu$. Hence, the vectors are linearly dependent.

2.9 The linear relation

$$\alpha\mathbf{A}+\beta\mathbf{B}+\gamma\mathbf{C}=\mathbf{0}$$

implies

$$2\alpha-\gamma=0, \quad -\alpha+\beta+\gamma=0, \quad \alpha-\beta=0$$

whose solution is trivial. Hence, the vectors are linearly independent.

2.10

(a) First check for linear independence. Set

$$\alpha\mathbf{A}+\beta\mathbf{B}+\gamma\mathbf{C}=\mathbf{0}$$

and find that

$$\alpha-4\beta+2\gamma=0, \quad 3\alpha+3\beta+\gamma=0, \quad -\alpha-5\beta+\gamma=0.$$

The solution of these equations is $\alpha=-2\beta$ and $\gamma=3\beta$. Hence, the set is linearly dependent and does not span $\Re^3$.

(b) Set

$$\alpha\mathbf{A}+\beta\mathbf{B}+\gamma\mathbf{C}=\mathbf{0}$$

and find that

$$\alpha+\beta+\gamma=0, \quad \alpha+\beta=0, \quad -2\beta-\gamma=0.$$

The solution of these equations is trivial $\alpha=\beta=\gamma=0$. Hence, the set is linearly independent. Next, express an arbitrary vector $\mathbf{x}=x_1\hat{\mathbf{e}}_1+x_2\hat{\mathbf{e}}_2+x_3\hat{\mathbf{e}}_3$ in $\Re^3$ as a linear combination of the given vectors

$$\mathbf{x}=c_1\mathbf{A}+c_2\mathbf{B}+c_3\mathbf{C}$$

and show that c_i will not imply a relation among the components x_i of the vector (because they are arbitrary). We have

$$c_1+c_2+c_3=x_1, \quad c_1+c_2=x_2, \quad -2c_1-c_3=x_3,$$

whose solution is

$$c_3=x_1-x_2, \quad c_1=\frac{1}{2}(x_2-x_1-x_3), \quad c_2=\frac{1}{2}(x_1+x_2+x_3).$$

In particular, the vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ are represented by the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ as

$$\hat{\mathbf{e}}_1=-\frac{1}{2}\mathbf{A}+\frac{1}{2}\mathbf{B}+\mathbf{C}, \quad \hat{\mathbf{e}}_2=\frac{1}{2}\mathbf{A}+\frac{1}{2}\mathbf{B}-\mathbf{C}, \quad \hat{\mathbf{e}}_3=-\frac{1}{2}\mathbf{A}+\frac{1}{2}\mathbf{B}.$$

Hence, the set $(\mathbf{A},\ \mathbf{B},\ \mathbf{C})$ spans $\Re^3$.

2.11 Follows as outlined in the problem statement.

2.12

(a) Let $\hat{\mathbf{e}}_i$ be the unit base vectors in the current orthogonal system, and $\hat{\mathbf{e}}'_i$ be the unit base vectors in the new coordinate system. The vector $\hat{\mathbf{e}}'_1$ has the same direction as the vector $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ but its magnitude must be unity

$$\hat{\mathbf{e}}'_1 = \frac{\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{|\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3|} = \frac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3).$$

The vector $\hat{\mathbf{e}}'_2$ is along the normal to the plane $2x_1 + 3x_2 + x_3 - 5 = 0$. Hence, $\hat{\mathbf{e}}'_2 = \hat{\mathbf{n}}$, the unit normal to the plane, which is given by

$$\hat{\mathbf{e}}'_2 = \frac{\nabla(2x_1 + 3x_2 + x_3 - 5)}{|\nabla(2x_1 + 3x_2 + x_3 - 5)|} = \frac{2\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{\sqrt{(2)^2 + (3)^2 + (1)^2}}$$
$$= \frac{1}{\sqrt{14}}(2\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3).$$

The third basis vector in an orthonormal system is related to the other two vectors by

$$\hat{\mathbf{e}}'_3 = \hat{\mathbf{e}}'_1 \times \hat{\mathbf{e}}'_2 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{vmatrix}$$
$$= \hat{\mathbf{e}}_1(-\frac{1}{\sqrt{42}} - \frac{3}{\sqrt{42}}) - \hat{\mathbf{e}}_2(\frac{1}{\sqrt{42}} - \frac{2}{\sqrt{42}}) + \hat{\mathbf{e}}_3(\frac{3}{\sqrt{42}} + \frac{2}{\sqrt{42}})$$
$$= \frac{1}{\sqrt{42}}(-4\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 5\hat{\mathbf{e}}_3).$$

Thus, the two coordinate systems are related by (note the matrix of direction cosines)

$$\begin{Bmatrix} \hat{\mathbf{e}}'_1 \\ \hat{\mathbf{e}}'_2 \\ \hat{\mathbf{e}}'_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{-4}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{5}{\sqrt{42}} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}.$$

(b)

$$\hat{\mathbf{e}}'_1 = \frac{\text{vector connecting point } (1,-1,3) \text{ to point}(2,-2,4)}{\text{vector magnitude}}$$
$$= \frac{(2\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3) - (\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3)}{\text{magnitude}} = \frac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)$$
$$\hat{\mathbf{e}}'_3 = \frac{1}{\sqrt{6}}(-\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3) \quad \text{(given)}$$
$$\hat{\mathbf{e}}'_2 = \hat{\mathbf{e}}'_3 \times \hat{\mathbf{e}}'_1 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix}$$
$$= \hat{\mathbf{e}}_1\left(\frac{1}{\sqrt{18}} + \frac{2}{\sqrt{18}}\right) - \hat{\mathbf{e}}_2\left(\frac{-1}{\sqrt{18}} - \frac{2}{\sqrt{18}}\right) + \hat{\mathbf{e}}_3\left(\frac{1}{\sqrt{18}} - \frac{1}{\sqrt{18}}\right)$$
$$= \hat{\mathbf{e}}_1\left(\frac{1}{\sqrt{18}} + \frac{2}{\sqrt{18}}\right) - \hat{\mathbf{e}}_2\left(\frac{-1}{\sqrt{18}} - \frac{2}{\sqrt{18}}\right) + \hat{\mathbf{e}}_3\left(\frac{1}{\sqrt{18}} - \frac{1}{\sqrt{18}}\right).$$

Transformation matrix relating $(\hat{\mathbf{e}}_1', \hat{\mathbf{e}}_2', \hat{\mathbf{e}}_3')$ to $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ is given by

$$[A] = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \quad (a_{ij} = \hat{\mathbf{e}}_i' \cdot \hat{\mathbf{e}}_j)$$

(c)

$$\hat{\mathbf{e}}_1' = \hat{\mathbf{e}}_1 \cos 30° + \hat{\mathbf{e}}_2 \sin 30°$$
$$\hat{\mathbf{e}}_2' = \hat{\mathbf{e}}_1(-\sin 30°) + \hat{\mathbf{e}}_2 \cos 30°$$
$$\hat{\mathbf{e}}_3' = \hat{\mathbf{e}}_3 \quad \text{(given)}$$
$$[A] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.13 Follows from the definition

$$[A] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.14 Follows from the definition

$$[A] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

2.15 The vector is given by

$$\mathbf{A} = \frac{1}{2}(6\hat{\mathbf{e}}_1 - 6\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3) - (\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3) = 2\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3.$$

The direction cosines are $(2/3, -2/3, -1/3)$.

2.16

(a) The orthogonal projection of $\mathbf{A}$ onto $\mathbf{B}$ is $\mathbf{A} \cdot \mathbf{B} = 6 - 4 = 2$.

(b) The angle between $\mathbf{A}$ and $\mathbf{B}$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|}\right) = \cos^{-1}\left(\frac{-2}{5 \times 3}\right) \to \theta = 82.34°.$$

2.17

(a) This is obvious.

(b) We have

$$\frac{d}{dt}\left[\mathbf{A}\frac{d\mathbf{A}}{dt}\frac{d^2\mathbf{A}}{dt^2}\right] = \left[\frac{d\mathbf{A}}{dt}\frac{d\mathbf{A}}{dt}\frac{d^2\mathbf{A}}{dt^2}\right] + \left[\mathbf{A}\frac{d^2\mathbf{A}}{dt^2}\frac{d^2\mathbf{A}}{dt^2}\right] + \left[\mathbf{A}\frac{d\mathbf{A}}{dt}\frac{d^3\mathbf{A}}{dt^3}\right]$$
$$= \left[\mathbf{A}\frac{d\mathbf{A}}{dt}\frac{d^2\mathbf{A}}{dt^2}\right].$$

2.18

(a) We have

$$\operatorname{grad}(r) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} (x_j x_j)^{\frac{1}{2}} = \hat{\mathbf{e}}_i \frac{1}{2} (x_j x_j)^{\frac{1}{2}-1} 2x_i$$
$$= \hat{\mathbf{e}}_i x_i (x_j x_j)^{-\frac{1}{2}} = \frac{\mathbf{r}}{r}.$$

(b) We have

$$\operatorname{grad}(r^n) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} (x_j x_j)^{\frac{n}{2}} = \hat{\mathbf{e}}_i \frac{n}{2} (x_j x_j)^{\frac{n}{2}-1} 2x_i$$
$$= n\hat{\mathbf{e}}_i x_i (x_j x_j)^{\frac{n-2}{2}} = nr^{n-2}\mathbf{r}.$$

Alternatively,

$$\nabla(r^n) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} (r^n) = nr^{n-1} \hat{\mathbf{e}}_i \frac{\partial r}{\partial x_i}$$
$$= nr^{n-2} x_i \hat{\mathbf{e}}_i = nr^{n-2}\mathbf{r}.$$

(c) We have, in view of the result in (b)

$$\nabla^2(r^n) = \frac{\partial^2}{\partial x_i \partial x_i} (r^n) = \frac{\partial}{\partial x_i} (nr^{n-2} x_i)$$
$$= n(n-2)r^{n-3} \frac{\partial r}{\partial x_i} x_i + nr^{n-2}\delta_{ii} = n(n-2)r^{n-3}\frac{x_i}{r} x_i + 3nr^{n-2}$$
$$= [n(n-2) + 3n]r^{n-2} = n(n+1)r^{n-2}.$$

(d) We have (note that **A** is a constant vector)

$$\operatorname{grad}(\mathbf{r}\cdot\mathbf{A}) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} (x_j A_j) = \hat{\mathbf{e}}_i \left(\delta_{ij} A_j + x_j \frac{\partial A_j}{\partial x_i}\right)$$
$$= \hat{\mathbf{e}}_i (A_i + 0) = \mathbf{A}.$$

(e)

$$\operatorname{div}(\mathbf{r}\times\mathbf{A}) = \hat{\mathbf{e}}_i \cdot \frac{\partial}{\partial x_i} (\varepsilon_{jk\ell} x_j A_k \hat{\mathbf{e}}_\ell) = \varepsilon_{jk\ell}\delta_{i\ell} \left(\frac{\partial x_j}{\partial x_i} A_k + x_j \frac{\partial A_k}{\partial x_i}\right)$$
$$= (0+0) = 0.$$

(f) Carrying out the indicated operation, we obtain

$$\operatorname{curl}(\mathbf{r}\times\mathbf{A}) = \hat{\mathbf{e}}_i \times \frac{\partial}{\partial x_i} (\varepsilon_{rst} x_r A_s \hat{\mathbf{e}}_t)$$
$$= \varepsilon_{rst} \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_t \left(\delta_{ir} A_s + x_j \frac{\partial A_s}{\partial x_i}\right)$$
$$= \varepsilon_{rst}\varepsilon_{jit} \hat{\mathbf{e}}_j (\delta_{ir} A_s + 0) = \varepsilon_{ist}\varepsilon_{jit} \hat{\mathbf{e}}_j A_s$$
$$= -2\hat{\mathbf{e}}_j \delta_{sj} A_s = -2\mathbf{A}.$$