CHAPTER 3 - Second-Order Linear Differential Equations - SOLUTIONS MANUAL

3.1 Homogeneous Differential Equations with Constant Coefficients

1P

Let $y = e^{rt}$, so that $y' = r\,e^{rt}$ and $y'' = r^2\,e^{rt}$. Direct substitution into the differential equation yields $(r^2 + 3r - 4)e^{rt} = 0$. Canceling the exponential, the characteristic equation is $r^2 + 3r - 4 = 0$. The roots of the equation are $r = -4\,,1$. Hence the general solution is $y = c_1e^t + c_2e^{-4t}$.

2P

Let $y = e^{rt}$. Substitution of the assumed solution results in the characteristic equation $r^2 + 5r + 6 = 0$. The roots of the equation are $r = -3\,,-2$. Hence the general solution is $y = c_1e^{-2t} + c_2e^{-3t}$.

3P

Let $y = e^{rt}$, so that $y' = r\,e^{rt}$ and $y'' = r^2\,e^{rt}$. Direct substitution into the differential equation yields $(12r^2 - r - 1)e^{rt} = 0$. Since $e^{rt} \neq 0$, the characteristic equation is $12r^2 - r - 1 = 0$. The roots of the equation are $r = -1/4\,,1/3$. Hence the general solution is $y = c_1e^{-t/4} + c_2e^{t/3}$.

4P

Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + 6r = 0$. The roots of the equation are $r = 0\,,-6$. Hence the general solution is $y = c_1e^{0t} + c_2e^{-6t} = c_1 + c_2e^{-6t}$.

5P

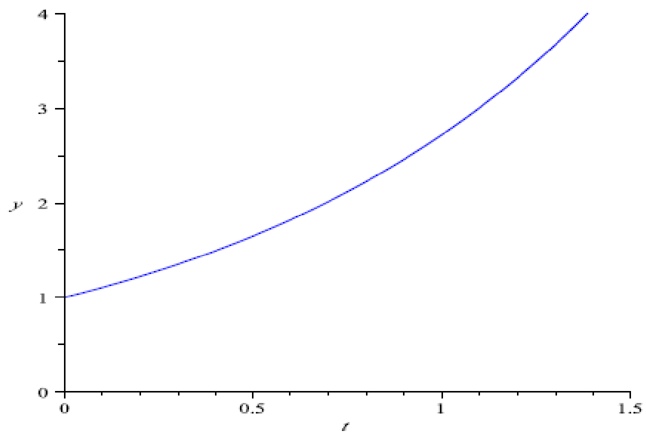
The characteristic equation is $9r^2 - 16 = 0$, with roots $r = \pm 4/3$. Therefore the general solution is $y = c_1e^{-4t/3} + c_2e^{4t/3}$.

6P

The characteristic equation is $r^2 - 4r - 4 = 0$, with roots $r = 2 \pm 2\sqrt{2}$. Hence the general solution is $y = c_1e^{(2-2\sqrt{2})t} + c_2e^{(2+2\sqrt{2})t}$.

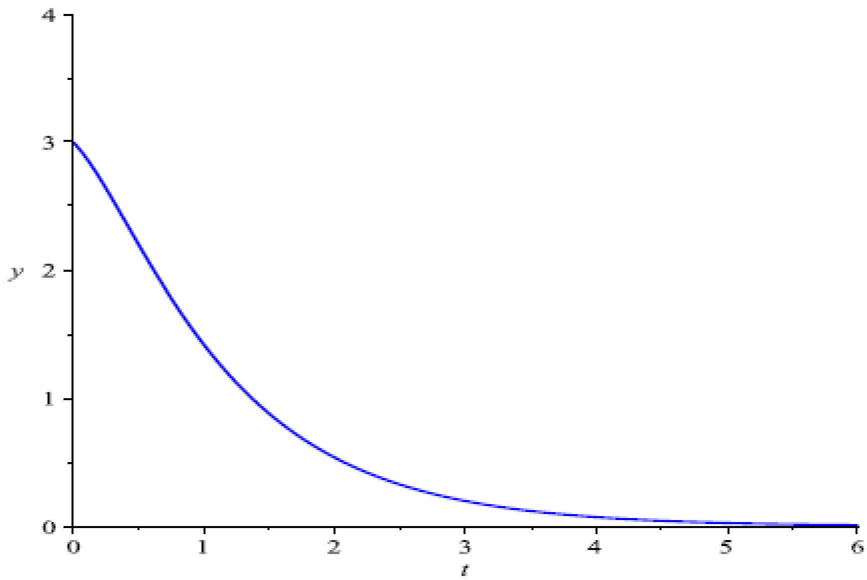
7P

Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + 2r - 3 = 0$. The roots of the equation are $r = -3, 1$. Hence the general solution is $y = c_1 e^{-3t} + c_2 e^t$. Its derivative is $y' = -3c_1 e^{-3t} + c_2 e^t$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 1$. In order to satisfy $y'(0) = 1$, we find that $-3c_1 + c_2 = 1$. Solving for the constants, $c_1 = 0$ and $c_2 = 1$. Hence the specific solution is $y(t) = e^t$. It clearly increases without bound as $t \to \infty$.



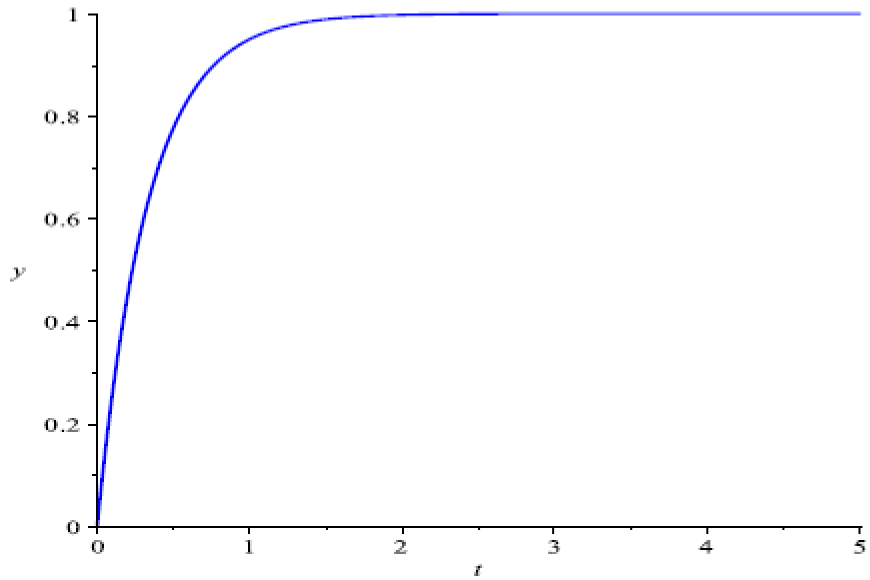
8P

Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + 4r + 3 = 0$. The roots of the equation are $r = -1, -3$. Hence the general solution is $y = c_1e^{-t} + c_2e^{-3t}$. Its derivative is $y' = -c_1e^{-t} - 3c_2e^{-3t}$. Based on the first condition, $y(0) = 2$, we require that $c_1 + c_2 = 3$. In order to satisfy $y'(0) = -1$, we find that $-c_1 - 3c_2 = -1$. Solving for the constants, $c_1 = 4$ and $c_2 = -1$. Hence the specific solution is $y(t) = 4e^{-t} - e^{-3t}$. It clearly converges to 0 as $t \to \infty$.



9P

The characteristic equation is $r^2 + 3r = 0$, with roots $r = -3, 0$. Therefore the general solution is $y = c_1 + c_2 e^{-3t}$, with derivative $y' = -3\,c_2 e^{-3t}$. In order to satisfy the initial conditions, we find that $c_1 + c_2 = 0$, and $-3\,c_2 = 3$. Hence the specific solution is $y(t) = 1 - e^{-3t}$. This converges to 1 as $t \to \infty$.



10P

The characteristic equation is $2r^2 + r - 4 = 0$, with roots $r = (-1 \pm \sqrt{33})/4$. The general solution is $y = c_1 e^{(-1-\sqrt{33})t/4} + c_2 e^{(-1+\sqrt{33})t/4}$, with derivative

$$y' = \frac{-1-\sqrt{33}}{4} c_1 e^{(-1-\sqrt{33})t/4} + \frac{-1+\sqrt{33}}{4} c_2 e^{(-1+\sqrt{33})t/4}.$$

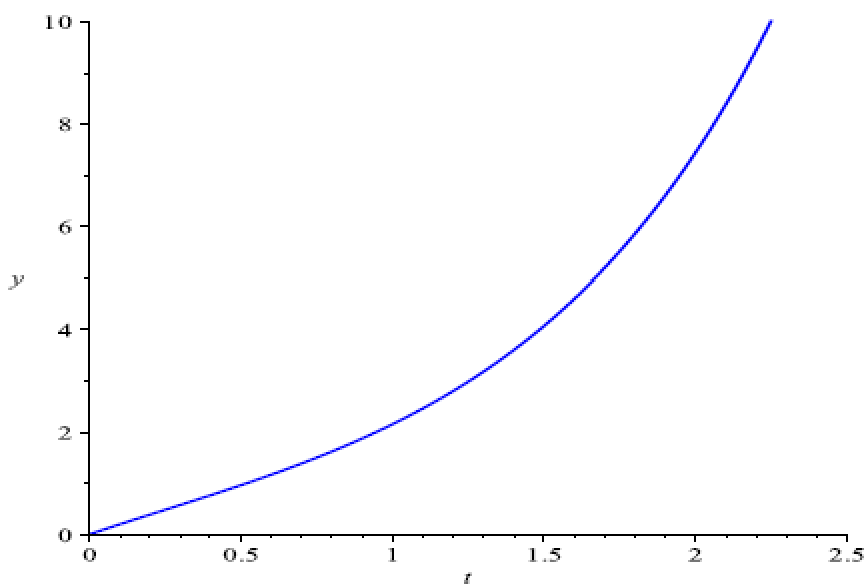
In order to satisfy the initial conditions, we require that

$$c_1 + c_2 = 0 \quad \text{and} \quad \frac{-1-\sqrt{33}}{4} c_1 + \frac{-1+\sqrt{33}}{4} c_2 = 2.$$

Solving for the coefficients, $c_1 = -4/\sqrt{33}$ and $c_2 = 4/\sqrt{33}$. The specific solution is

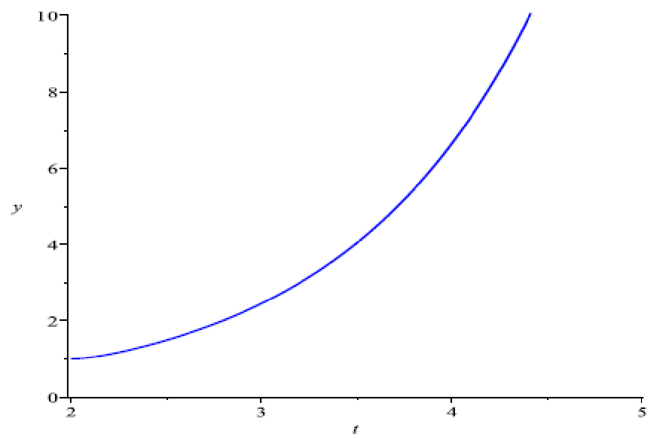
$$y(t) = -4\left[e^{(-1-\sqrt{33})t/4} - e^{(-1+\sqrt{33})t/4}\right]/\sqrt{33}.$$

It clearly increases without bound as $t \to \infty$.



11P

Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + 8r - 9 = 0$. The roots of the equation are $r = 1, -9$. Hence the general solution is $y = c_1 e^t + c_2 e^{-9t}$. Its derivative is $y' = c_1 e^t - 9c_2 e^{-9t}$. Based on the first condition, $y(2) = 1$, we require that $c_1 e^2 + c_2 e^{-18} = 1$. In order to satisfy the condition $y'(2) = 0$, we find that $c_1 e^2 - 9c_2 e^{-18} = 0$. Solving for the constants, $c_1 = 9e^{-2}/10$ and $c_2 = e^{18}/10$. Hence the specific solution is $y(t) = 9e^{t-2}/10 + e^{18-9t}/10 = 9e^{(t-2)}/10 + e^{-9(t-2)}/10$. (Observe the shift on the time axis.) It clearly increases without bound as $t \to \infty$.

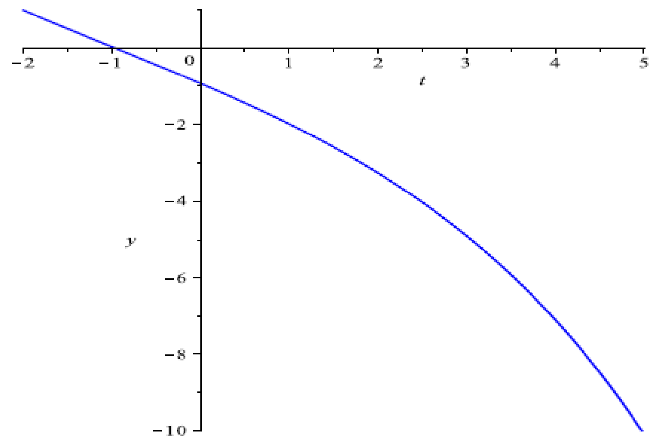


12P

The characteristic equation is $9r^2 - 1 = 0$, with roots $r = \pm 1/3$. Therefore the general solution is $y = c_1 e^{-t/3} + c_2 e^{t/3}$. Since the initial conditions are specified at $t = -2$, is more convenient to write $y = d_1 e^{-(t+2)/3} + d_2 e^{(t+2)/3}$. The derivative is given by $y' = -\left[d_1 e^{-(t+2)/3}\right]/3 + \left[d_2 e^{(t+2)/3}\right]/3$. In order to satisfy the initial conditions, we find that $d_1 + d_2 = 1$, and $-d_1/3 + d_2/3 = -1$. Solving for the coefficients, $d_1 = 2$, and $d_2 = -1$. The specific solution is

$$y(t) = 2e^{-(t+2)/3} - e^{(t+2)/3} = 2e^{-2/3}e^{-t/3} - e^{2/3}e^{t/3}.$$

It clearly decreases without bound as $t \to \infty$.



13P

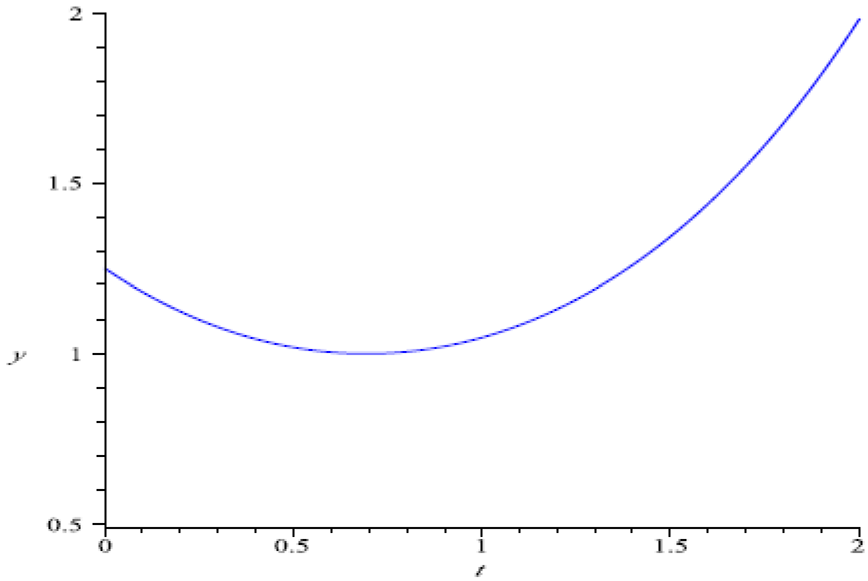
An algebraic equation with roots 4 and -3 is $(r-4)(r+3) = r^2 - r - 12 = 0$. This is the characteristic equation for the differential equation $y'' - y' - 12y = 0$.

14P

The characteristic equation is $2r^2 - 3r + 1 = 0$, with roots $r = 1/2, 1$. Therefore the general solution is $y = c_1 e^{t/2} + c_2 e^{t}$, with derivative $y' = c_1 e^{t/2}/2 + c_2 e^{t}$. In order to satisfy the initial conditions, we require $c_1 + c_2 = 2$ and $c_1/2 + c_2 = 1/2$. Solving for the coefficients, $c_1 = 3$, and $c_2 = -1$. The specific solution is $y(t) = 3e^{t/2} - e^{t}$. To find the stationary point, set $y' = 3e^{t/2}/2 - e^{t} = 0$. There is a unique solution, with $t_1 = \ln(9/4)$. The maximum value is then $y(t_1) = 9/4$. To find the x-intercept, solve the equation $3e^{t/2} - e^{t} = 0$. The solution is readily found to be $t_2 = \ln 9 \approx 2.1972$.

15P

The characteristic equation is $r^2 - 1 = 0$, with roots $r = 1, -1$. Therefore the general solution is $y = c_1e^t + c_2e^{-t}$, with derivative $y' = c_1e^t - c_2e^{-t}$. To satisfy the initial conditions, we require that $c_1 + c_2 = 5/4$ and $c_1 - c_2 = -3/4$. Solving for the coefficients, $c_1 = 1/4$ and $c_2 = 1$. This means that the specific solution is $y(t) = e^t/4 + e^{-t}$. From this, $y' = e^t/4 - e^{-t} = 0$ when $e^{2t} = 4$ or $t = \ln 2$. The value here is $y(\ln 2) = 2/4 + 1/2 = 1$. Since $y'' = y$ is positive at $t = \ln 2$, this is a minimum.



16P

The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Hence the general solution is $y = c_1e^{-t/2} + c_2e^{t/2}$ and $y' = -c_1e^{-t/2}/2 + c_2e^{t/2}/2$. Invoking the initial conditions, we require that $c_1 + c_2 = 2$ and $-c_1 + c_2 = 2\beta$. The specific solution is $y(t) = (1-\beta)e^{-t/2} + (1+\beta)e^{t/2}$. Based on the form of the solution, it is evident that as $t \to \infty$, $y(t) \to 0$ as long as $\beta = -1$.

17P

The characteristic equation is $r^2 + (3-\alpha)r - 2(\alpha-1) = 0$. Solving this equation, we see that the roots are $r = \alpha - 1, -2$. Therefore, the general solution is $y(t) = c_1 e^{(\alpha-1)t} + c_2 e^{-2t}$. In order for all solutions to tend to zero, we need $\alpha - 1 < 0$. Therefore, the solutions will all tend to zero as long as $\alpha < 1$. Due to the term $c_2 e^{-2t}$, we can never guarantee that all solutions will become unbounded as $t \to \infty$.

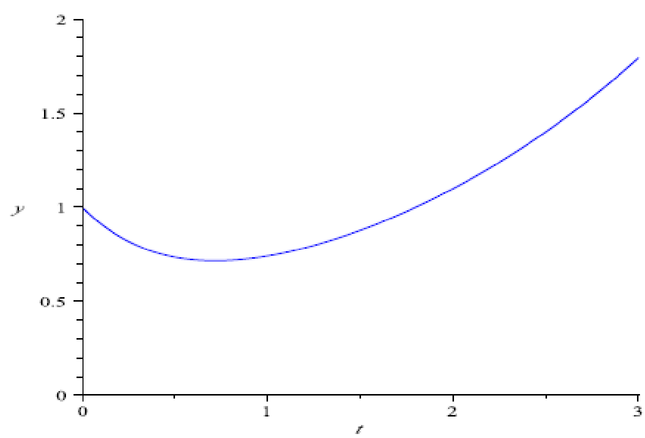
18P

The characteristic equation is $r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0$. Examining the coefficients, the roots are $r = \alpha\,,\ \alpha - 1$. Hence the general solution of the differential equation is $y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$. Assuming $\alpha \in \mathbb{R}$, all solutions will tend to zero as long as $\alpha < 0$. On the other hand, all solutions will become unbounded as long as $\alpha - 1 > 0$, that is, $\alpha > 1$.

19P

(a) The characteristic equation is $2r^2 + 3r - 2 = 0$, with roots $r = 1/2$ and $r = -2$. The initial conditions give $y(t) = (2\beta + 1)e^{-2t}/5 + (4 - 2\beta)e^{t/2}/5$.

(b) $y(t) = 2\,e^{t/2}/5 + 3\,e^{-2t}/5\,.$



We obtain that $y' = (-6e^{-2t} + e^{t/2})/5$. Setting this equal to zero and solving for t yields $t_0 = (2\ln 6)/5$. At this point, $y_0 = \sqrt[5]{3/16} \approx 0.715485$.

(c) From part (a), if $\beta = 2$ then $y = e^{-2t}$ and the solution simply decays to zero. For $\beta > 2$, the solution becomes unbounded negatively, and again there is no minimum point. For $0 < \beta < 2$ there is always a minimum point, as found in part (b).

20P

(a) The roots of the characteristic equation are $r = (-b \pm \sqrt{b^2 - 4ac})/2a$. For the roots to be real and different we must have $b^2 - 4ac > 0$. If they are to be negative, then we must have $b > 0$ (since we are given that $a > 0$) and $c > 0$. This latter condition comes from the fact that if $c \leq 0$ then $\sqrt{b^2 - 4ac} \geq b$ and hence the numerator of r would give both positive and negative values, or a zero if $c = 0$.

(b) From part (a), this will happen when $b^2 - 4ac > 0$ and $c < 0$.

(c) Similarly to part (a), this happens when $b^2 - 4ac > 0$ and $b < 0$ and $c > 0$.

21P

(a) Assuming that y is a constant, the differential equation reduces to $cy = d$. Hence the only equilibrium solution is $y = d/c$.

(b) Setting $y = Y + d/c$, substitution into the differential equation results in the equation $aY'' + bY' + c(Y + d/c) = d$. The equation satisfied by Y is $aY'' + bY' + cY = 0$.

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

1P

$$W(e^{4t}, e^{-3t/2}) = \begin{vmatrix} e^{4t} & e^{-3t/2} \\ 4e^{4t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{11}{2}e^{5t/2}.$$

2P

$$W(x, xe^{2x}) = \begin{vmatrix} x & xe^{2x} \\ 1 & e^{2x} + 2xe^{2x} \end{vmatrix} = xe^{2x} + 2x^2e^{2x} - xe^{2x} = 2x^2e^{2x}.$$

3P

$$W(e^{-3t}, te^{-3t}) = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & (1-3t)e^{-3t} \end{vmatrix} = e^{-6t}.$$

4P

$$W(e^{2t}\sin t\,, e^{2t}\cos t) = \begin{vmatrix} e^{2t}\sin t & e^{2t}\cos t \\ e^{2t}(2\sin t + \cos t) & e^{2t}(2\cos t - \sin t) \end{vmatrix} = -e^{4t}\,.$$

5P

$$W(\sin^2\theta\,, 1-\cos 2\theta) = \begin{vmatrix} \sin^2\theta & 1-\cos 2\theta \\ 2\sin\theta\cos\theta & 2\sin 2\theta \end{vmatrix} = 0\,.$$

6P

Write the equation as $y'' + (3/t)y' = 1$. $p(t) = 3/t$ is continuous for all $t > 0$. Since $t_0 > 0$, the IVP has a unique solution for all $t > 0$.

7P

Write the equation as $y'' + (3/(t-4))y' + (5/t(t-4))y = 2/t(t-4)$. The coefficients are not continuous at $t = 0$ and $t = 4$. Since $t_0 \in (0, 4)$, the largest interval is $0 < t < 4$.

8P

The coefficient $3\ln|t|$ is discontinuous at $t = 0$. Since $t_0 > 0$, the largest interval of existence is $0 < t < \infty$.

9P

Write the IVP as

$$y'' + \frac{1}{x-2}y' + (\tan x)y = 0.$$

Since the coefficient functions are continuous for all x such that $x \neq 2$, $n\pi + \pi/2$ and $x_0 = 4$, the IVP is guaranteed to have a unique solution for all x such that $2 < x < 3\pi/2$.

10P

No. Substituting $y = \sin(t^2)$ into the differential equation,

$$-4t^2\sin(t^2) + 2\cos(t^2) + 2t\cos(t^2)p(t) + \sin(t^2)q(t) = 0\,.$$

At $t = 0$, this equation becomes $2 = 0$ (if we suppose that $p(t)$ and $q(t)$ are continuous), which is impossible.

11P

$y_1'' = 2$. We see that $t^2(2) - 2(t^2) = 0$. $y_2'' = 2t^{-3}$, with $t^2(y_2'') - 2(y_2) = 0$. Let $y_3 = c_1 t^2 + c_2 t^{-1}$, then $y_3'' = 2c_1 + 2c_2 t^{-3}$. It is evident that y_3 is also a solution.

12P

For $y = 1$, $y' = 0$ and $y'' = 0$, so $yy'' + (y')^2 = 0$. For $y = t^{1/2}$, $y' = t^{-1/2}/2$ and $y'' = -t^{-3/2}/4$, thus $yy'' + (y')^2 = -t^{-1}/4 + t^{-1}/4 = 0$. If $y = c_1 \cdot 1 + c_2 t^{1/2}$ is substituted into the differential equation, we get $(c_1 + c_2 t^{1/2})(-c_2 t^{-3/2}/4) + (c_2 t^{-1/2}/2)^2 = -c_1 c_2 t^{-3/2}/4$, which is zero only if $c_1 = 0$ or $c_2 = 0$. Thus the linear combination of two solutions is not, in general, a solution. Theorem 3.2.2 is not contradicted however, since the differential equation is not linear.

13P

$y = \phi(t)$ is a solution of the differential equation, so $L[\phi](t) = g(t)$. Since L is a linear operator, $L[c\phi](t) = cL[\phi](t) = cg(t)$. But, since $g(t) \neq 0$, $cg(t) = g(t)$ if and only if $c = 1$. This is not a contradiction of Theorem 3.2.2 since the linear differential equation is not homogeneous.

14P

$W(e^{3t}, g(t)) = e^{3t}g'(t) - 3e^{3t}g(t) = 2e^{6t}$. Dividing both sides by e^{3t}, we find that g must satisfy the ODE $g' - 3g = 2e^{3t}$. Hence $g(t) = 2t\,e^{3t} + c\,e^{3t}$.

15P

$W(t, g(t)) = tg'(t) - g(t) = 2t^2e^t$. Dividing both sides of the equation by t, we have $g' - g/t = 2te^t$. This a linear equation for g with an integrating factor $1/t$. Therefore, $g(t) = 2te^t + ct$.

16P

We compute

$$W(a_1y_1 + a_2y_2, b_1y_1 + b_2y_2) = \begin{vmatrix} a_1y_1 + a_2y_2 & b_1y_1 + b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} =$$

$$= (a_1y_1 + a_2y_2)(b_1y_1' + b_2y_2') - (b_1y_1 + b_2y_2)(a_1y_1' + a_2y_2') =$$

$$= a_1b_2(y_1y_2' - y_1'y_2) - a_2b_1(y_1y_2' - y_1'y_2) = (a_1b_2 - a_2b_1)W(y_1, y_2).$$

This now readily shows that y_3 and y_4 form a fundamental set of solutions if and only if $a_1b_2 - a_2b_1 \neq 0$.