Chapter 1

FUNDAMENTAL PRINCIPLES

Problem 1.1 We are told that the scales of the two major terms in the two groups of terms in eq. (1.5) or eq. (1.6) are measured experimentally:

$$\underbrace{\frac{D\rho}{Dt}}_{\left(\sim u\frac{\partial \rho}{\partial x}\right)} + \underbrace{\rho\nabla\cdot\mathbf{v}}_{\left(\sim \rho\frac{\partial u}{\partial x}\right)} = 0 \quad \leftarrow \text{scales}$$

Therefore, if eq. (1.8) is to apply, then the first scale must be negligible,

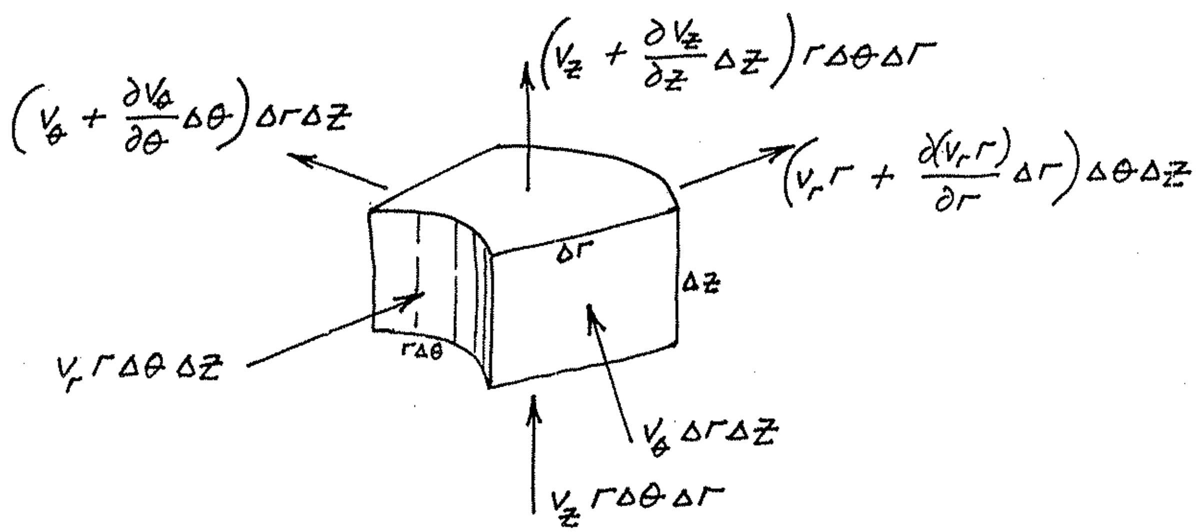
$$u\frac{\partial \rho}{\partial x} < \rho\frac{\partial u}{\partial x}$$

in other words, the relationship between $\partial\rho/\partial x$ and $\partial u/\partial x$ must be

$$\frac{\partial\rho/\partial x}{\partial u/\partial x} < \frac{\rho}{u}$$

Note that "<" means "less than, in an order-of-magnitude sense", or "negligible with respect to". The scale analysis literature often uses "<<" to say the same thing; in the present treatment I use "<", because one sign is enough when we compare orders of magnitude (the use of multiple signs such as "<<" leads to the temptation to read too much in the length of the sign, for example, by using something like "<<<" to stress the word "negligible").

Problem 1.2. Consider the control volume $(\Delta r)(r\Delta\theta)(\Delta z)$ drawn around the point (r,θ,z) in Fig. 1.1. Around this control volume we write graphically eq. (1.1):



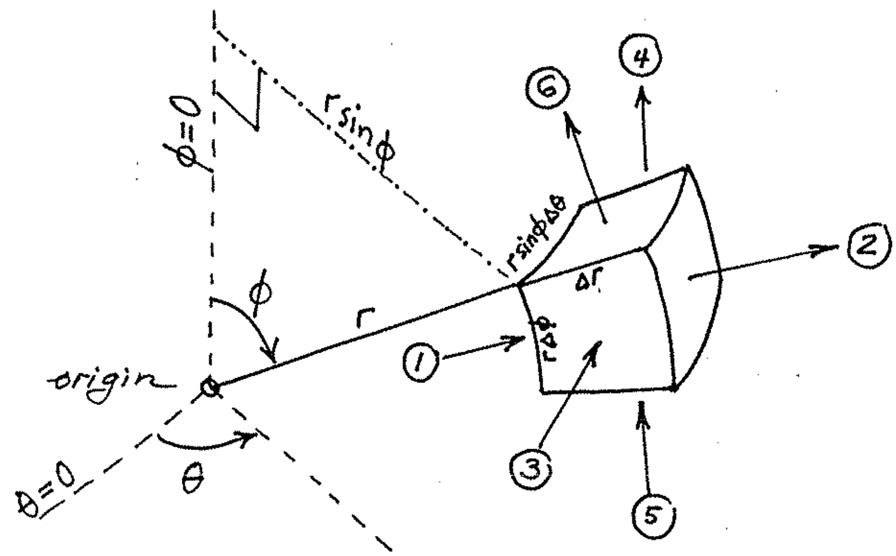
The term $\partial M_{cv}/\partial t$ is zero because ρ is constant. Note also that the "in" arrows cancel, respectively, the leading terms of the "out" arrows. Dividing the three surviving terms by the control volume $r\Delta\theta\Delta r\Delta z$, we are left with

$$\frac{1}{r}\frac{\partial}{\partial r}(v_r r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0,$$

which is the same as eq. (1.9),

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Problem 1.3. Consider the control volume described by the point (r,θ,ϕ) in Fig. 1.1, as r, θ and ϕ change by Δr, $\Delta\theta$ and $\Delta\phi$, respectively



mass flowrates "in":

(1) $v_r\, r\Delta\phi\, r\sin\phi\Delta\theta$

(3) $v_\theta\, r\Delta\phi\, \Delta r$

(5) $v_\phi\, \Delta r\, r\sin\phi\, \Delta\theta$

mass flowrates "out":

(2) $\left(v_r r^2 + \frac{\partial}{\partial r}(v_r r^2)\Delta r\right)\sin\phi\, \Delta\phi\, \Delta\theta$

(4) $\left(v_\theta + \frac{\partial}{\partial \theta}(v_\theta)\Delta\theta\right) r\Delta\phi\, \Delta r$

(6) $\left(v_\phi \sin\phi + \frac{\partial}{\partial \phi}(v_\phi \sin\phi)\Delta\phi\right) r\, \Delta r\, \Delta\theta$

Since $\frac{\partial M_{cv}}{\partial t} = 0$, the six flowrates add up to

$$\frac{1}{r}\frac{\partial}{\partial r}(v_r r^2) + \frac{1}{\sin\phi}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}(v_\phi \sin\phi) = 0,$$

which is the same as eq. (1.10).

Problem 1.4. The mass conservation equation for constant-density flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \text{or} \qquad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

With this property in mind, the x-momentum equation (1.17) can be simplified:

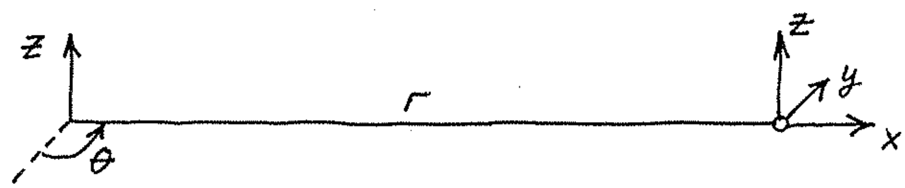
$$\rho \frac{Du}{dt} = -\frac{\partial P}{\partial x} + \underbrace{\frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x}\right.}_{\mu \frac{\partial^2 u}{\partial x^2} - \underbrace{\mu \frac{\partial^2 v}{\partial x \partial y}}} - \frac{2\mu}{3}\underbrace{\left.\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]}_{0} + \underbrace{\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]}_{\mu \frac{\partial^2 u}{\partial y^2} + \underbrace{\mu \frac{\partial^2 v}{\partial y \partial x}}} + X$$

$$\longrightarrow 0 \longleftarrow$$

In conclusion, we obtain

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + X \qquad (1.18)$$

Problem 1.5. Graphically, the limit $r \to \infty$ and the transformation $\Delta r \to \Delta x$, $r\Delta\theta \to \Delta y$, $\Delta z \to \Delta z$ can be sketched as follows:



In eq. (1.9) we have

$$\underbrace{\frac{\partial v_r}{\partial r}}_{\partial x} + \underbrace{\frac{v_r}{r}}_{\infty} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

or, since $v_r \to u$, $v_\theta \to v$ and $v_z \to w$,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1.8)$$

The momentum equations (1.21) have the same property; for example, the r equation (1.21a) can be written as

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\underbrace{\frac{\partial v_r}{\partial r}}_{\partial x}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{\partial y}-\underbrace{\frac{v_\theta^2}{r}}_{\infty}+v_z\frac{\partial v_r}{\partial z}\right)=$$

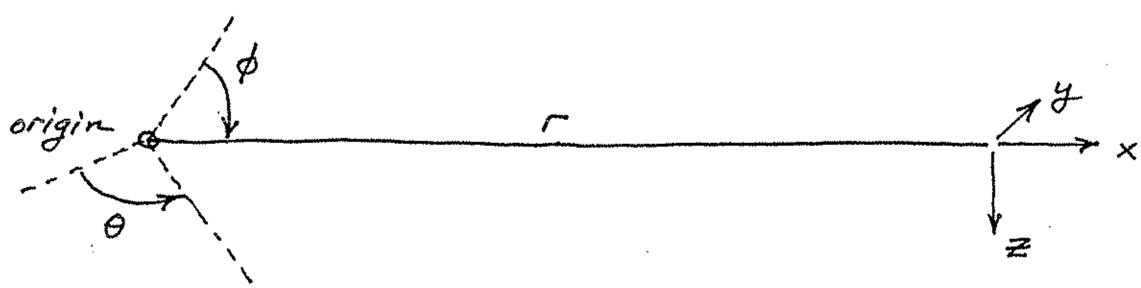
$$=-\underbrace{\frac{\partial P}{\partial r}}_{\partial x}+\mu\left(\underbrace{\frac{\partial^2 v_r}{\partial r^2}}_{\partial x^2}+\underbrace{\frac{1}{r}}_{\infty}\underbrace{\frac{\partial v_r}{\partial r}}_{\partial x}-\underbrace{\frac{v_r}{r^2}}_{\infty^2}+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{\partial y^2}-\underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{\infty\,\partial y}+\frac{\partial^2 v_r}{\partial z^2}\right)+F_r,$$

in other words,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+F_r \qquad (1.19a)$$

The message of this exercise is that, through a simple transformation, the validity of equations in cylindrical coordinates may be tested based on the considerably more familiar Cartesian forms.

Problem 1.6. In the $r \to \infty$ limit, the spherical coordinates sketched in Fig. 1.1 become



in other words, $\Delta r \to \Delta x$, $r \sin\phi\,\Delta\theta \to \Delta y$ and $r\Delta\phi \to \Delta z$. The mass continuity equation (1.10) can be expanded as:

$$\underbrace{\frac{\partial v_r}{\partial r}}_{\partial x}+\underbrace{\frac{2\,v_r}{r}}_{\infty}+\underbrace{\frac{1}{r}\frac{\partial v_\phi}{\partial \phi}}_{\partial z}+\underbrace{\frac{\text{cotan}\,\phi}{r}}_{\infty}+\underbrace{\frac{1}{r\sin\phi}\frac{\partial v_\theta}{\partial \theta}}_{\partial y}=0$$

Noting that $v_r \to u$, $v_\theta \to v$ and $v_\phi \to w$, the above equation reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}+\frac{\partial v}{\partial y}=0 \qquad (1.8)$$

Following the same procedure, the momentum equation (1.22a) reduces to eq. (1.19a).

Problem 1.7. Taking the specific kinetic energy into account, e is replaced by $e + V^2/2$ in eq. (1.25). This substitution generates two more terms:

$$\{\ \}_1^* = \Delta x\, \Delta y \frac{\partial}{\partial t}\left(\rho \frac{V^2}{2}\right)$$

$$\{\ \}_2^* = -\Delta x\, \Delta y \left[\frac{\partial}{\partial x}\left(\rho u \frac{V^2}{2}\right) + \frac{\partial}{\partial y}\left(\rho v \frac{V^2}{2}\right)\right]$$

Together, these additional terms combine into one new term on the left-hand-side ("LHS") of eq. (1.25),

$$LHS_* = \Delta x\, \Delta y\, \rho \frac{D\left(V^2/2\right)}{Dt}$$

where we have assumed a constant-ρ flow. This contribution, i.e. LHS_*, is cancelled by the last four terms in the work transfer group $\{\ \}_5$: to see this, we combine the constitutive relations (1.15) and (1.16) with the last four work terms, and get

$$\frac{1}{\Delta x \Delta y}\left\{\begin{matrix}\text{the last}\\ \text{four terms}\end{matrix}\right\}_5 = u\left[\frac{\partial P}{\partial x} - 2\mu \frac{\partial^2 u}{\partial x^2} + \frac{2}{3}\mu \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{0,\ \text{cf. eq. (1.8)}}\right]$$

$$-\mu u \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$+ v\left[\frac{\partial P}{\partial y} - 2\mu \frac{\partial^2 v}{\partial y^2} + \frac{2}{3}\mu \frac{\partial}{\partial y}\overbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}^{0}\right]$$

$$-\mu v \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

In the above expression we eliminate the pressure gradients using the momentum equations (1.19a,b)

$$\frac{\partial P}{\partial x} = -\rho \frac{Du}{Dt} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial P}{\partial y} = -\rho \frac{Dv}{Dt} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

After some algebra, the last four work terms reduce to

$$\frac{1}{\Delta x \Delta y}\left\{\begin{matrix}\text{the last}\\ \text{four terms}\end{matrix}\right\}_5 = -\rho \frac{D(u^2/2)}{Dt} - \rho \frac{D(v^2/2)}{Dt} = -\frac{D(V^2/2)}{Dt}$$

which amounts to a new term on the right-hand-side of eq. (25)

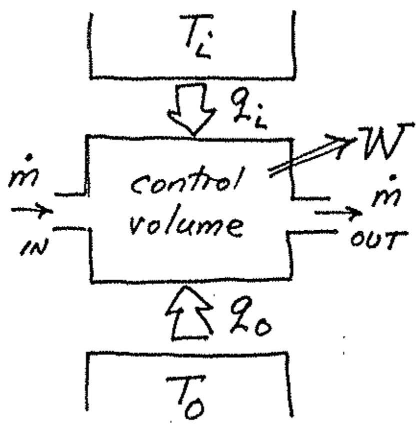
$$RHS_* = \Delta x \Delta y \, \rho \frac{D(V^2/2)}{Dt}$$

This additional term cancels LHS*. In conclusion, the energy equation (1.39) is valid even when the specific kinetic energy is accounted for in the derivation.

Problem 1.8. Consider the general flow system shown in the sketch. The first and second laws of thermodynamics state, respectively,

$$W = \sum q_i + q_o + \sum_{in} \dot{m}h - \sum_{out} \dot{m}h - \frac{\partial E_{cv}}{\partial t}$$

$$S_{gen} = \frac{\partial S_{cv}}{\partial t} - \sum \frac{q_i}{T_i} - \frac{q_o}{T_o} - \sum_{in} \dot{m}s + \sum_{out} \dot{m}s \geq 0$$



The reason for distinguishing q_o from the remaining heat transfer interactions q_i is that, if W is to vary with the degree of irreversibility (S_{gen}), then, according to the first law, at least one additional energy interaction must vary. Let q_o be the energy interaction that varies when W varies. In the extreme case of reversible operation, $S_{gen} = 0$, the second law requires

$$q_{o,rev} = T_o \left(-\frac{\partial S_{cv}}{\partial t} - \sum \frac{q_i}{T_i} - \sum_{in} \dot{m}s + \sum_{out} \dot{m}s \right)$$

In the same limit, the first law reads

$$W_{rev} = \sum q_i + q_{o,rev} + \sum_{in} \dot{m}h - \sum_{out} \dot{m}h - \frac{\partial E_{cv}}{\partial t}$$

The difference between the work output in the reversible limit, W_{rev}, and the actual work W is simply

$$W_{rev} - W = q_{o,rev} - q_o$$

However, from the second law for the general case ($S_{gen} > 0$), we recall that

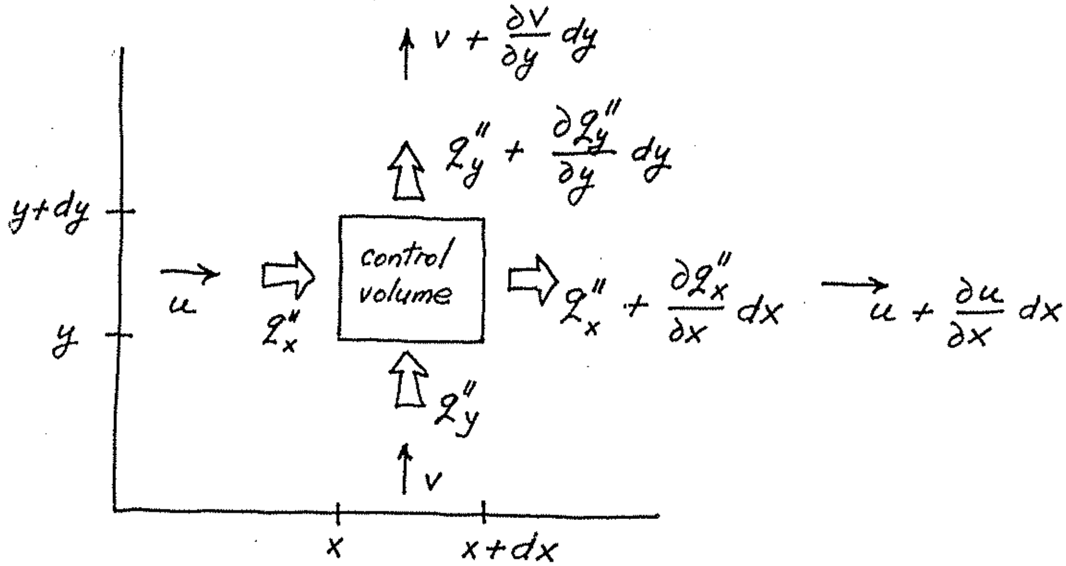
$$q_o = T_o \left(-S_{gen} + \frac{\partial S_{cv}}{\partial t} - \sum \frac{q_i}{T_i} - \sum_{in} \dot{m}s + \sum_{out} \dot{m}s \right)$$

Therefore, $q_o - q_{o,rev} = -T_o S_{gen}$, in other words

$$\underbrace{W_{rev} - W}_{W_{lost}} = T_o S_{gen} > 0$$

Note that T_o is the absolute temperature of the heat reservoir whose heat transfer rate q_o varies (floats) as W and the degree of irreversibility (S_{gen}) vary.

Problem 1.9. Consider the two-dimensional control volume of size dxdy shown below. The entropy generation rate in this system is S'''_{gen} dxdy: this quantity can be calculated by applying eq. (1.47) to the control volume dxdy,



$$S'''_{gen}dxdy = \frac{q''_x + \frac{\partial q''_x}{\partial x}dx}{T + \frac{\partial T}{\partial x}dx}dy + \frac{q''_y + \frac{\partial q''_y}{\partial y}dy}{T + \frac{\partial T}{\partial y}dy}dx - \frac{q''_x}{T}dy - \frac{q''_y}{T}dx$$

$$+\left(s + \frac{\partial s}{\partial x}dx\right)\left(u + \frac{\partial u}{\partial x}dx\right)\left(\rho + \frac{\partial \rho}{\partial x}dx\right)dy$$

$$+\left(s + \frac{\partial s}{\partial y}dy\right)\left(v + \frac{\partial v}{\partial y}dy\right)\left(\rho + \frac{\partial \rho}{\partial y}dy\right)dx$$

$$- su\rho\, dy - sv\rho\, dx + \frac{\partial(\rho s)}{\partial t}dxdy$$

Neglecting the terms smaller than (dxdy), we obtain

$$S'''_{gen} = \frac{1}{T}\left(\frac{\partial q''_x}{\partial x} + \frac{\partial q''_y}{\partial y}\right) - \frac{1}{T^2}\left(q''_x\frac{\partial T}{\partial x} + q''_y\frac{\partial T}{\partial y}\right) +$$

$$+\rho\left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y}\right)$$

$$+s\underbrace{\left[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]}_{\text{zero, cf. eq. (1.5)}}$$

In conclusion,

$$S'''_{gen} = \frac{1}{T} \nabla \cdot \mathbf{q}'' - \frac{1}{T^2} \mathbf{q}'' \cdot \nabla T + \rho \frac{Ds}{Dt}$$

From the canonical relation $du = Tds - P\, d\left(\frac{1}{\rho}\right)$ we deduce that

$$\rho \frac{Ds}{Dt} = \frac{\rho}{T} \frac{Du}{Dt} - \frac{P}{\rho T} \frac{D\rho}{Dt}$$

and from the first law of thermodynamics, eq. (1.26),

$$\rho \frac{Du}{Dt} = - \nabla \cdot \mathbf{q}'' - P\, \nabla \cdot \mathbf{v} + \mu \Phi$$

Combining the last three equations to eliminate Ds/Dt and Du/Dt, we find

$$S'''_{gen} = - \frac{1}{T^2} \mathbf{q}'' \cdot \nabla T + \frac{\mu}{T} \Phi$$

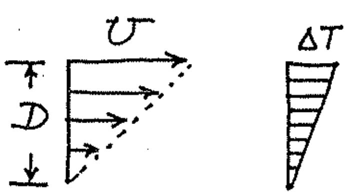
or, using the Fourier law of heat conduction, $\mathbf{q}'' = - k\, \nabla T$, in which k is constant,

$$S'''_{gen} = \frac{k}{T^2} (\nabla T)^2 + \frac{\mu}{T} \Phi \qquad (1.49)$$

Problem 1.10. In plane Couette flow we have

$$\frac{\partial u}{\partial y} = \frac{U}{D}, \quad \frac{\partial T}{\partial y} = \frac{\Delta T}{D}$$

so that eq. (1.49) becomes



$$S'''_{gen} = \underbrace{\frac{k}{T^2}\left(\frac{\Delta T}{D}\right)^2}_{\text{heat transfer irreversibility}} + \underbrace{\frac{\mu}{T}\left(\frac{U}{D}\right)^2}_{\text{fluid friction irreversibility}}$$

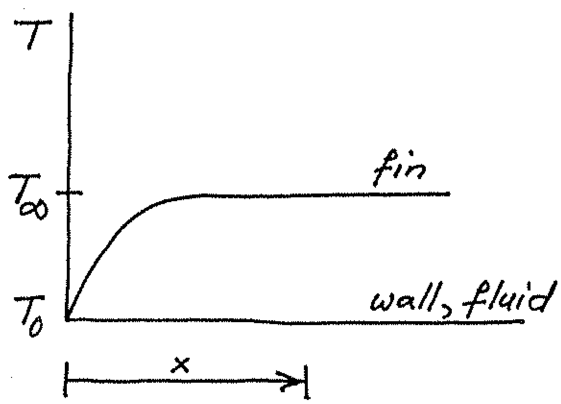
Thus, S'''_{gen} is dominated by fluid friction when

$$\frac{k\,(\Delta T)^2}{T^2 D^2} < \frac{\mu U^2}{T D^2},$$

or when

$$\frac{U}{\Delta T} \left(\frac{\mu T}{k}\right)^{1/2} > 1$$

Problem 1.11. In general, the energy equation represents the competition between three effects, longitudinal conduction, lateral convection and internal heat generation,



$$\underbrace{kA\frac{d^2T}{dx^2}}_{\text{conduction}} - \underbrace{hP(T-T_0)}_{\substack{\text{convection}\\ \text{(plays the role of}\\ \text{"heat sink" term)}}} + \underbrace{q'''A = 0}_{\text{generation}}$$

a) In the system of length x the three competing scales are $kA\frac{\Delta T}{x^2}$, $hP\,\Delta T$ and $q'''A$, where $\Delta T = T_\infty - T_0$. In the limit $x \to \infty$, the conduction scale becomes negligible relative to convection and generation.

b) Sufficiently far from the wall the balance is $hP\,\Delta T \sim q'''A$, and this means that

$$T_\infty - T_0 \sim \frac{q'''A}{hP}$$

c) Sufficiently close to the wall we have $kA\frac{\Delta T}{\delta^2} \sim q'''A$, therefore we conclude that

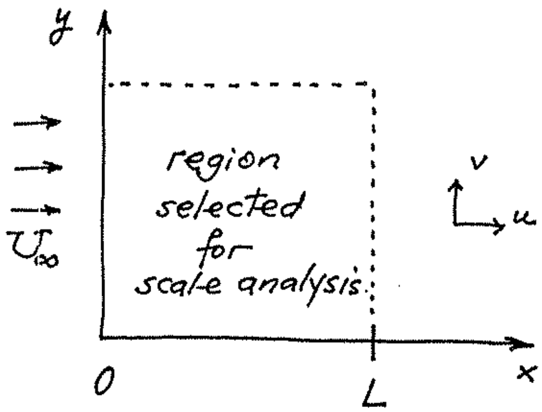
$$\delta \sim \left(\frac{k\,\Delta T}{q'''}\right)^{1/2}$$

d) The heat transfer rate into the wall is

$$q_B \sim kA\frac{\Delta T}{\delta} \sim$$
$$\sim A\,(q'''k\,\Delta T)^{1/2}.$$

Problem 1.12. The momentum equation (2.26) is

$$\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{inertia}} = \underbrace{-\frac{1}{\rho}\frac{dP_\infty}{dx}}_{\text{pressure}} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\text{friction}}$$



In the space selected for scale analysis, we have

$$x \sim L, \quad y \sim L, \quad u \sim U_\infty$$

in other words,

$$\frac{U_\infty^2}{L}, \quad \frac{1}{\rho}\frac{dP_\infty}{dx}, \quad \frac{\nu U_\infty}{L^2}$$

inertia pressure friction

The ratio inertia/friction is of the order of

$$\frac{\text{inertia}}{\text{friction}} \sim \frac{U_\infty^2}{L}\frac{L^2}{\nu U_\infty} = \frac{U_\infty L}{\nu} = Re_L$$

If $Re_L = 10^3$, then inertia > friction, meaning that the friction effect can be neglected in the $L \times L$ region. In that region, the flow is ruled by the balance between the remaining effects, namely, inertia ~ pressure.

Observations: – the $L \times L$ region is not the Boundary Layer region;
– the "inertia/friction ~ Re_L" interpretation of the Reynolds number *does not apply* in the boundary layer region discussed in chapter 2.

Problem 1.13. The total volume and heat leak are

$$V = t_H A_H + t_L A_L \tag{1}$$

$$q = q_H + q_L = kA_H \frac{\Delta T_H}{t_H} + kA_L \frac{\Delta T_L}{t_L} \tag{2}$$

1. To minimize q subject to V = constant is the same as finding the extremum of the function

$$\Phi = kA_H \frac{\Delta T_H}{t_H} + kA_L \frac{\Delta T_L}{t_L} + \lambda\left(t_H A_H + t_L A_L\right) \tag{3}$$

Solving the system $\frac{\partial \Phi}{\partial t_H} = 0$ and $\frac{\partial \Phi}{\partial t_L} = 0$, and eliminating λ we obtain

$$\frac{t_H}{t_L} = \left(\frac{\Delta T_H}{\Delta T_L}\right)^{1/2} \tag{4}$$

2. The heat leak q_H is more damaging than q_L because it is potentially more valuable: one can produce more power with a heat stream that comes from a higher temperature.

3. The imagined Carnot power outputs are

$$W_H = q_H \left(1 - \frac{T_0}{T_H}\right) = q_H \frac{\Delta T_H}{T_H} \tag{5}$$

$$W_L = q_L \left(1 - \frac{T_0}{T_L}\right) = q_L \frac{\Delta T_L}{T_L} \tag{6}$$

4. The total power lost because of the two heat leaks is

$$W = W_H + W_L = kA_H \Delta T_H \frac{\Delta T_H / T_H}{t_H} + kA_L \Delta T_L \frac{\Delta T_L / T_L}{t_L} \tag{7}$$

The optimization procedure is the same as in section 1. Note that Eq. (7) replaces Eq. (2), and that instead of ΔT_H and ΔT_L we now have $(\Delta T_H)^2 / T_H$ and $(\Delta T_L)^2 / T_L$. In conclusion, the optimal distribution of insulation is

$$\frac{t_H}{t_L} = \frac{\Delta T_H}{\Delta T_L} \cdot \left(\frac{T_L}{T_H}\right)^{1/2} \tag{8}$$

Comparing Eq. (8) with Eq. (4), we discover that

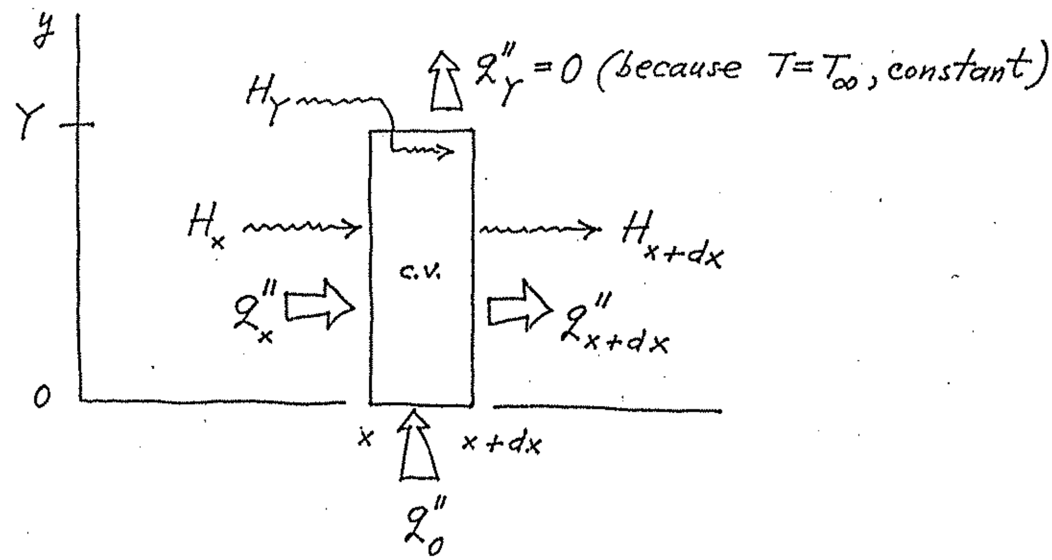
$$\frac{\left(t_H / t_L\right)_{\min q}}{\left(t_H / t_L\right)_{\min W}} = \left(\frac{\Delta T_L / T_L}{\Delta T_H / T_H}\right)^{1/2} \tag{9}$$

One cannot say which t_H/t_L ratio is larger: the answer depends on how $(\Delta T / T)_H$ compares with $(\Delta T / T)_L$. The recommended result for practice is Eq. (8).

Chapter 2

LAMINAR BOUNDARY LAYER FLOW

Problem 2.1. In order to derive eq. (2.52), construct the energy-equivalent of the bottom portion of Fig. 2.3:



In the steady-state, the first law of thermodynamics for the control volume (c.v.) is

$$H_x + \int_0^Y q''_x \, dy - H_{x+dx} - \int_0^Y q''_{x+dx} \, dy + q''_0 \, dx + H_Y = 0$$

where

$$H_Y = c_P T_\infty \, d\dot{m}; \qquad \dot{m} = \int_0^Y \rho u \, dy; \qquad H_x = \int_0^Y \rho c_P u T \, dy$$

We assume T_∞ = constant. Dividing the first law by dx, we obtain

$$-\frac{d}{dx}\int_0^Y \rho c_P u T \, dy + c_P T_\infty \frac{d}{dx}\int_0^Y \rho u \, dy - \frac{d}{dx}\int_0^Y q''_x \, dy + q''_0 = 0$$

The scales of the last two terms in the last equation are, respectively,

$$\frac{1}{x} k \frac{\Delta T}{x} Y, \qquad k \frac{\Delta T}{Y}$$

Clearly, if $x > Y$ [i.e. if the $(x) \times (Y)$ region is slender] the last scale dominates, and the third term in the energy equation can be neglected. We are left with

$$\frac{d}{dx}\int_0^Y \rho\, c_P\, u\,(T_\infty - T)\, dy = k\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

or, assuming $\rho\, c_P$ = constant,

$$\frac{d}{dx}\int_0^Y u\,(T_\infty - T)\, dy = \alpha\left(\frac{\partial T}{\partial y}\right)_{y=0} \qquad (2.52)$$

Problem 2.2. The Blasius profile problem reduces to solving

$$2f''' + ff'' = 0 \qquad (2.82)$$

subject to

$$f(0) = f'(0) = 0 \qquad (2.83)$$

$$f'(\infty) = 1 \qquad (2.84)$$

Assuming the small-η expansion

$$f(\eta) = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + ...$$

$$f'(\eta) = a_1 + 2a_2\eta + 3a_3\eta^2 + 4a_4\eta^3 + ...$$

from the $\eta = 0$ conditions (2.83) we learn that

$$a_0 = a_1 = 0$$

The Blasius equation (2.82) becomes

$$2(3\times 2\times 1\, a_3 + 4\times 3\times 2\, a_4\,\eta + 5\times 4\times 3\, a_5\,\eta^2 + 6\times 5\times 4\, a_6\,\eta^3 + ...)$$

$$+ (a_2\,\eta^2 + a_3\,\eta^3 + ...)\,(2\times 1\, a_2 + 3\times 2\, a_3\,\eta + 4\times 3\, a_4\,\eta^2 + ...) = 0$$

or, in the form of a table to collect the same powers of η,

$$\left.\begin{array}{ccccccccc} 12\, a_3 & + & 48\, a_4\,\eta & + & 120\, a_5\,\eta^2 & + & 240\, a_6\,\eta^3 & + & .... \\ & & & & 2\, a_2^2\,\eta^2 & + & 2\, a_3\, a_2\,\eta^3 & + & .... \\ & & & & & & 6\, a_3\, a_2\,\eta^3 & + & .... \\ & & & & & & & & .... \end{array}\right\} = 0$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$a_3 = 0 \qquad a_4 = 0 \qquad 120\, a_5 + 2a_2^2 = 0 \qquad 240\, a_6 + 8\, a_2 a_3 = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$a_5 = -\frac{a_2^2}{60} \qquad\qquad a_6 = 0$$

The coefficients of higher order, a_7, a_8,..., are determined in the same manner. The resulting series can be written as

$$f = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \frac{\alpha^{n+1} C_n}{(3n+2)!} \eta^{3n+2}$$

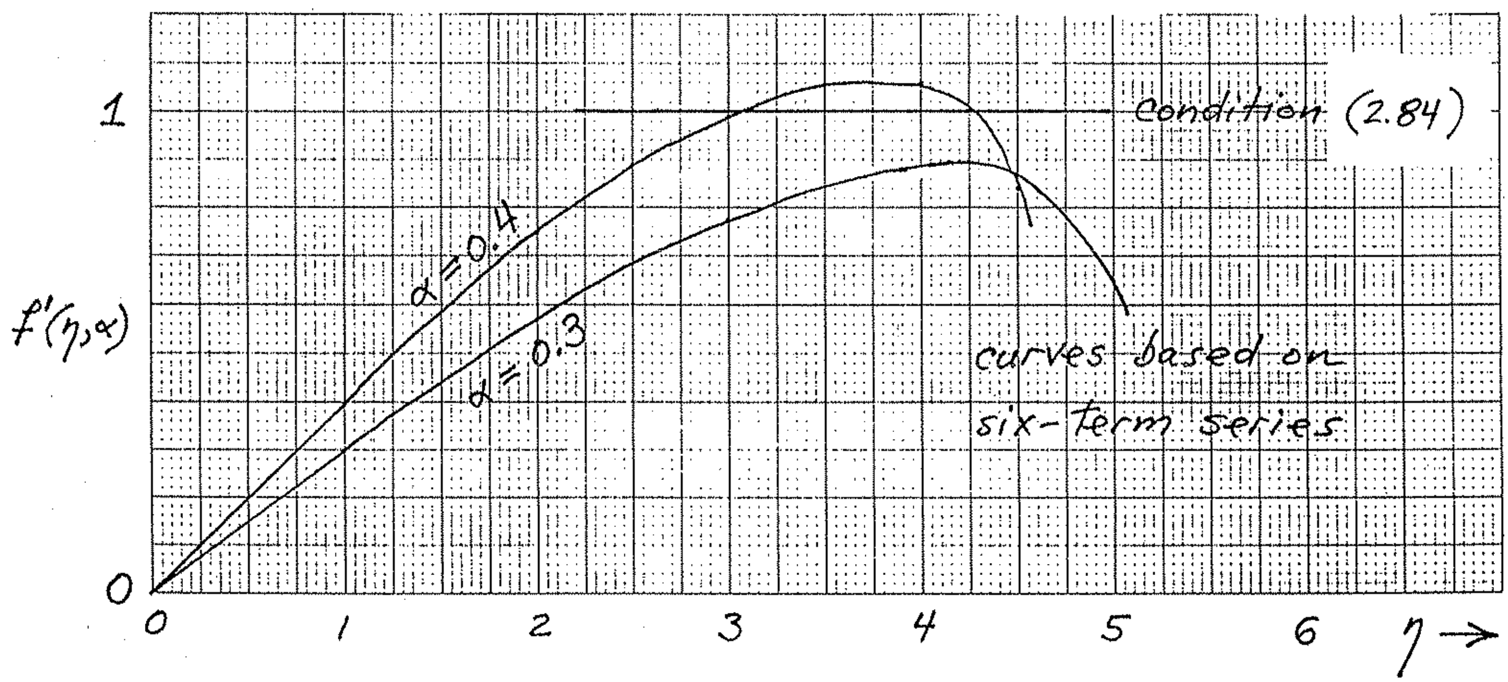
where $\alpha = f''(0)$ is the unknown curvature at the wall, and [13]

$$C_0 = 1 \qquad C_3 = 375$$

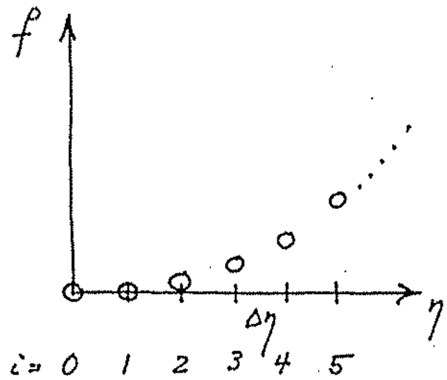
$$C_1 = 1 \qquad C_4 = 27{,}897$$

$$C_2 = 11 \qquad C_5 = 3{,}817{,}137$$

The unknown α follows from the remaining condition, eq. (2.84). Unfortunately, the six-term Blasius series listed above is acceptable only when η is small, say, $\eta < 2$ as illustrated in the sketch below. Nevertheless, the sketch of $f'(\alpha,\eta)$ shows that [$f' = 1$ at large η] is achieved when α is a number between 0.3 and 0.4. The correct value ($\alpha = 0.332$) is found by matching the small-η series with another expansion valid in the domain $\eta > O(1)$: this procedure is outlined in Ref. [13].



Problem 2.3. We divide the η space into slices of thickness $\Delta\eta$. At any node i, the Blasius equation (2.82) requires



$$2f_i''' + f_i f_i'' = 0$$

where $$f_i'' = \frac{f_{i+1} + f_{i-1} - 2f_i}{(\Delta\eta)^2}$$

$$f_i''' = \frac{f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}}{(\Delta\eta)^3}$$

Thus, the Blasius equation allows us to calculate f_{i+2} based on the values of f at the preceding three nodes,

$$f_{i+2} = 3f_{i+1} - 3f_i + f_{i-1} - \frac{\Delta\eta}{2} f_i (f_{i+1} + f_{i-1} - 2f_i)$$

The values at the first three nodes are

$f_0 = 0$, cf. condition (2.83)

$f_1 = 0$, so that $f_0' = \dfrac{f_1 - f_0}{\Delta\eta} = 0$; condition (2.83)

$f_2 = (\Delta\eta)^2\alpha$, where $\alpha = f_0''$ is the unknown curvature at the wall, which must be determined from condition (2.84)

$$f'(\infty) = 1$$

As indicated in the second part of the problem statement, it is sufficient to shoot once. Taking $\alpha = 1$, we find that the slope f' at large η's ($\eta = 10$) approaches the value

$$f'(10) = a \cong 2.088,$$

Therefore, the correct guess for the initial curvature is

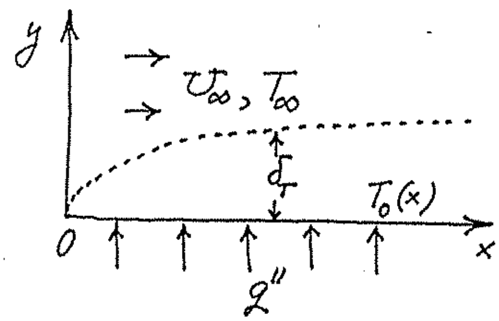
$$\alpha = 2.088^{-3/2} = 0.3314$$

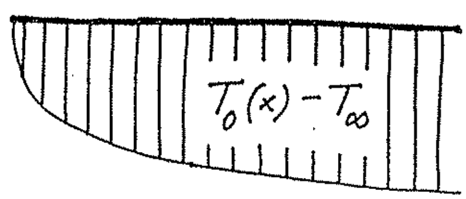
when the integration step is $\Delta\eta = 0.0125$. This value agrees very well with Blasius' well known number

$$f''(0) = 0.332$$

step size $\Delta\eta$	f'(10)
0.2	2.12113563
0.1	2.1019154
0.05	2.09333778
0.025	2.08946109
0.0125	2.08886862

Problem 2.4. The special feature of the heat transfer configuration in this problem is that the wall temperature T_0 must increase in the x-direction, in order for the constant flux q" to overcome the growing thermal boundary layer thickness, $\delta_T(x)$.





We are interested in the local Nusselt number

$$Nu = \frac{q''}{T_0(x) - T_\infty} \frac{x}{k},$$

hence, in $T_0 - T_\infty$ or, from the graph, in $\delta_T(x)$. The energy integral for T_∞ = constant is

$$\frac{d}{dx} \int_0^Y u(T_\infty - T)\, dy = \alpha \left(\frac{\partial T}{\partial y}\right)_{y=0} \qquad (2.53)$$

subject to

$$q'' = -k \left(\frac{\partial T}{\partial y}\right)_{y=0} = \text{constant}$$

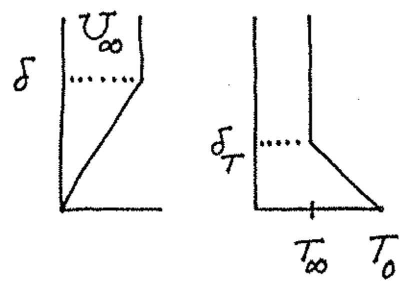
Assuming the simplest velocity and temperature profiles,

$$u = U_\infty \frac{y}{\delta} \qquad \text{and} \qquad \frac{T_0 - T}{T_0 - T_\infty} = \frac{y}{\delta_T},$$

we distinguish the following two limits:

$\delta_T < \delta$ (or $Pr > 1$). The energy equation becomes

$$\frac{d}{dx}\int_0^{\delta_T} U_\infty \frac{y}{\delta}(T_\infty - T_0)\left(1 - \frac{y}{\delta_T}\right) dy = \alpha \frac{T_\infty - T_0}{\delta_T}$$



where $$q'' = -k \frac{T_\infty - T_0}{\delta_T}, \text{ const.}$$

and, from Table 2.1,

$$\delta = 3.46\, x\, Re_x^{-1/2}$$

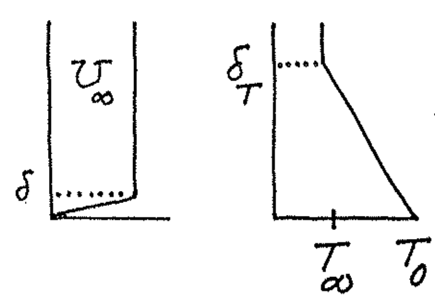
Integrating the energy equation, and noting that its right-hand-side is a constant, we obtain

$$\frac{\delta_T}{x} = 2.75\, Pr^{-1/3}\, Re_x^{-1/2}$$

In conclusion,

$$Nu = \frac{q''}{T_0 - T_\infty}\frac{x}{k} = \frac{x}{\delta_T} = 0.364\, Pr^{1/3}\, Re_x^{1/2} \qquad \text{(see Table 2.1)}$$

$\delta_T > \delta$ (or $Pr < 1$). In the extreme we have $\delta/\delta_T \to 0$, so that u may be taken as equal to U_∞ in the layer of thickness δ_T. The energy equation becomes



$$\frac{d}{dx}\int_0^{\delta_T} U_\infty (T_\infty - T_0)\left(1 - \frac{y}{\delta_T}\right) dy = \alpha \frac{T_\infty - T_0}{\delta_T}$$

where the right-hand-side is again constant. The solution is

$$\delta_T = \left(\frac{2\alpha x}{U_\infty}\right)^{1/2},$$

In other words,

$$Nu = \frac{x}{\delta_T} = 0.707\, Pr^{1/2}\, Re_x^{1/2}$$

Note that results for Pr << 1 are not listed in Table 2.1.

Problem 2.5. Since Pr > 1, the local Nusselt number given by Table 2.1 is

$$Nu = 0.332\, Pr^{1/3}\, Re_x^{1/2}$$

The Reynolds number follows from $C_{f,x} = 0.0066$ and Table 2.1,

$$C_{f,x} = 0.664\, Re_x^{-1/2} = 0.0066,$$

namely,

$Re_x = 1.01 \times 10^4$ (note that the boundary layer is entirely laminar, marginally; see Table 6.1).

In conclusion,

$$Nu = (0.332)\,(7^{1/3})\,(1.01 \times 10^4)^{1/2} = 63.9$$

Problem 2.6. The analysis that follows is a generalization of the analysis presented in the text, eqs. (2.93)-(2.107). The difference is that the boundary condition (2.95) is replaced by

$$\theta = J\frac{\partial \theta}{\partial \eta} \quad \text{at } \eta = 0 \qquad (1)$$

where

$$J = \frac{k}{k_w}\left(\frac{U_\infty t^2}{\nu x}\right)^{1/2} \qquad (2)$$

The J parameter is a constant if t(x) varies as

$$t = C\, x^{1/2} \qquad (3)$$

The C constant is proportional to J,

$$C = J\,\frac{k_w}{k}\left(\frac{\nu}{U_\infty}\right)^{1/2} \tag{4}$$

The analysis up to eq. (2.97) is still valid. The new boundary condition (1) replaces eq. (2.98) with

$$\theta(\eta) - \theta(0) = \theta'(0)\int_0^\eta \exp[\,....\,]\,d\gamma \tag{5}$$

in which, cf. eq. (1),

$$\theta(0) = J\,\theta'(0) \tag{6}$$

If we set $\eta \to \infty$ and $\theta(\infty) = 1$ in eq. (5) we obtain

$$1 = \theta'(0)\left\{J + \int_0^\infty \exp[\,....\,]\,d\eta\right\} \tag{7}$$

The integral that appears above has the value, cf. eqs. (2.99) and (2.102),

$$\int_0^\infty \exp[\,....\,]\,d\eta = \frac{1}{0.332\,Pr^{1/3}}, \qquad (Pr > 0.5) \tag{8}$$

The overall heat transfer rate through the wall of length L is, cf. eq. (2.101)

$$q' = q''_{0-L}\,L = k\,(T_\infty - T_0)\,2\theta'(0)\,Re_L^{1/2} \tag{9}$$

The J = 0 limit of this result is obtained by integrating the local heat flux calculated with eq. (2.103),

$$q'\,(J = 0) = k\,(T_\infty - T_0)\,0.664\,Pr^{1/3}\,Re_L^{1/2} \tag{10}$$

To see the effect of the solid coating (J), we estimate the ratio

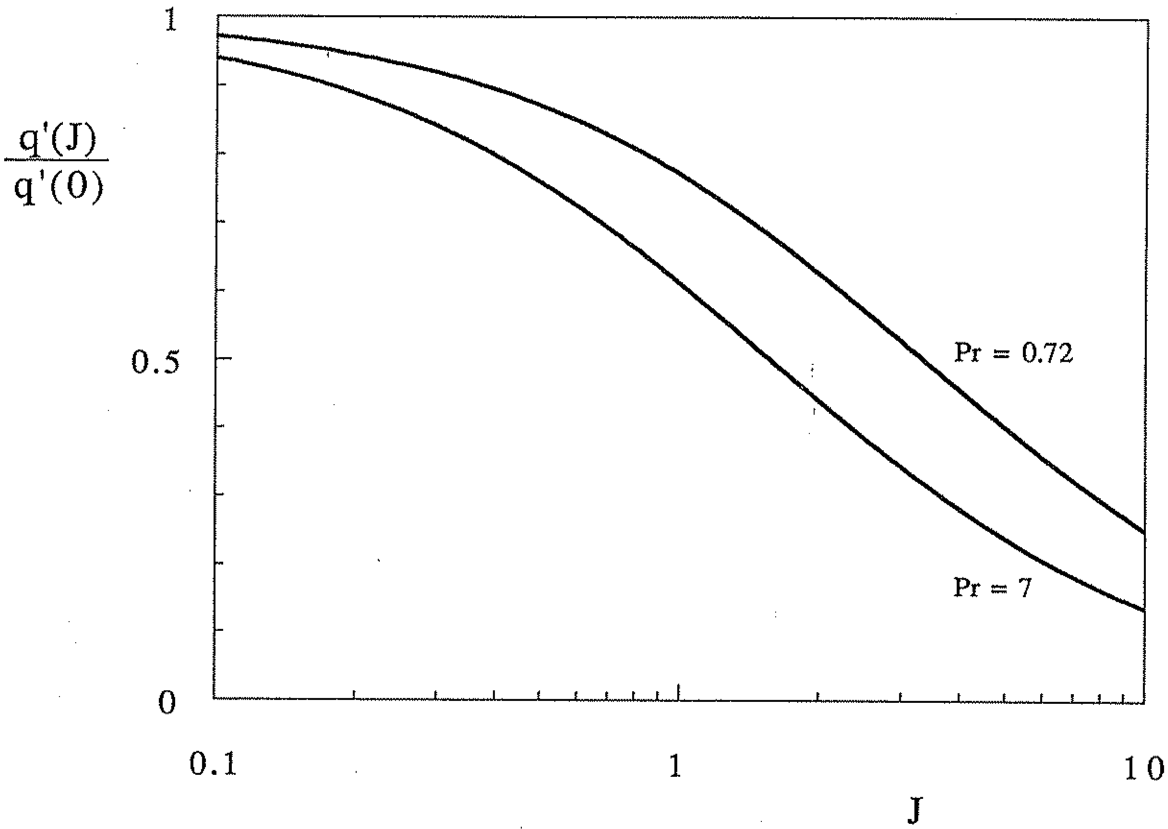
$$\frac{q'(J)}{q'(J=0)} = \frac{\theta'(0)}{0.332\,Pr^{1/3}} \tag{11}$$

$$= \frac{1}{0.332\,Pr^{1/3}\left\{J + \int_0^\infty \exp[\,....\,]\,d\eta\right\}} \tag{12}$$

If we accept the high-Pr limit (8) this ratio becomes

$$\frac{q'(J)}{q'(J=0)} = \frac{1}{0.332\,Pr^{1/3}\,J + 1} \tag{13}$$

which is shown in the attached graph. The coating reduces the heat transfer rate when J exceeds the order of 1.



Problem 2.7. The problem statement for the thermal boundary layer over a constant-q" flat plate is

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{1}$$

$$k \frac{\partial T}{\partial y} = -q'' \quad \text{at} \quad y = 0 \tag{2}$$

$$T \to T_\infty \quad \text{as} \quad y \to \infty \tag{3}$$

The flow solution is known:

$$u = U_\infty f', \tag{4}$$

$$v = \frac{1}{2}\left(\frac{\alpha U_\infty}{x}\right)^{1/2} (\eta f' - f) \tag{5}$$

$$\eta = y \left(\frac{U_\infty}{\nu x}\right)^{1/2} \tag{6}$$

To construct a similarity solution to problem (1)-(3), we must choose the dimensionless temperature θ in such a way that the transformed eqs. (1)-(3) depend solely on θ, η, $f(\eta)$ and Pr. The beginning of writing θ is easy,

$$\theta(\eta, Pr) = \frac{T - T_\infty}{\Delta T_{scale}} \tag{7}$$

The temperature difference scale ΔT_{scale} must be chosen such that the wall boundary condition (2) reduces to a statement involving only the similarity variables. We rewrite eq. (2) in an order of magnitude sense,

$$k \frac{\partial T}{\partial y} \sim k \frac{\Delta T_{scale}}{(\nu x/U_\infty)^{1/2}} \sim q'' \tag{8}$$

and draw the key conclusion that

$$\Delta T_{scale} = \frac{q''}{k} \left(\frac{\nu x}{U_\infty}\right)^{1/2} \tag{9}$$

Combining eqs. (7) and (9), we find the needed transformation from T to θ:

$$T = T_\infty + \theta \cdot \left(\frac{\nu x}{U_\infty}\right)^{1/2} \cdot \frac{q''}{k} \tag{10}$$

Beyond this point, the transformation of eqs. (1)-(3) is routine. The partial derivatives of eq. (1) are

$$\frac{\partial T}{\partial y} = \frac{q''}{k} \theta' \tag{11}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{q''}{k} \theta'' \left(\frac{U_\infty}{\nu x}\right)^{1/2} \tag{12}$$

$$\frac{\partial T}{\partial x} = \frac{q''}{2k} \left(\frac{\nu}{U_\infty x}\right)^{1/2} (\theta - \eta\, \theta') \tag{13}$$

Equations (1)-(3) become, in order,

$$\theta'' + \frac{Pr}{2} (f\, \theta' - f'\, \theta) = 0 \tag{14}$$

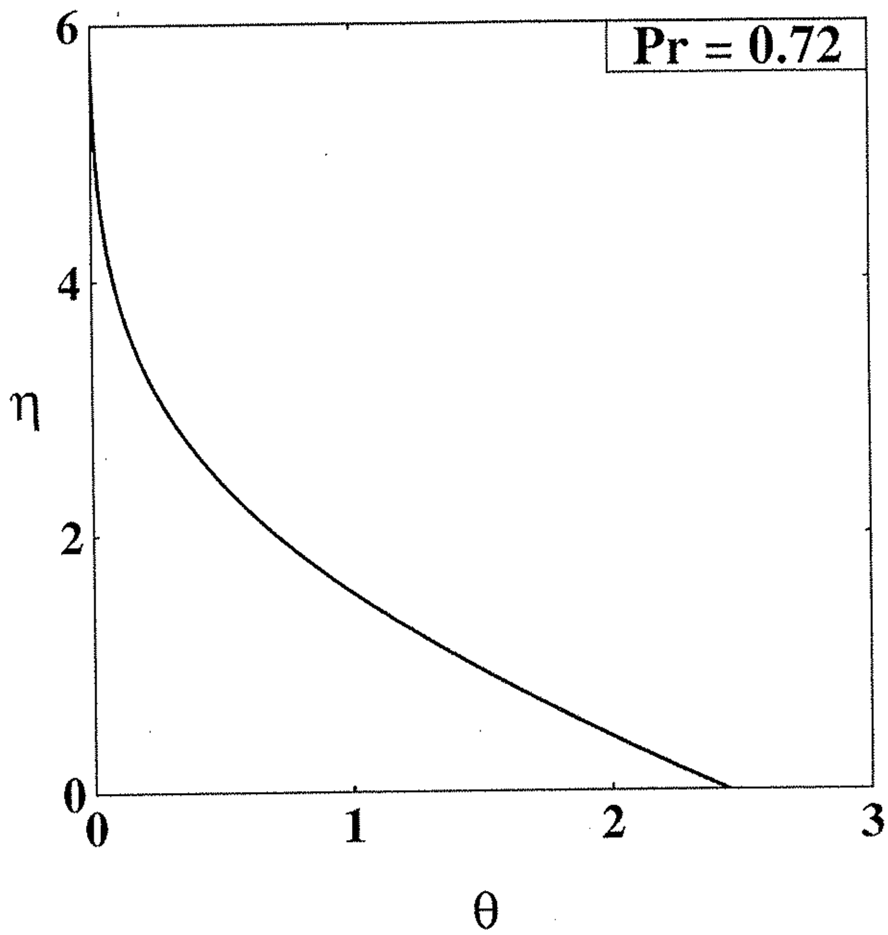
$$\theta''(0) = -1 \tag{15}$$

$$\theta(\infty) = 0 \tag{16}$$

The problem represented by eqs. (14)-(16) was solved numerically for $\theta(\eta, Pr)$. The similarity temperature profile for Pr = 0.72 is shown in the attached figure.

The key heat transfer result is the local Nusselt number

$$Nu = \frac{q''}{T_0(x) - T_\infty} \cdot \frac{x}{k} \tag{17}$$



which is proportional to the inverse of the local temperature difference, cf. eq. (10),

$$Nu = \frac{1}{\theta(0)} Re_x^{1/2} \tag{18}$$

For Pr > 0.5 we expect a scaling of type $Nu \sim Pr^{1/3} Re_x^{1/2}$, which would mean that $\theta(0)$ must vary as $Pr^{-1/3}$. Therefore we solve eqs. (14)-(16) for several large Pr values, and calculate the product $\theta(0)\, Pr^{1/3}$, to find the constant C in the proportionality

$$\theta(0) = C\, Pr^{-1/3} \tag{19}$$

The result is C = 1/0.463 if Pr > 10 [36].

For liquid metals, Pr << 0.5, we expect $Nu \sim Pr^{1/2} Re_x^{1/2}$, which means

$$\theta(0) = K\, Pr^{-1/2} \tag{20}$$

We solve eqs. (14)-(16) for two small Prandtl numbers, and obtain $K \to 1/0.886$ as $Pr \to 0$ [38].

In conclusion, the local Nusselt number asymptotes for laminar boundary layer flow over a constant-q" flat plate are

$$Nu = 0.463\, Pr^{1/3}\, Re_x^{1/2} \qquad (Pr > 0.5)$$

$$Nu = 0.886\, Pr^{1/2}\, Re_x^{1/2} \qquad (Pr << 0.5)$$

Problem 2.8. a) We recognize, in order, the L-averaged shear stress, the total tangential force experienced by the wall, and the mechanical power spent on dragging the wall through the fluid:

$$\bar{\tau} = 0.664\, \rho\, U^2 \left(\frac{UL}{\nu}\right)^{-1/2}$$

$$F' = \bar{\tau}\, L$$

$$\mathbf{P} = F'U = 0.664\, \rho\, U^3 L \left(\frac{\nu}{UL}\right)^{1/2}$$

$$= 0.664\, \rho\, \nu^{1/2}\, U^{5/2}\, L^{1/2}$$

If $(\ )_c$ and $(\ )_h$ represent the "cold" and "hot" flow conditions, the dissipated power changes according to the ratio:

$$\frac{\mathbf{P}_h}{\mathbf{P}_c} = \frac{\rho_h}{\rho_c}\left(\frac{\nu_h}{\nu_c}\right)^{1/2} = \left(\frac{\rho_h\, \mu_h}{\rho_c\, \mu_c}\right)^{1/2}$$

In the cold case (no heating), $T_c = 10°C$ and

$$\rho_c \cong 1\, \frac{g}{cm^3} \qquad \mu_c = 0.013\, \frac{g}{cm \cdot s}$$

In the hot case, the heat transfer from the 90°C wall to the 10°C water takes place across a boundary layer with the film temperature $T_h = 50°C$. The properties corresponding to T_h are

$$\rho_h \cong 1\, \frac{g}{cm^3} \qquad \mu_h = 0.00548\, \frac{g}{cm \cdot s}$$

The power ratio

$$\frac{\mathbf{P}_h}{\mathbf{P}_c} = \left(\frac{1}{1} \cdot \frac{0.00548}{0.013}\right)^{1/2} = 0.65$$

shows that the heating of the boundary layer decreases the dissipated power (as well as F' and $\bar{\tau}$) by 35 percent.

b) By retaining the observation that the water density is practically constant, we find that the power that has been saved by heating the boundary layer is

$$\mathbf{P_c} - \mathbf{P_h} = 0.664\ \rho\ U^{5/2}\ L^{1/2}\ \nu_c \left[1 - \left(\frac{\nu_h}{\nu_c}\right)^{1/2}\right] \qquad (1)$$

The electric power spent on heating the wall is given sequentially by

$$\frac{\bar{h}L}{k} = \frac{\overline{q''}}{\Delta T}\ \frac{L}{k} = 0.664\ Pr^{1/3}\ Re_L^{1/2}$$
$$\overline{q''}L = 0.664\ k_h\ \Delta T\ Pr_h^{1/3} \left(\frac{UL}{\nu_h}\right)^{1/2} \qquad (2)$$

in which the properties (k, Pr, ν) are evaluated at the film temperature $T_h = 50°C$,

$$k_h = 0.64\ \frac{W}{m \cdot K}, \qquad Pr_h = 3.57$$

By dividing equations (1) and (2), we obtain (after some manipulation) a dimensionless measure of how effectively the heating of the wall has been converted into saved mechanical power:

$$\frac{\mathbf{P_c} - \mathbf{P_h}}{\overline{q''}L} = \frac{U^2}{c_h\ \Delta T}\ Pr_h^{2/3} \left(\frac{\nu_c}{\nu_h}\right)^{1/2} \left[1 - \left(\frac{\nu_h}{\nu_c}\right)^{1/2}\right]$$

In this expression we substitute $c_h = 4.18$ kJ/kg·K, $\Delta T = 90°C - 10°C = 80°C$ and $(\nu_h/\nu_c)^{1/2} = 0.65$, and obtain

$$\frac{\mathbf{P_c} - \mathbf{P_h}}{\overline{q''}L} = \left(\frac{U}{516\ m/s}\right)^2$$

This ratio is appealing (greater than 1) only when U > 516 m/s. The length L, however, must be sufficiently small if the boundary layer is to remain laminar

$$\frac{UL}{\nu_h} < 5 \times 10^5$$

$$L < 5 \times 10^5\ \frac{\nu_h}{U} < 5 \times 10^5\ \frac{\nu_h}{516\ m/s}$$

$$< 5 \times 10^5\ \frac{0.00554\ cm^2/s}{51600\ cm/s}$$

$$< 0.54\ mm$$

In conclusion, when L > 0.54 mm and the flow is laminar the electric power used to heat the wall is greater than the fluid-drag power saved.

Problem 2.9. The film temperature in this configuration is (40°C + 20°C) = 30°C. The calculation of the total force F proceeds in this order:

$$Re_x = \frac{U_\infty x}{\nu} \; 0.5 \frac{m}{s} \; 10m \frac{s}{0.16\ cm^2} = \; 3.1 \times 10^5$$
(laminar, just barely)

$$\bar{C}_{f,x} = 1.328\ Re_x^{-1/2} = 0.00238$$

$$A = 10m\ 20m = 200\ m^2 \qquad \text{(roof area)}$$

$$F = A\ \bar{\tau}_{w,x} = A\ \bar{C}_{f,x}\ \frac{1}{2}\ \rho\ U_\infty^2$$

$$= 200\ m^2\ 0.00238\ \frac{1}{2}\ 1.165\ \frac{kg}{m^3}\left(0.5\ \frac{m}{s}\right)^2$$

$$= 0.069\ N \cong 0.016\ lbf$$

In order to calculate the total heat transfer rate q, we must first evaluate the average heat flux:

$$\overline{Nu}_x = 0.664\ Pr^{1/3}\ Re_x^{1/2}$$
$$= 0.664\ (0.72)^{1/3}\left(3.1 \times 10^5\right)^{1/2} = 331.4$$

$$\bar{h}_x = \overline{Nu}_x \frac{k}{x} = 331.4 \times 0.026 \frac{W}{m \cdot K} \frac{1}{10m}$$
$$= 0.862 \frac{W}{m^2\ K}$$

$$\bar{q}_x'' = \bar{h}_x\left(T_w - T_\infty\right) = 0.862 \frac{W}{m^2\ K}\ (40 - 20)\ K$$
$$= 17.2 \frac{W}{m^2}$$
$$q = \bar{q}_x''\ A = 17.2 \frac{W}{m^2}\ 200\ m^2 = 3.4\ kW$$

Problem 2.10. a) The local heat flux varies as

$$q_{w,x}'' = C\ x^n$$

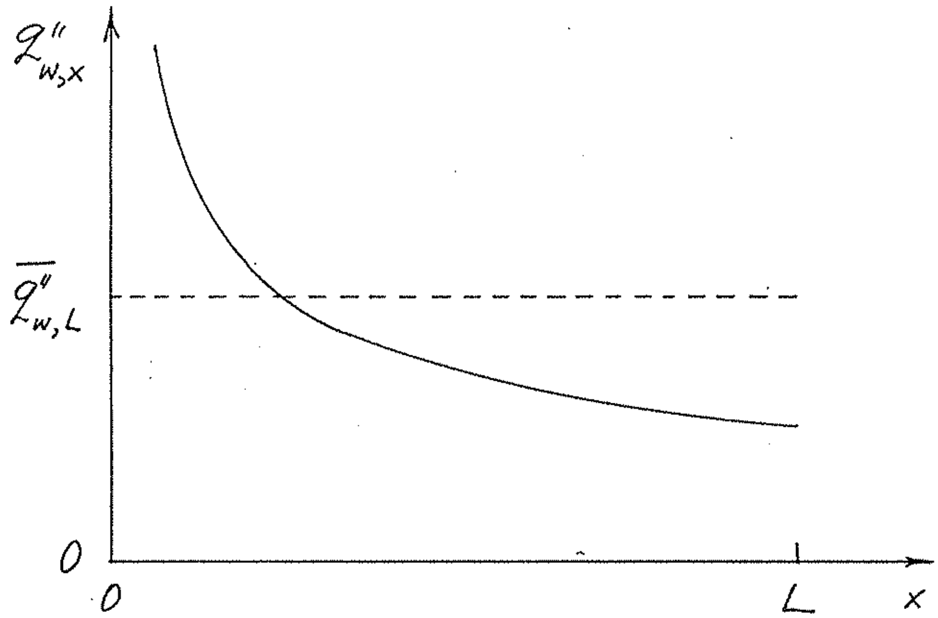
in which n = – 1/2, and C is a constant. The L-averaged heat flux is therefore

$$\overline{q''_{w,L}} = \frac{1}{1+n}\, q''_{w,L} = 2q''_{w,L} = 2\ CL^{-1/2}$$

The position x where the local flux matches the L-averaged value is obtained by writing

$$Cx^{-1/2} = 2\ CL^{-1/2}$$

and this yields x = L/4. The attached figure confirms this finding, because it is drawn to scale.



b) The relationship between the local heat flux in the middle and the L-averaged flux is obtained by first writing

$$q''_{w,L/2} = C\left(\frac{L}{2}\right)^{-1/2}$$

$$\overline{q''_{w,L}} = 2\ CL^{-1/2}$$

Dividing side by side, we conclude that the mid-point heat flux is about 30 percent smaller than the average flux:

$$\frac{q''_{w,L/2}}{\overline{q''_{w,L}}} = \frac{C\ 2^{1/2}\ L^{-1/2}}{2\ CL^{-1/2}} = \frac{1}{2^{1/2}} = 0.707$$

Problem 2.11. When the boundary layer is much thinner than the pipe diameter D, the wall friction force F and the total heat transfer rate q can be estimated based on the formulas for the flat wall:

$$F = \pi DL\, \tau_{w,L} = \pi DL\, \overline{C}_{f,L} \frac{1}{2} \rho U_\infty^2$$

$$= \pi DL\, 1.328\, Re_L^{-1/2} \frac{1}{2} \rho U_\infty^2 \tag{1}$$

$$= 2.086\, \rho U_\infty^2\, DL\, Re_L^{-1/2}$$

$$q = \pi DL\, \bar{q}''_{w,L} = \pi DL\, \Delta T\, \bar{h}_L$$

$$= \pi DL\, \Delta T \frac{k}{L}\, 0.664\, Pr^{1/3} Re_L^{1/2} \tag{2}$$

$$= 2.086\, k\, \Delta T\, D\, Pr^{1/3} Re_L^{1/2}$$

Dividing (1) and (2) side by side we obtain

$$\frac{q}{F} = Pr^{-2/3} \frac{c_P\, \Delta T}{U_\infty}$$

Problem 2.12. We evaluate the properties of water at the film temperature (20°C + 50°C)/2 = 35°C, and calculate in order:

$$Re_x = \frac{U_\infty x}{\nu} = 5 \frac{cm}{s}\, 100\, cm \frac{s}{0.00725\, cm^2} = 6.9 \times 10^4 \text{ (laminar)}$$

$$Nu_x = 0.332\, Pr^{1/3} Re_x^{1/2}$$

$$= 0.332\, (4.87)^{1/3} (6.9 \times 10^4)^{1/2} = 148$$

$$Nu_x = \frac{h_x x}{k}$$

$$h_x = 148 \frac{k}{x} = 148 \times 0.62 \frac{W}{m \cdot K} \frac{1}{1m} = 91.6 \frac{W}{m^2 K}$$

$$\bar{h}_x = 2\, h_x = 183.3 \frac{W}{m^2\, K}$$

$$\bar{q}''_{w,x} = \bar{h}_x \Delta T = 183.3 \frac{W}{m^2 K}\, 30\, K = 5500 \frac{W}{m^2}$$

Since the total area of the wall of length x = 1m is

$$A = 4 \times 20\, cm \times 1m = 0.8\, m^2$$

the total heat transfer rate is

$$q = \bar{q}''_{w,x}\, A = 4400\, W$$

The velocity boundary layer thickness at the same location (x)

$$\delta = 4.92 \times Re_x^{-1/2} =$$
$$= 4.92 \times 1\text{m} \left(6.9 \times 10^4\right)^{-1/2} = 1.9 \text{ cm}$$

is much smaller than the 20 cm-side of the duct cross-section. In conclusion, the use of the flat wall boundary layer results is justified.

Problem 2.13. The narrow-strip configuration is a case of wall with non-uniform heat flux, in which

$$q_w''(\xi) = \begin{cases} 0, & 0 < \xi < x_1 \\ q_w'/\Delta x, & x_1 < \xi < x_1 + \Delta x \\ 0, & \xi > x_1 + \Delta x \end{cases}$$

Substituting this expression in eq. (2.122) we obtain, in order:

$$T_w(x) - T_\infty = \frac{0.623}{k\,Pr^{1/3}\,Re_x^{1/2}} \left(\int_0^{x_1} + \int_{x_1}^{x_1+\Delta x} + \int_{x_1+\Delta x}^{x} \right)$$
$$= \frac{0.623}{k\,Pr^{1/3}\,Re_x^{1/2}} \int_{x_1}^{x_1+\Delta x} \left[1 - \left(\frac{\xi}{x}\right)^{3/4}\right]^{-2/3} \frac{q_w'}{\Delta x}\, d\xi$$
$$= \frac{0.623}{k\,Pr^{1/3}\,Re_x^{1/2}} \left[1 - \left(\frac{x_1}{x}\right)^{3/4}\right]^{-2/3} q_w'$$

Problem 2.14. The flux q" covers only the front half of the plate length L. We use eq. (2.122)

$$T_w(L) - T_\infty = \frac{0.623}{k\,Pr^{1/3}\,Re_x^{1/2}} \int_{\xi=0}^{L/2} \left[1 - \left(\frac{\xi}{L}\right)^{3/4}\right]^{-2/3} q''\, d\xi \qquad (1)$$
$$= \frac{q'' L}{k\,Pr^{1/3}\,Re_L^{1/2}}\, 0.623 \underbrace{\int_0^{1/2} \left(1 - m^{3/4}\right)^{-2/3} dm}_{C = 0.682}$$

The alternative calculation is based on eq. (2.121), in which we assume that the uniform flux q"/2 is spread over the entire length L:

$$\left[T_w(L) - T_\infty\right]_{approx.} = \frac{\frac{q''}{2} L}{0.453\, k\, Pr^{1/3}\, Re_x^{1/2}} \qquad (2)$$

The relative goodness of this approximation can be seen by dividing eq. (2) by eq. (1):

$$\frac{[T_w(L) - T_\infty]_{approx.}}{T_w(L) - T_\infty} = \frac{\dfrac{1}{2 \times 0.453}}{0.623 \times 0.682} = 2.60$$

In conclusion, the approximation is not very good.

Problem 2.15. The relevant properties of water at 20°C and atmospheric pressure are

$$\rho = 0.997 \frac{g}{cm^3} \qquad Pr = 7.07$$

$$\nu = 0.01 \frac{cm^2}{s} \qquad k = 0.59 \frac{W}{mK}$$

The L-averaged shear stress $\tau_{w,L}$ can be calculated in the following order,

$$\begin{aligned} \tau_{w,L} &= \left(\frac{1}{2} \rho U_\infty^2\right) \bar{C}_{f,L} \\ &= \left(\frac{1}{2} \rho U_\infty^2\right) 1.328 \, Re_L^{-1/2} \end{aligned} \qquad (a)$$

where the Reynolds number has a value that falls in the laminar range:

$$Re_L = \frac{U_\infty L}{\nu} = \frac{0.5 \, m}{s} \frac{1 \, cm}{0.01 \, cm^2/s} = 5000 \qquad (b)$$

Combining (a) and (b) we obtain

$$\begin{aligned} \tau_{w,L} &= \frac{1}{2} 0.997 \frac{g}{cm^3} (0.5)^2 \frac{m^2}{s^2} 1.328 \, (5000)^{-1/2} \\ &= 0.00234 \frac{g \cdot m^2}{cm^3 \, s^2} = 2340 \frac{N}{m^2} \end{aligned} \qquad (c)$$

The L-averaged heat flux can be calculated based on a similar sequence:

$$\bar{q}''_{w,L} = \frac{k \, \Delta T}{L} \overline{Nu}_L$$

$$= \frac{k \, \Delta T}{L} 0.664 \, Pr^{1/3} \, Re_L^{1/2}$$

$$= 0.59 \frac{W}{m \, K} \frac{1 \, K}{1 \, cm} 0.664 \, (7.07)^{1/3} \, (5000)^{1/2}$$

$$= 53.17 \frac{W}{m \cdot cm} = 5317 \frac{W}{m^2} \qquad (d)$$
