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Chapter 0

0.1 Homogeneous Linear Equations

1. You should be able to write out the solution without going through any algebra

$$\phi(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$$

3.(a) Treat this as a constant-coefficients equation. The characteristic equation is $m^2 = 0$, with double root $m = 0$. Therefore the solution of the differential equation is $u(t) = c_1 + c_2 t$.

(b) Because there is no u or du/dt term, you can integrate directly, twice: $du/dt = c_2$, $u = c_2 t + c_1$.

5. Do the indicated differentiation.

$$\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} - \frac{\lambda^2}{r^2}W = 0.$$

This is a Cauchy-Euler equation. Guess $w = r^m$ so the characteristic equation is $m(m-1) + m - \lambda^2 = 0$ or $m^2 - \lambda^2 = 0$, with solutions $m = \lambda$, $m = -\lambda$. The general solution is

$$w(r) = c_1 r^{\lambda} + c_2 r^{-\lambda}.$$

7. This differential equation is best solved by integrating (since there is no term in v).

$$(h + kx)\frac{dv}{dx} = c_1$$

$$\frac{dv}{dx} = \frac{c_1}{h + kx}$$

$$v = \frac{c_1}{k}\ln(h + kx) + c_2$$

9. Solve by integrating, since there is no term in u:

$$x^3\frac{du}{dx} = c_1; \quad \frac{du}{dx} = c_1 x^{-3};$$

$$u = -\frac{c_1}{2}x^{-2} + c_2.$$

11. Solve by integrating, since there is no term in u.

$$r\frac{du}{dr} = c_1; \quad \frac{du}{dr} = \frac{c_1}{r}; \quad u = c_1\ln(r) + c_2.$$

13. The characteristic polynomial is $(m^4 - \lambda^4) = (m^2 + \lambda^2)(m^2 - \lambda^2)$ with roots $m = \pm\lambda$, $\pm i\lambda$. The general solution of the differential equation is

$$u(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + c_3\cos(\lambda x) + c_4\sin(\lambda x).$$

15. Guess $u = e^{mx}$. Then the characteristic equation is $m^4 + 2\lambda^2 m^2 + \lambda^4 = 0$. This is a biquadratic polynomial equation: $(m^2+\lambda^2)^2 = 0$. The roots are $m = \pm i\lambda$, both having multiplicity 2. The general solution is

$$u = (c_1 + c_2 x)\cos(\lambda x) + (c_3 + c_4 x)\sin(\lambda x).$$

17. Assume $u_2 = t^b \cdot v$. Then

$$u_2' = bt^{b-1}v + t^b v'; \quad u_2'' = b(b-1)t^{b-2}v + 2bt^{b-1}v' + t^b v''.$$

Substituting gives

$$b(b-1)t^b v + 2bt^{b+1}v' + t^{b+2}v'' + (1-2b)(bt^b v + t^{b+1}v') + b^2 t^b v = 0.$$

Sort terms by derivatives

$$t^{b+2}v'' + (2bt^{b+1} + (1-2b)t^{b+1})v' + \left(b(b-1) + b(1-2b) + b^2\right)t^b v = 0.$$

The coefficient of v *has to be* 0. If not, there was an error. The differential equation reduces to $tv'' + v' = 0$, which is a Cauchy-Euler equation. The characteristic equation is $m^2 = 0$, with double root $m = 0$. Thus $v(t) = c_1 + c_2 \ln(t)$ and the general solution of the original equation is $u(t) = (c_1 + c_2 \ln(t))\, t^b$. Choosing $c_1 = 0$, $c_2 = 1$ gives a second solution.

19. Replace each R by u/ρ.

$$\frac{d}{d\rho}\left(\rho^2 \frac{d}{d\rho}\left(\frac{u}{\rho}\right)\right) + \lambda^2\rho^2\frac{u}{\rho} = 0.$$

Differentiate $\dfrac{d}{d\rho}\left(\dfrac{u}{\rho}\right) = \dfrac{\rho u' - u}{\rho^2}$ substitute and clean up

$$\frac{d}{d\rho}\left(\rho\frac{du}{d\rho} - u\right) + \lambda^2\rho u = 0.$$

Differentiate

$$\rho\frac{d^2u}{d\rho^2} + \frac{du}{d\rho} - \frac{du}{d\rho} + \lambda^2\rho u = 0.$$

Simplify

$$\frac{d^2u}{d\rho^2} + \lambda^2 u = 0.$$

The equation comes up in physical problems of waves in 3d.

21. Use the chain rule:

$$\frac{du}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot\frac{1}{t}$$

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{dv}{dx}\cdot\frac{1}{t}\right) = \frac{dv}{dx}\left(-\frac{1}{t^2}\right) + \frac{d}{dx}\left(\frac{dv}{dx}\right)\cdot\frac{1}{t^2}.$$

Now substitute:

$$\frac{d^2v}{dx^2} - \frac{dv}{dx} + k\frac{dv}{dx} + pv = 0.$$

This is a constant-coefficient equation with characteristic equation $m^2 + (k-1)m + p = 0$.

23. The characteristic equation is $m^2 + 2\alpha m + \sigma^2 = 0$, with roots $m = -\alpha \pm \sqrt{\alpha^2 - \sigma^2}$. We know that the system is underdamped, so $\sigma^2 > \alpha^2$ and we set $\sqrt{\alpha^2 - \sigma^2} = i\beta$. The solution of the differential equaiton is

$$y(t) = e^{-\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right).$$

The initial conditions determine c_1, c_2.

$$y(0) = -.001h : c_1 = -.001h$$

$$y'(0) = 0 : -\alpha c_1 + \beta c_2 = 0$$

The complete solution is

$$y(t) = -.001h\left(\cos(\beta t) + \frac{\alpha}{\beta}\sin(\beta t)\right).$$

25. Damping is critical when

$$\left(\frac{2\alpha}{2}\right)^2 = \sigma^2 \quad \text{or} \quad \alpha/\sigma = 1.$$

This occurs at $v = 2.62 m/s$.

Chapter 0

0.2 Nonhomogeneous Linear Equations

1. In standard form the equation is $u' + au = aT$. The inhomogeneity is aT, assumed constant. Guess $u_p(t) = A$ and substitute to find $0 + aA = aT$. Hence $u_p = T$, and the general solution is $u(t) = T + c_1 e^{-at}$.

3. Guess $u_p(t) = Ae^{-at}$. Substituting in the differential equation gives $-Aae^{-at} + aAe^{-at} = e^{-at}$. This is impossible, since the left-hand side is 0, because Ae^{-at} is a solution of the homogeneous equation. Now apply the revision rule: guess $u_p(t) = Ate^{-at}$. Substituting in the differential equation gives $A(-ate^{-at} + e^{-at}) + aAte^{-at} = e^{-at}$. The two terms containing t cancel, leaving $Ae^{-at} = e^{-at}$. Hence $A = 1$ and the general solution is $u(t) = te^{-at} + ce^{-at}$, with c arbitrary.

5. The solution to guess is $u_p(t) = A\cos(t) + B\sin(t)$. However, this is (for any A, B) a solution of the homogeneous equation $u'' + u = 0$. Hence, we must apply the revision rule and guess $u_p(t) = t(A\cos(t) + B\sin(t))$. Substituting into the differential equation gives $t(-A\cos(t) - B\sin(t)) + 2(-A\sin(t) + B\cos(t)) + t(A\cos(t) + B\sin(t)) = \cos(t)$. Terms containing t cancel; matching coefficients gives $-2A = 0$, $-2B = 1$. Thus the general solution is $u(t) = -\frac{1}{2}t\sin(t) + c_1\cos(t) + c_2\sin(t)$.

7. This is a deceptive equation, because (1) $\cosh(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$ and (2) the general solution of the homogeneous equation is $c_1e^{-t} + c_2e^{-2t}$. Thus, we need to break up the cosh, solve for two particular solutions, and use the revision rule for one of them.

To find the particular solution of $u'' + 3u' + 2u = \frac{1}{2}e^t$, guess $u_{p1}(t) = Ae^t$ and find $(A + 3A + 2A)e^t = \frac{1}{2}e^t$, so $A = 1/12$. To find the particular solution of $u'' + 3u' + 2u = \frac{1}{2}e^{-t}$, apply the revision rule and guess $u_{p2}(t) = Bte^{-t}$. We find

$$B(te^{-t} - 2e^{-t}) + 3B(-te^{-t} + e^{-t}) + 2Bte^{-t} = \frac{1}{2}e^{-t}.$$

The terms with t drop out, leaving $(-2+3)Be^{-t} = \frac{1}{2}e^{-t}$, so $B = \frac{1}{2}$. Finally, the general solution is

$$u(t) = \frac{1}{12}e^t + \frac{1}{2}te^{-t} + c_1e^{-t} + c_2e^{-2t}.$$

9. Recall that the method of undetermined coefficients is not guaranteed to work if the differential equation has variable coefficients, as this one has. This equation is best solved by algebra and integration:

$$(\rho^2u')' = -\rho^2; \quad \rho^2u' = -\frac{1}{3}\rho^3 + c_1;$$

$$u' = -\frac{1}{3}\rho + c_1\rho^{-2}; \quad u = -\frac{1}{6}\rho^2 - c_1\rho^{-1} + c_2.$$

11. Divide through by M, and let $K/M = b$. Then

$$\frac{d^2h}{dt^2} + b\frac{dh}{dt} = -g.$$

The equation is linear, **non**homogeneous, with constant coefficients. The characteristic equation is $m^2 + bm = 0$ with roots $-b$ and 0. Thus $h_c(t) = c_1 e^{-bt} + c_2$. Because the r.h.s. is constant, **and** constant is a solution of the homogeneous equation, the guessed form for the particular solution is (see p. 16) $h_p(t) = At$ (not just A).

The general solution of the differential equation is

$$h(t) = c_1 e^{-bt} + c_2 - \frac{gt}{b}.$$

The initial conditions require

$$c_1 = -\frac{g}{b^2}, \quad c_2 = h_0 + \frac{g}{b^2},$$

and finally

$$h(t) = h_0 - \frac{gt}{b} + \frac{g}{b^2}\left(1 - e^{-bt}\right).$$

13. Seek a solution in the form $u(t) = v(t)e^{-at}$. Substitute to find

$$v(-ae^{-at}) + v'e^{-at} + ave^{-at} = e^{-at},$$

which leads to $v' = 1$ or $v(t) = t$. Thus $u_p(t) = te^{-at}$.

15. From Eq. (12') and (15) we have

$$\left.\begin{array}{r} v_1' \cos(x) + v_2' \sin(x) = 0 \\ -v_1' \sin(x) + v_2' \cos(x) = \tan(x) \end{array}\right\}$$

with solution

$$v_1'(x) = -\sin^2(x)/\cos(x), \quad v_2'(x) = \sin(x).$$

Change $v_1'(x)$ to $(\cos^2(x) - 1)/\cos(x) = \cos(x) + \sec(x)$ to integrate. Then $v_1(x) = \sin(x) - \ln|\tan(x) + \sec(x)|$, $v_2(x) = -\cos(x)$. This problem could *not* be solved by undetermined coefficients.

17. The system of equations to solve is

$$\left.\begin{array}{r} v_1' + v_2' t = 0 \\ v_2' = -1 \end{array}\right\}$$

with solution $v_2' = -1, v_1' = t$. Hence $v_1 = t^2/2$, $v_2 = -t$ and $u_p(t) = t^2/2 - t^2 = -t^2/2$. The same solution could be found by integration.

19. Note that the differential equation has to be divided by t^2 to be in standard form. The system of equations to solve is

$$\left.\begin{array}{r} v_1' t + v_2'/t = 0 \\ v_1' - v_2'/t^2 = 1/t^2 \end{array}\right\}$$

with solution $v_1' = 1/2t^2$, $v_2' = -1/2$. Hence $v_1 = -1/2t$, $v_2 = -t/2$ and $u_p(t) = -1/2 - 1/2 = -1$.

21. Here we must follow the development of the theorem. Assume $u_p(t) = e^{-at}v(t)$. Then $(e^{-at}v' - ae^{-at}v) + ae^{-at}v = f(t)$ or $e^{-at}v' = f(t)$. Because the equation is first order, there is only one equation for v'. The solution is $v' = e^{at}f(t)$, from which $v(t) = \int_0^t e^{az}f(z)dz$ and $u_p(t) = e^{-at}\int_0^t e^{az}f(z)dz$.

23. To get the multiplier α out of the parentheses, choose $\beta = 1/\alpha$. Then collect multipliers to find $K = R\alpha/\rho c$.

25. The easiest way to solve is by integration. Divide through the equation by $(\beta + T)$:

$$\frac{1}{\beta + T}\frac{dT}{dt} = KI_{max}^2 e^{-2\lambda t}$$

$$\ln(\beta + T) = KI_{max}^2 \frac{e^{-2\lambda t}}{-2\lambda} + c.$$

From the initial condition, find

$$c = \ln\beta + KI_{max}^2/2\lambda.$$

Then $\ln(\beta + T) = \ln\beta + KI_{max}^2(1 - e^{-2\lambda t})/2\lambda$. Now, use each side as an exponent:

$$\beta + T = \beta\exp\left(KI_{max}^2(1 - e^{-2\lambda t})/2\lambda\right).$$

Subtract β from each side to get $T(t)$.

Chapter 0

0.3 Boundary Value Problems

1.a. The solution of the differential equation is $u(x) = c_1 \cos(x) + c_2 \sin(x)$. Applying the boundary conditions gives two equations:

$$u(0) = 0: \qquad c_1 = 0$$

$$u(\pi) = 0: \qquad -c_1 = 0$$

Therefore $c_1 = 0$ and c_2 is arbitrary, giving infinitely many solutions.

b. Solution $u(x) = 1 + c_1 \cos(x) + c_2 \sin(x)$. Boundary conditions:

$$u(0) = 0: \quad 1 + c_1 = 0$$

$$u(1) = 0: \quad 1 + c_1 \cos(1) + c_2 \sin(1) = 0$$

The solution is $c_1 = -1$, $c_2 = \dfrac{-1 + \cos(1)}{\sin(1)}$. Unique solution.

c. Solution $u(x) = c_1 \cos(x) + c_2 \sin(x)$. Boundary conditions:

$$u(0) = 0: \quad c_1 = 0$$

$$u(\pi) = 1: \quad -c_1 = 1$$

These are inconsistent, so no solution is possible.

3. In all cases, the general solution of the differential equation is $u(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$ if $\lambda \neq 0$. The two boundary conditions give two simultaneous equations for c_1 and c_2.

a. $c_1 = 0$; $-\lambda c_1 \sin(\lambda a) + \lambda c_2 \cos(\lambda a) = 0$. Therefore $\cos(\lambda a) = 0$ and $\lambda = \pm\pi/2a$, $\pm 3\pi/2a$, $\pm 5\pi/2a, \cdots$.

b. $\lambda c_2 = 0$; $c_1 \cos(\lambda a) + c_2 \sin(\lambda a) = 0$. Therefore $\cos(\lambda a) = 0$; solutions as in a.

c. $\lambda c_2 = 0$; $-\lambda c_1 \sin(\lambda a) + \lambda c_2 \cos(\lambda a) = 0$. Therefore $\sin(\lambda a) = 0$ and $\lambda = \pm\pi/a$, $\pm 2\pi/a$, $\pm 3\pi/a, \cdots$. For this case $\lambda = 0$ is a possibility as well, with solution $u(x) = c_1$.

5. The two boundary conditions become:

$$u(0) = h: \quad c' + \frac{1}{\mu}\cosh(\mu c) = h.$$

$$u(a) = h: \quad c' + \frac{1}{\mu}\cosh(\mu(c + a)) = h.$$

Deduce that $\cosh(\mu c) = \cosh(\mu(c + a))$. By the symmetry of the cosh function, $c + a = -c$, or $c = -a/2$. Then $c' = h - \frac{1}{\mu}\cosh(\mu a/2)$.