Problem 2.1

[Difficulty: 1]

2.1 For the velocity fields given below, determine:

a. whether the flow field is one-, two-, or three-dimensional, and why.

b. whether the flow is steady or unsteady, and why.

(The quantities a and b are constants.)

(1) $\vec{V} = [(ax+t)e^{by}]\hat{i}$ (2) $\vec{V} = (ax - by)\hat{i}$

(3) $\vec{V} = ax\hat{i} + [e^{bx}]\hat{j}$ (4) $\vec{V} = ax\hat{i} + bx^2\hat{j} + ax\hat{k}$

(5) $\vec{V} = ax\hat{i} + [e^{bt}]\hat{j}$ (6) $\vec{V} = ax\hat{i} + bx^2\hat{j} + ay\hat{k}$

(7) $\vec{V} = ax\hat{i} + [e^{bt}]\hat{j} + ay\hat{k}$ (8) $\vec{V} = ax\hat{i} + [e^{by}]\hat{j} + az\hat{k}$

Given: Velocity fields

Find: Whether flows are 1, 2 or 3D, steady or unsteady.

Solution:

(1)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(2)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} \neq \vec{V}(t)$	Steady
(3)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(4)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(5)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(6)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} \neq \vec{V}(t)$	Steady
(7)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(8)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady

Problem 2.2

[Difficulty: 1]

2.2 For the velocity fields given below, determine:
a. whether the flow field is one-, two-, or three-dimensional, and why.
b. whether the flow is steady or unsteady, and why.
(The quantities a and b are constants.)

(1) $\vec{V} = [ay^2e^{-bt}]\hat{i}$ (2) $\vec{V} = ax^2\hat{i} + bx\hat{j} + c\hat{k}$
(3) $\vec{V} = axy\hat{i} - byt\hat{j}$ (4) $\vec{V} = ax\hat{i} - by\hat{j} + ct\hat{k}$
(5) $\vec{V} = [ae^{-bx}]\hat{i} + bt^2\hat{j}$ (6) $\vec{V} = a(x^2 + y^2)^{1/2}(1/z^3)\hat{k}$
(7) $\vec{V} = (ax + t)\hat{i} - by^2\hat{j}$ (8) $\vec{V} = ax^2\hat{i} + bxz\hat{j} + cy\hat{k}$

Given: Velocity fields

Find: Whether flows are 1, 2 or 3D, steady or unsteady.

Solution:

(1)	$\vec{V} = \vec{V}(y)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(2)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(3)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(4)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(5)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(6)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady
(7)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(8)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady

Problem 2.3

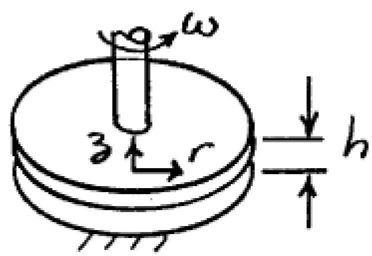
[Difficulty: 2]

2.3 A viscous liquid is sheared between two parallel disks; the upper disk rotates and the lower one is fixed. The velocity field between the disks is given by $\vec{V} = \hat{e}_\theta r\omega z/h$. (The origin of coordinates is located at the center of the lower disk; the upper disk is located at $z = h$.) What are the dimensions of this velocity field? Does this velocity field satisfy appropriate physical boundary conditions? What are they?

Given: Viscous liquid sheared between parallel disks.

Upper disk rotates, lower fixed.

Velocity field is: $\vec{V} = \hat{e}_\theta \dfrac{r\omega z}{h}$



Find:

a. Dimensions of velocity field.

b. Satisfy physical boundary conditions.

Solution: To find dimensions, compare to $\vec{V} = \vec{V}(x, y, z)$ form.

The given field is $\vec{V} = \vec{V}(r, z)$. Two space coordinates are included, so the field is 2-D.

Flow must satisfy the no-slip condition:

1. At lower disk, $\vec{V} = 0$ since stationary.

   $z = 0$, so $\vec{V} = \hat{e}_\theta \dfrac{r\omega 0}{h} = 0$, so satisfied.

2. At upper disk, $\vec{V} = \hat{e}_\theta r\omega$ since it rotates as a solid body.

   $z = h$, so $\vec{V} = \hat{e}_\theta \dfrac{r\omega h}{h} = \hat{e}_\theta r\omega$, so satisfied.

Problem 2.4

[Difficulty: 1]

2.4 For the velocity field $\vec{V} = Ax^2y\hat{i} + Bxy^2\hat{j}$, where $A = 2\ \text{m}^{-2}\text{s}^{-1}$ and $B = 1\ \text{m}^{-2}\text{s}^{-1}$, and the coordinates are measured in meters, obtain an equation for the flow streamlines. Plot several streamlines in the first quadrant.

Given: Velocity field

Find: Equation for streamlines

Solution:

For streamlines

$$\frac{v}{u} = \frac{dy}{dx} = \frac{B \cdot x \cdot y^2}{A \cdot x^2 \cdot y} = \frac{B \cdot y}{A \cdot x}$$

So, separating variables

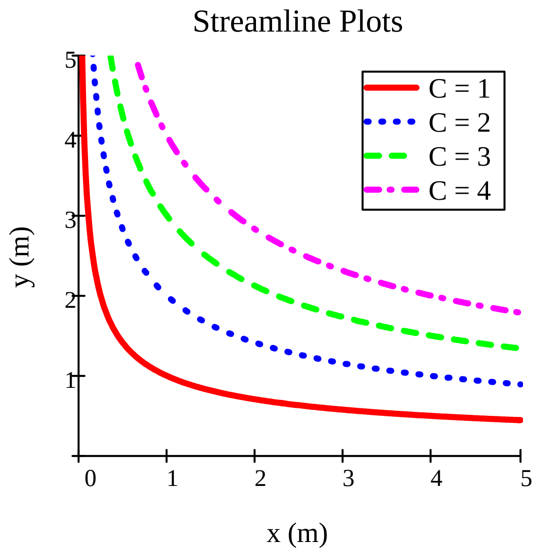
$$\frac{dy}{y} = \frac{B}{A} \cdot \frac{dx}{x}$$

Integrating

$$\ln(y) = \frac{B}{A} \cdot \ln(x) + c = -\frac{1}{2} \cdot \ln(x) + c$$

The solution is

$$y = \frac{C}{\sqrt{x}}$$



The plot can be easily done in *Excel*.

Problem 2.5

[Difficulty: 2]

2.5 The velocity field $\vec{V} = Ax\hat{i} - Ay\hat{j}$, where $A = 2\ s^{-1}$, can be interpreted to represent flow in a corner. Find an equation for the flow streamlines. Explain the relevance of A. Plot several streamlines in the first quadrant, including the one that passes through the point $(x, y) = (0, 0)$.

Given: Velocity field

Find: Equation for streamlines; Plot several in the first quadrant, including one that passes through point (0,0)

Solution:

Governing equation: For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

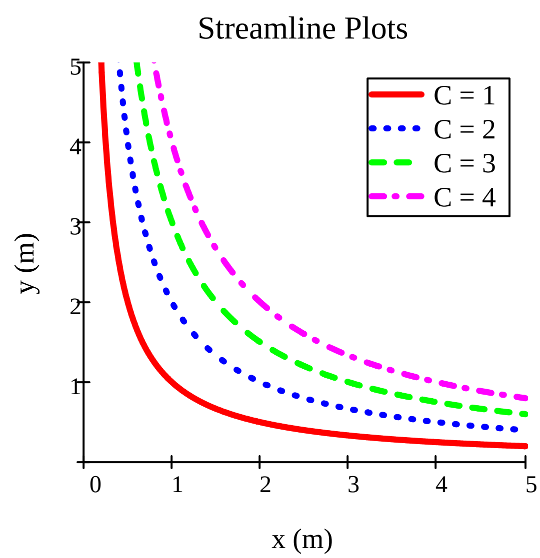
Hence $\frac{v}{u} = \frac{dy}{dx} = -\frac{A \cdot y}{A \cdot x} = -\frac{y}{x}$

So, separating variables $\frac{dy}{y} = -\frac{dx}{x}$

Integrating $\ln(y) = -\ln(x) + c$

The solution is $\ln(x \cdot y) = c$

or $y = \frac{C}{x}$



The plot can be easily done in *Excel*.

The streamline passing through (0,0) is given by the vertical axis, then the horizontal axis.
The value of A is irrelevant to streamline shapes but IS relevant for computing the velocity at each point.

Problem 2.6

[Difficulty: 1]

2.6 A velocity field is specified as $\vec{V} = axy\hat{i} + by^2\hat{j}$, where $a = 2\ \text{m}^{-1}\text{s}^{-1}$, $b = -6\ \text{m}^{-1}\text{s}^{-1}$, and the coordinates are measured in meters. Is the flow field one-, two-, or three-dimensional? Why? Calculate the velocity components at the point (2, ½). Develop an equation for the streamline passing through this point. Plot several streamlines in the first quadrant including the one that passes through the point (2, ½).

Given: Velocity field

Find: Whether field is 1D, 2D or 3D; Velocity components at (2,1/2); Equation for streamlines; Plot

Solution:

The velocity field is a function of *x* and *y*. It is therefore 2D.

At point (2,1/2), the velocity components are

$$u = a\cdot x\cdot y = 2\cdot\frac{1}{m\cdot s}\times 2\cdot m\times\frac{1}{2}\cdot m \qquad u = 2\cdot\frac{m}{s}$$

$$v = b\cdot y^2 = -6\cdot\frac{1}{m\cdot s}\times\left(\frac{1}{2}\cdot m\right)^2 \qquad v = -\frac{3}{2}\cdot\frac{m}{s}$$

For streamlines

$$\frac{v}{u} = \frac{dy}{dx} = \frac{b\cdot y^2}{a\cdot x\cdot y} = \frac{b\cdot y}{a\cdot x}$$

So, separating variables

$$\frac{dy}{y} = \frac{b}{a}\cdot\frac{dx}{x}$$

Integrating

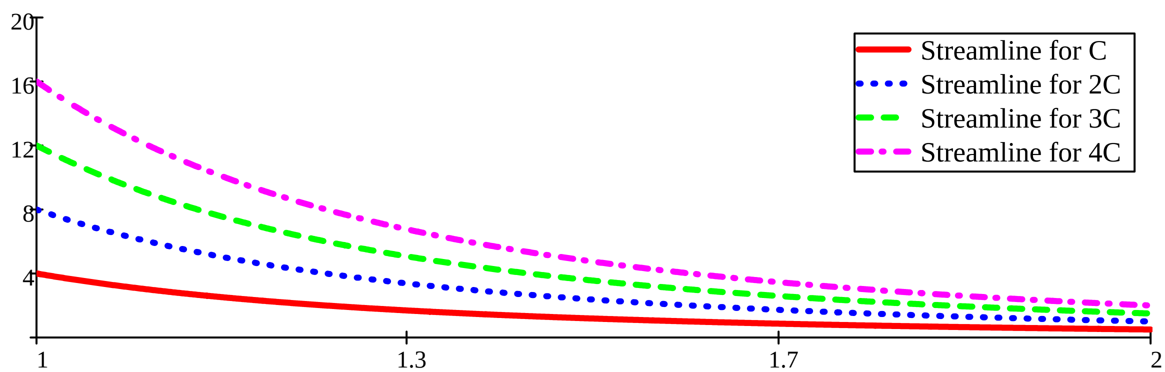
$$\ln(y) = \frac{b}{a}\cdot\ln(x) + c \qquad y = C\cdot x^{\frac{b}{a}}$$

The solution is

$$y = C\cdot x^{-3}$$

The streamline passing through point (2,1/2) is given by

$$\frac{1}{2} = C\cdot 2^{-3} \qquad C = \frac{1}{2}\cdot 2^3 \qquad C = 4 \qquad y = \frac{4}{x^3}$$



This can be plotted in *Excel*.

Problem 2.7

[Difficulty: 2]

2.7 A velocity field is given by $\vec{V} = ax\hat{i} - bty\hat{j}$, where $a = 1\ s^{-1}$ and $b = 1\ s^{-2}$. Find the equation of the streamlines at any time t. Plot several streamlines in the first quadrant at $t = 0$ s, $t = 1$ s, and $t = 20$ s.

Given: Velocity field

Find: Equation for streamlines; Plot streamlines

Solution:

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{-b \cdot t \cdot y}{a \cdot x}$

So, separating variables $\quad \frac{dy}{y} = \frac{-b \cdot t}{a} \cdot \frac{dx}{x}$

Integrating $\quad \ln(y) = \frac{-b \cdot t}{a} \cdot \ln(x)$

The solution is $\quad y = c \cdot x^{\frac{-b}{a} \cdot t}$

For $t = 0$ s $\quad y = c \qquad$ For $t = 1$ s $\quad y = \frac{c}{x} \qquad$ For $t = 20$ s $\quad y = c \cdot x^{-20}$

t = 0

x	y (c = 1)	y (c = 2)	y (c = 3)
0.05	1.00	2.00	3.00
0.10	1.00	2.00	3.00
0.20	1.00	2.00	3.00
0.30	1.00	2.00	3.00
0.40	1.00	2.00	3.00
0.50	1.00	2.00	3.00
0.60	1.00	2.00	3.00
0.70	1.00	2.00	3.00
0.80	1.00	2.00	3.00
0.90	1.00	2.00	3.00
1.00	1.00	2.00	3.00
1.10	1.00	2.00	3.00
1.20	1.00	2.00	3.00
1.30	1.00	2.00	3.00
1.40	1.00	2.00	3.00
1.50	1.00	2.00	3.00
1.60	1.00	2.00	3.00
1.70	1.00	2.00	3.00
1.80	1.00	2.00	3.00
1.90	1.00	2.00	3.00
2.00	1.00	2.00	3.00

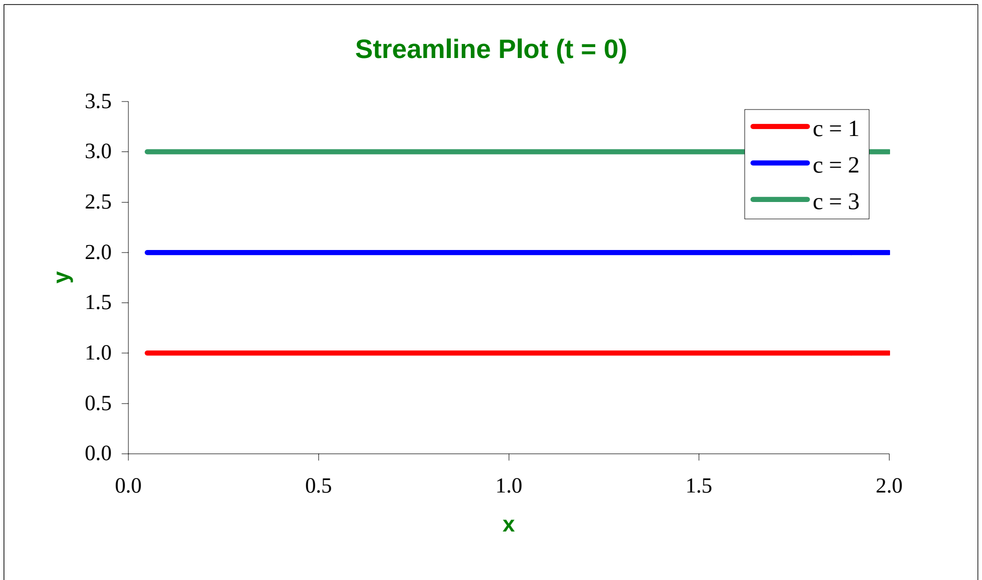
t =1 s

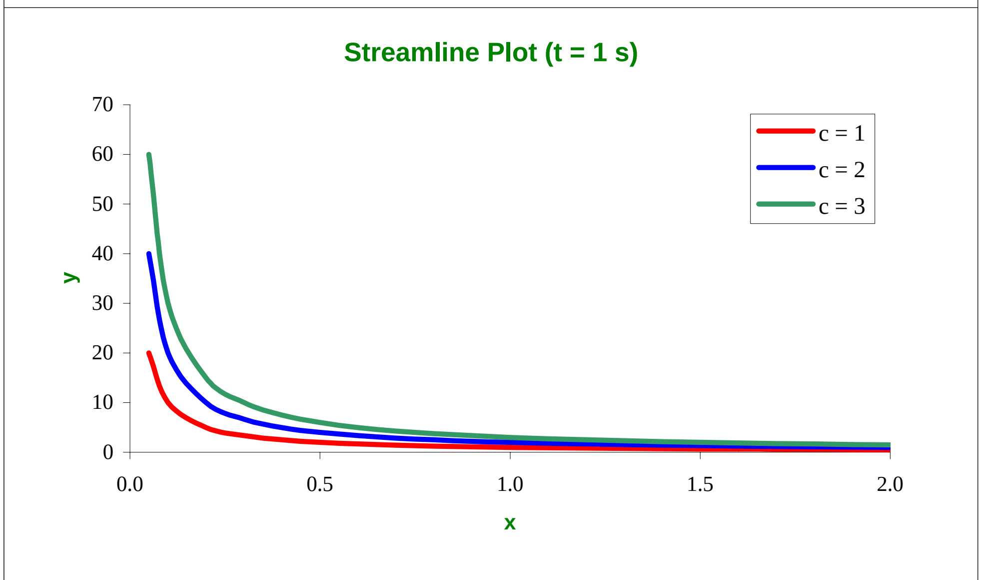
(### means too large to view)

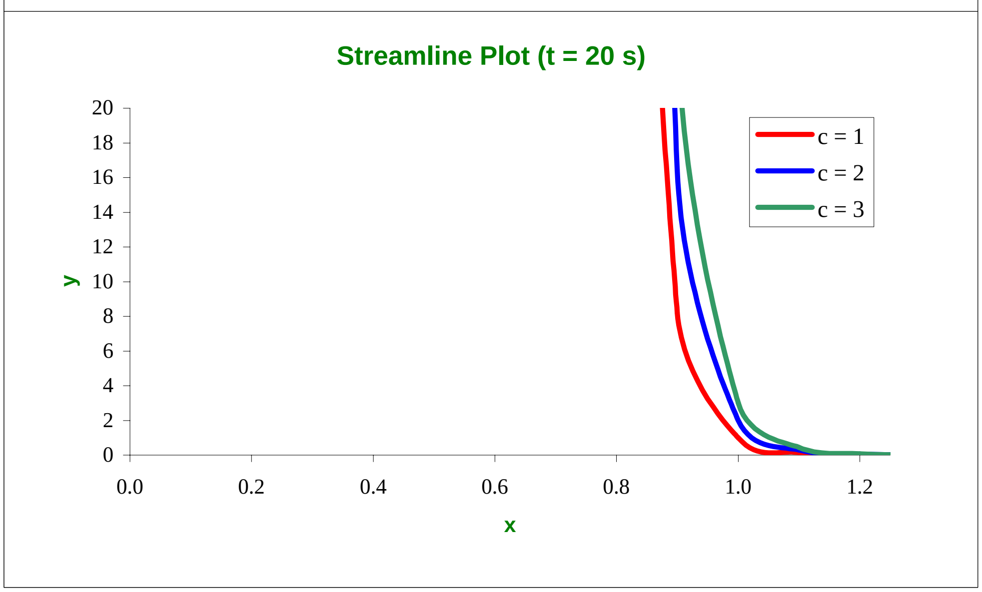
x	y (c = 1)	y (c = 2)	y (c = 3)
0.05	20.00	40.00	60.00
0.10	10.00	20.00	30.00
0.20	5.00	10.00	15.00
0.30	3.33	6.67	10.00
0.40	2.50	5.00	7.50
0.50	2.00	4.00	6.00
0.60	1.67	3.33	5.00
0.70	1.43	2.86	4.29
0.80	1.25	2.50	3.75
0.90	1.11	2.22	3.33
1.00	1.00	2.00	3.00
1.10	0.91	1.82	2.73
1.20	0.83	1.67	2.50
1.30	0.77	1.54	2.31
1.40	0.71	1.43	2.14
1.50	0.67	1.33	2.00
1.60	0.63	1.25	1.88
1.70	0.59	1.18	1.76
1.80	0.56	1.11	1.67
1.90	0.53	1.05	1.58
2.00	0.50	1.00	1.50

t = 20 s

x	y (c = 1)	y (c = 2)	y (c = 3)
0.05	######	######	######
0.10	######	######	######
0.20	######	######	######
0.30	######	######	######
0.40	######	######	######
0.50	######	######	######
0.60	######	######	######
0.70	######	######	######
0.80	86.74	173.47	260.21
0.90	8.23	16.45	24.68
1.00	1.00	2.00	3.00
1.10	0.15	0.30	0.45
1.20	0.03	0.05	0.08
1.30	0.01	0.01	0.02
1.40	0.00	0.00	0.00
1.50	0.00	0.00	0.00
1.60	0.00	0.00	0.00
1.70	0.00	0.00	0.00
1.80	0.00	0.00	0.00
1.90	0.00	0.00	0.00
2.00	0.00	0.00	0.00


Streamline Plot (t = 0)
c = 1
c = 2
c = 3
y
x
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5


Streamline Plot (t = 1 s)
c = 1
c = 2
c = 3
y
x
0
10
20
30
40
50
60
70
0.0
0.5
1.0
1.5
2.0


Streamline Plot (t = 20 s)
c = 1
c = 2
c = 3
y
x
0
2
4
6
8
10
12
14
16
18
20
0.0
0.2
0.4
0.6
0.8
1.0
1.2

Problem 2.8

[Difficulty: 2]

2.8 A velocity field is given by $\vec{V} = ax^3\hat{i} + bxy^3\hat{j}$, where $a = 1\ \text{m}^{-2}\text{s}^{-1}$ and $b = 1\ \text{m}^{-3}\text{s}^{-1}$. Find the equation of the streamlines. Plot several streamlines in the first quadrant.

Given: Velocity field

Find: Equation for streamlines; Plot streamlines

Solution:

Streamlines are given by $$\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot x \cdot y^3}{a \cdot x^3}$$

So, separating variables $$\frac{dy}{y^3} = \frac{b \cdot dx}{a \cdot x^2}$$

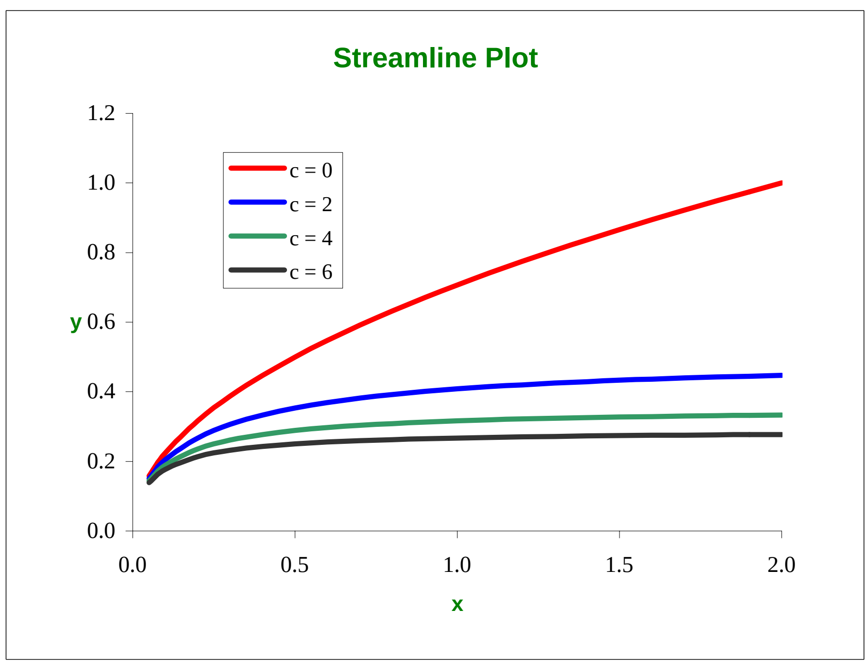
Integrating $$-\frac{1}{2 \cdot y^2} = \frac{b}{a} \cdot \left(-\frac{1}{x}\right) + C$$

The solution is $$y = \frac{1}{\sqrt{2 \cdot \left(\frac{b}{a \cdot x} + C\right)}}$$

Note: For convenience the sign of C is changed.

a = **1**
b = **1**

C =	0	2	4	6
x	**y**	**y**	**y**	**y**
0.05	0.16	0.15	0.14	0.14
0.10	0.22	0.20	0.19	0.18
0.20	0.32	0.27	0.24	0.21
0.30	0.39	0.31	0.26	0.23
0.40	0.45	0.33	0.28	0.24
0.50	0.50	0.35	0.29	0.25
0.60	0.55	0.37	0.30	0.26
0.70	0.59	0.38	0.30	0.26
0.80	0.63	0.39	0.31	0.26
0.90	0.67	0.40	0.31	0.27
1.00	0.71	0.41	0.32	0.27
1.10	0.74	0.41	0.32	0.27
1.20	0.77	0.42	0.32	0.27
1.30	0.81	0.42	0.32	0.27
1.40	0.84	0.43	0.33	0.27
1.50	0.87	0.43	0.33	0.27
1.60	0.89	0.44	0.33	0.27
1.70	0.92	0.44	0.33	0.28
1.80	0.95	0.44	0.33	0.28
1.90	0.97	0.44	0.33	0.28
2.00	1.00	0.45	0.33	0.28



Problem 2.9

[Difficulty: 2]

2.9 A flow is described by the velocity field $\vec{V} = (Ax + B)\hat{i} + (-Ay)\hat{j}$, where $A = 3$ m/s/m and $B = 6$ m/s. Plot a few streamlines in the xy plane, including the one that passes through the point $(x, y) = (0.3, 0.6)$.

Given: Velocity field.

Find: Plot streamlines.

Solution:

Streamlines are given by $$\frac{v}{u} = \frac{dy}{dx} = \frac{-A \cdot y}{A \cdot x + B}$$

So, separating variables $$\frac{dy}{-A \cdot y} = \frac{dx}{A \cdot x + B}$$

Integrating $$-\frac{1}{A}\ln(y) = \frac{1}{A}\cdot \ln\left(x + \frac{B}{A}\right)$$

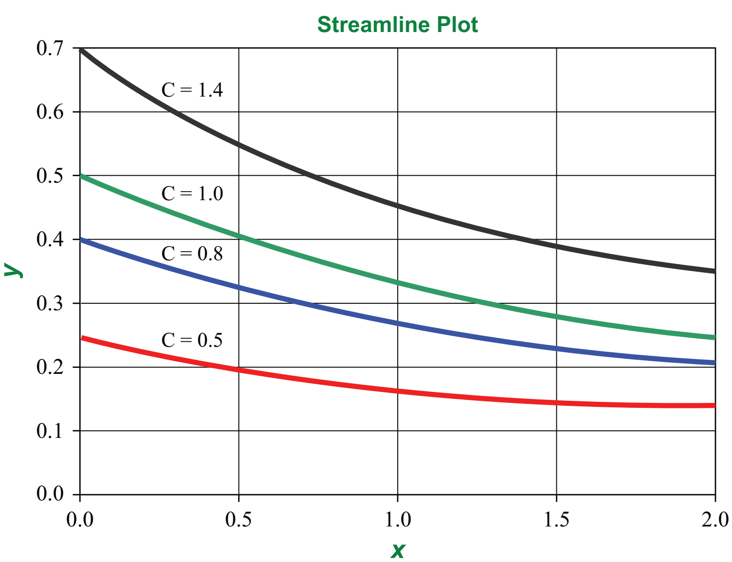
The solution is $$y = \frac{C}{x + \frac{B}{A}}$$

For the streamline that passes through point $(x, y) = (0.3, 0.6)$ $$C = y \cdot \left(x + \frac{B}{A}\right) = 0.6 \cdot \left(0.3 + \frac{6}{3}\right) = 1.4$$

$$y = \frac{1.4}{x + \frac{6}{3}}$$

A = 3
B = 6
C = 0.5, 0.8, 1.0, and 1.4

	0.5	0.8	1.0	1.4
x	y	y	y	y
0.00	0.25	0.40	0.50	0.70
0.10	0.24	0.38	0.48	0.67
0.20	0.23	0.36	0.45	0.64
0.30	0.22	0.35	0.43	0.61
0.40	0.21	0.33	0.42	0.58
0.50	0.20	0.32	0.40	0.56
0.60	0.19	0.31	0.38	0.54
0.70	0.18	0.30	0.37	0.52
0.80	0.18	0.29	0.36	0.50
0.90	0.17	0.28	0.34	0.48
1.00	0.17	0.27	0.33	0.47
1.10	0.16	0.26	0.32	0.45
1.20	0.16	0.25	0.31	0.44
1.30	0.15	0.24	0.29	0.42
1.40	0.15	0.24	0.29	0.41
1.50	0.14	0.23	0.29	0.40
1.60	0.14	0.22	0.28	0.39
1.70	0.14	0.22	0.27	0.38
1.80	0.13	0.21	0.26	0.37
1.90	0.13	0.21	0.26	0.36
2.00	0.13	0.20	0.25	0.35



Problem 2.10

[Difficulty: 2]

2.10 The velocity for a steady, incompressible flow in the xy plane is given by $\vec{V} = \hat{i}A/x + \hat{j}Ay/x^2$, where $A = 2\ \text{m}^2/\text{s}$, and the coordinates are measured in meters. Obtain an equation for the streamline that passes through the point $(x, y) = (1, 3)$. Calculate the time required for a fluid particle to move from $x = 1$ m to $x = 2$ m in this flow field.

Given: Velocity field

Find: Equation for streamline through (1,3)

Solution:

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{A \cdot \frac{y}{x^2}}{\frac{A}{x}} = \frac{y}{x}$

So, separating variables $\frac{dy}{y} = \frac{dx}{x}$

Integrating $\ln(y) = \ln(x) + c$

The solution is $y = C \cdot x$ which is the equation of a straight line.

For the streamline through point (1,3) $3 = C \cdot 1$ $C = 3$ and $y = 3 \cdot x$

For a particle $u_p = \frac{dx}{dt} = \frac{A}{x}$ or $x \cdot dx = A \cdot dt$ $x = \sqrt{2 \cdot A \cdot t + c}$ $t = \frac{x^2}{2 \cdot A} - \frac{c}{2 \cdot A}$

Hence the time for a particle to go from x = 1 to x = 2 m is

$\Delta t = t(x = 2) - t(x = 1)$ $\Delta t = \frac{(2 \cdot m)^2 - c}{2 \cdot A} - \frac{(1 \cdot m)^2 - c}{2 \cdot A} = \frac{4 \cdot m^2 - 1 \cdot m^2}{2 \times 2 \cdot \frac{m^2}{s}}$ $\Delta t = 0.75 \cdot s$

Problem 2.11

[Difficulty: 3]

2.11 The flow field for an atmospheric flow is given by

$$\vec{V} = -\frac{My}{2\pi}\hat{i} + \frac{Mx}{2\pi}\hat{j}$$

where $M = 1\ s^{-1}$, and the x and y coordinates are the parallel to the local latitude and longitude. Plot the velocity magnitude along the x axis, along the y axis, and along the line $y = x$, and discuss the velocity direction with respect to these three axes. For each plot use a range x or $y = 0$ km to 1 km. Find the equation for the streamlines and sketch several of them. What does this flow field model?

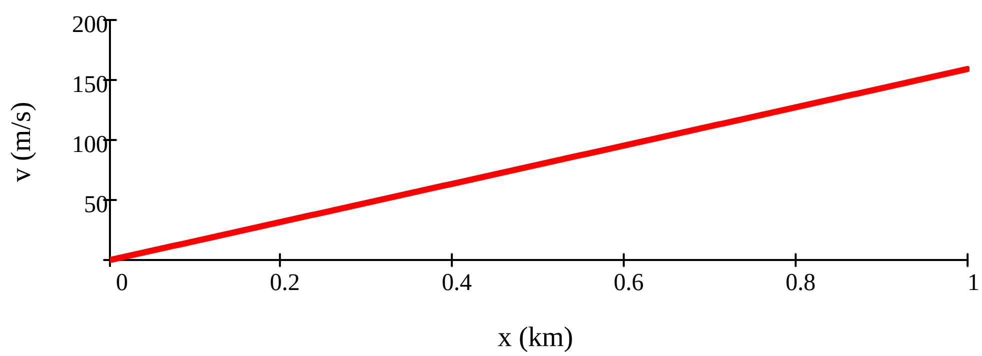
Given: Flow field

Find: Plot of velocity magnitude along axes, and y = x; Equation for streamlines

Solution:

On the x axis, y = 0, so $u = -\frac{M \cdot y}{2 \cdot \pi} = 0$ $v = \frac{M \cdot x}{2 \cdot \pi}$

Plotting

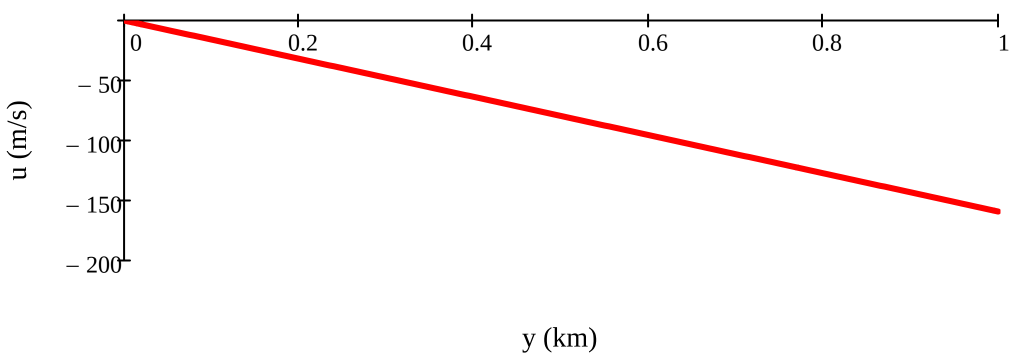


The velocity is perpendicular to the axis and increases linearly with distance x.

This can also be plotted in Excel.

On the y axis, x = 0, so $u = -\frac{M \cdot y}{2 \cdot \pi}$ $v = \frac{M \cdot x}{2 \cdot \pi} = 0$

Plotting



The velocity is perpendicular to the axis and increases linearly with distance y.

This can also be plotted in Excel.

On the y = x axis $u = -\frac{M \cdot y}{2 \cdot \pi} = -\frac{M \cdot x}{2 \cdot \pi}$ $v = \frac{M \cdot x}{2 \cdot \pi}$

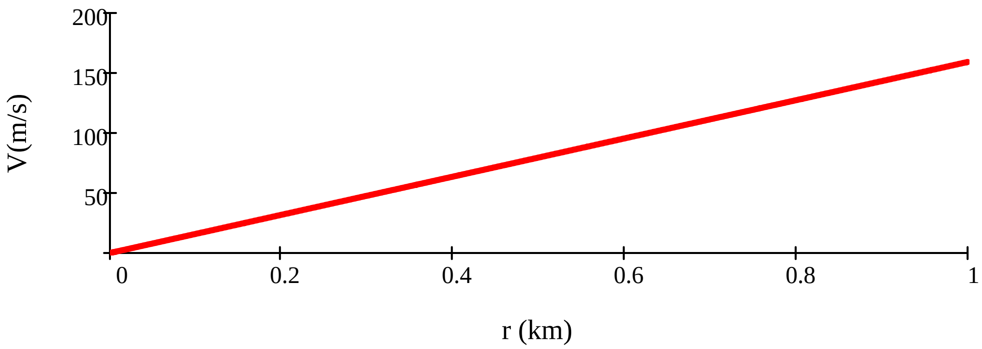
The flow is perpendicular to line y = x: Slope of line y = x: 1

Slope of trajectory of motion: $\frac{u}{v} = -1$

If we define the radial position: $r = \sqrt{x^2 + y^2}$ then along y = x $r = \sqrt{x^2 + x^2} = \sqrt{2} \cdot x$

Then the magnitude of the velocity along y = x is $V = \sqrt{u^2 + v^2} = \frac{M}{2 \cdot \pi} \cdot \sqrt{x^2 + x^2} = \frac{M \cdot \sqrt{2} \cdot x}{2 \cdot \pi} = \frac{M \cdot r}{2 \cdot \pi}$

Plotting



This can also be plotted in Excel.

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{\frac{M \cdot x}{2 \cdot \pi}}{-\frac{M \cdot y}{2 \cdot \pi}} = -\frac{x}{y}$

So, separating variables $y \cdot dy = -x \cdot dx$

Integrating $\frac{y^2}{2} = -\frac{x^2}{2} + c$

The solution is $x^2 + y^2 = C$ which is the equation of a circle.

The streamlines form a set of concentric circles.

This flow models a rigid body vortex flow. See Example 5.6 for streamline plots. Streamlines are circular, and the velocity approaches zero as we approach the center. In Problem 2.10, we see that the streamlines are also circular. In a real tornado, at large distances from the center, the velocities behave as in Problem 2.10; close to the center, they behave as in this problem.

Problem 2.12

[Difficulty: 3]

2.12 The flow field for an atmospheric flow is given by

$$\vec{V} = -\frac{Ky}{2\pi(x^2+y^2)}\hat{i} + \frac{Kx}{2\pi(x^2+y^2)}\hat{j}$$

where $K = 10^5\ \mathrm{m^2/s}$, and the x and y coordinates are parallel to the local latitude and longitude. Plot the velocity magnitude along the x axis, along the y axis, and along the line $y = x$, and discuss the velocity direction with respect to these three axes. For each plot use a range x or $y = -1$ km to 1 km, excluding $|x|$ or $|y| < 100$ m. Find the equation for the streamlines and sketch several of them. What does this flow field model?

Given: Flow field

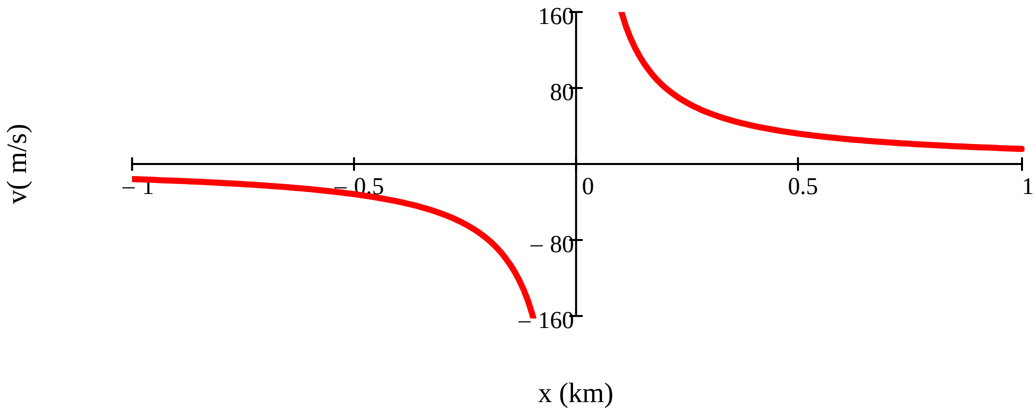
Find: Plot of velocity magnitude along axes, and y = x; Equation of streamlines

Solution:

On the x axis, y = 0, so

$$u = -\frac{K\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = 0 \qquad v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = \frac{K}{2\cdot\pi\cdot x}$$

Plotting



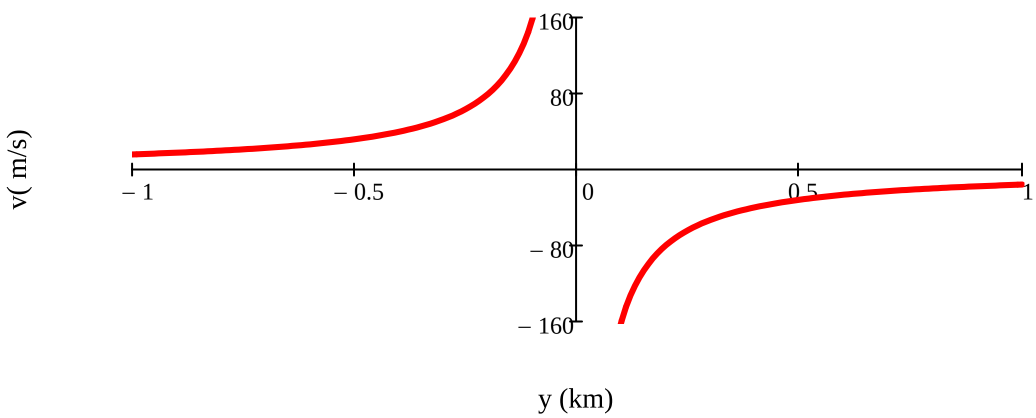
The velocity is perpendicular to the axis, is very high close to the origin, and falls off to zero.

This can also be plotted in Excel.

On the y axis, x = 0, so

$$u = -\frac{K\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = -\frac{K}{2\cdot\pi\cdot y} \qquad v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = 0$$

Plotting



The velocity is perpendicular to the axis, is very high close to the origin, and falls off to zero.
This can also be plotted in Excel.

On the y = x axis $\quad u = -\frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + x^2)} = -\frac{K}{4 \cdot \pi \cdot x} \qquad v = \frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + x^2)} = \frac{K}{4 \cdot \pi \cdot x}$

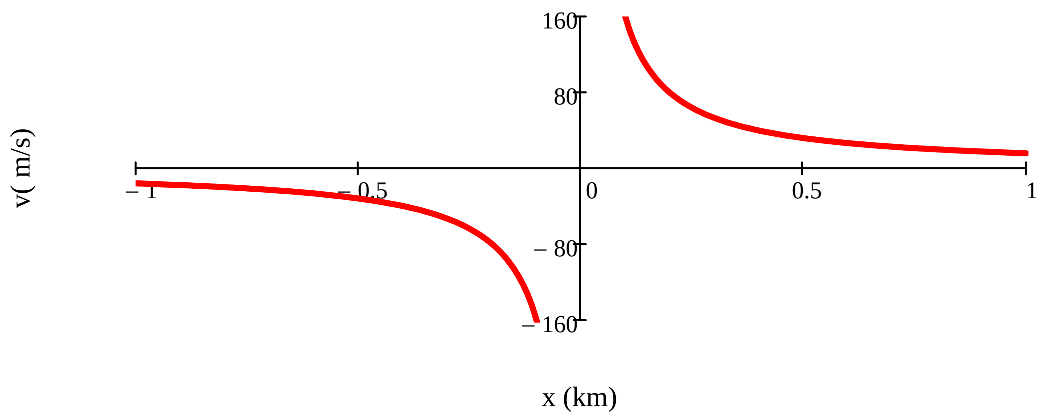
The flow is perpendicular to line y = x: Slope of line y = x: 1

Slope of trajectory of motion: $\frac{u}{v} = -1$

If we define the radial position: $r = \sqrt{x^2 + y^2}$ then along y = x $\quad r = \sqrt{x^2 + x^2} = \sqrt{2} \cdot x$

Then the magnitude of the velocity along y = x is $\quad V = \sqrt{u^2 + v^2} = \frac{K}{4 \cdot \pi} \cdot \sqrt{\frac{1}{x^2} + \frac{1}{x^2}} = \frac{K}{2 \cdot \pi \cdot \sqrt{2} \cdot x} = \frac{K}{2 \cdot \pi \cdot r}$

Plotting



This can also be plotted in Excel.

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{\frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + y^2)}}{-\frac{K \cdot y}{2 \cdot \pi \cdot (x^2 + y^2)}} = -\frac{x}{y}$

So, separating variables $\quad y \cdot dy = -x \cdot dx$

Integrating $\quad \frac{y^2}{2} = -\frac{x^2}{2} + c$

The solution is $\quad x^2 + y^2 = C \quad$ which is the equation of a circle.

Streamlines form a set of concentric circles.

This flow models a vortex flow. See Example 5.6 for streamline plots. Streamlines are circular, and the velocity approaches infinity as we approach the center. In Problem 2.11, we see that the streamlines are also circular. In a real tornado, at large distances from the center, the velocities behave as in this problem; close to the center, they behave as in Problem 2.11.

Problem 2.13

[Difficulty: 3]

2.13 A flow field is given by

$$\vec{V} = -\frac{qx}{2\pi(x^2+y^2)}\hat{i} - \frac{qy}{2\pi(x^2+y^2)}\hat{j}$$

where $q = 5 \times 10^4\ \mathrm{m^2/s}$. Plot the velocity magnitude along the x axis, along the y axis, and along the line $y = x$, and discuss the velocity direction with respect to these three axes. For each plot use a range x or $y = -1$ km to 1 km, excluding $|x|$ or $|y| < 100$ m. Find the equation for the streamlines and sketch several of them. What does this flow field model?

Given: Flow field

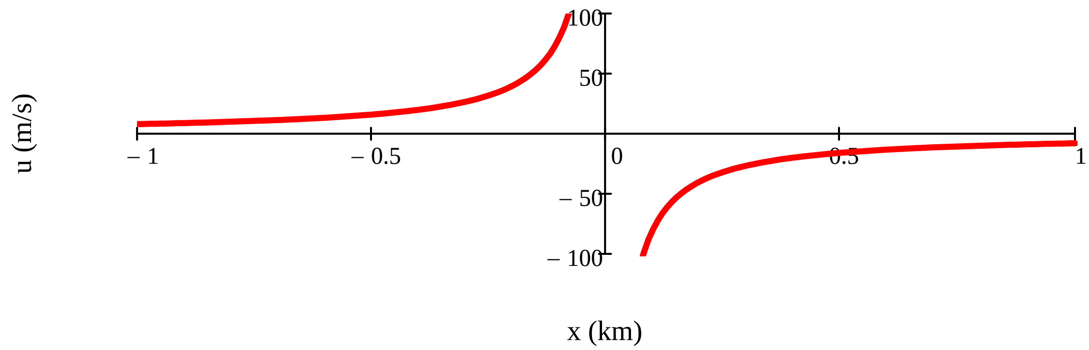
Find: Plot of velocity magnitude along axes, and y = x; Equations of streamlines

Solution:

On the x axis, y = 0, so

$$u = -\frac{q \cdot x}{2\cdot\pi\cdot\left(x^2 + y^2\right)} = -\frac{q}{2\cdot\pi\cdot x} \qquad v = -\frac{q \cdot y}{2\cdot\pi\cdot\left(x^2 + y^2\right)} = 0$$

Plotting

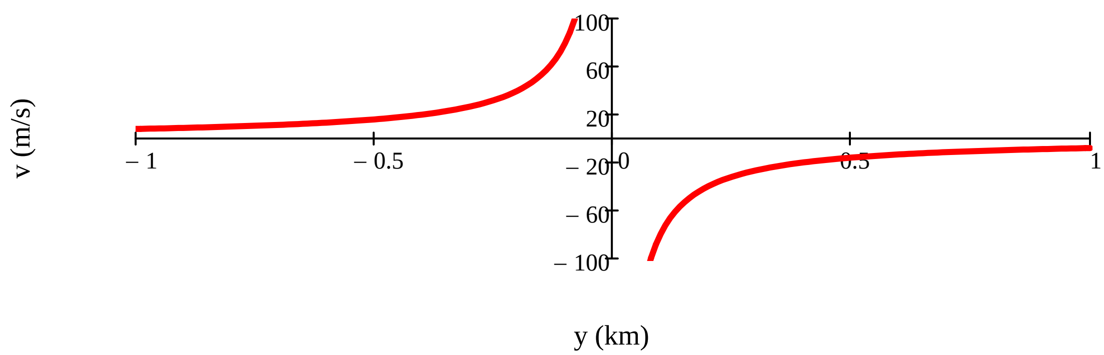


The velocity is very high close to the origin, and falls off to zero. It is also along the axis. This can be plotted in *Excel*.

On the y axis, x = 0, so

$$u = -\frac{q \cdot x}{2\cdot\pi\cdot\left(x^2 + y^2\right)} = 0 \qquad v = -\frac{q \cdot y}{2\cdot\pi\cdot\left(x^2 + y^2\right)} = -\frac{q}{2\cdot\pi\cdot y}$$

Plotting



The velocity is again very high close to the origin, and falls off to zero. It is also along the axis.

This can also be plotted in *Excel*.

On the y = x axis $u = -\frac{q\cdot x}{2\cdot\pi\cdot(x^2 + x^2)} = -\frac{q}{4\cdot\pi\cdot x}$ $v = -\frac{q\cdot x}{2\cdot\pi\cdot(x^2 + x^2)} = -\frac{q}{4\cdot\pi\cdot x}$

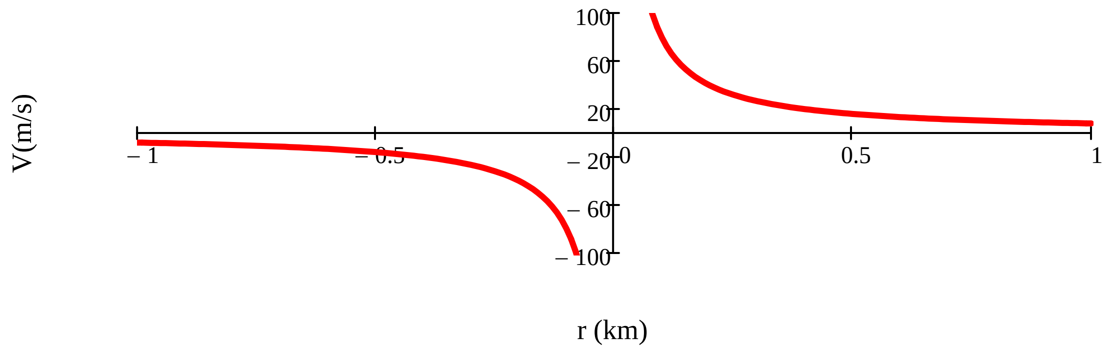
The flow is parallel to line y = x: Slope of line y = x: 1

Slope of trajectory of motion: $\frac{v}{u} = 1$

If we define the radial position: $r = \sqrt{x^2 + y^2}$ then along y = x $r = \sqrt{x^2 + x^2} = \sqrt{2}\cdot x$

Then the magnitude of the velocity along y = x is $V = \sqrt{u^2 + v^2} = \frac{q}{4\cdot\pi}\cdot\sqrt{\frac{1}{x^2} + \frac{1}{x^2}} = \frac{q}{2\cdot\pi\cdot\sqrt{2}\cdot x} = \frac{q}{2\cdot\pi\cdot r}$

Plotting



This can also be plotted in Excel.

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{-\frac{q\cdot y}{2\cdot\pi\cdot(x^2 + y^2)}}{-\frac{q\cdot x}{2\cdot\pi\cdot(x^2 + y^2)}} = \frac{y}{x}$

So, separating variables $\frac{dy}{y} = \frac{dx}{x}$

Integrating $\ln(y) = \ln(x) + c$

The solution is $y = C\cdot x$ which is the equation of a straight line.

This flow field corresponds to a sink (discussed in Chapter 6).

Problem 2.14

[Difficulty: 2]

2.14 Beginning with the velocity field of Problem 2.5, show that the parametric equations for particle motion are given by $x_p = c_1 e^{At}$ and $y_p = c_2 e^{-At}$. Obtain the equation for the pathline of the particle located at the point $(x, y) = (2, 2)$ at the instant $t = 0$. Compare this pathline with the streamline through the same point.

Given: Velocity field

Find: Proof that the parametric equations for particle motion are $x_p = c_1 \cdot e^{A \cdot t}$ and $y_p = c_2 \cdot e^{-A \cdot t}$; pathline that was at (2,2) at t = 0; compare to streamline through same point, and explain why they are similar or not.

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = A \cdot x$ $\qquad v_p = \frac{dy}{dt} = -A \cdot y$

So, separating variables $\frac{dx}{x} = A \cdot dt$ $\qquad \frac{dy}{y} = -A \cdot dt$

Integrating $\ln(x) = A \cdot t + C_1$ $\qquad \ln(y) = -A \cdot t + C_2$

$$x = e^{A \cdot t + C_1} = e^{C_1} \cdot e^{A \cdot t} = c_1 \cdot e^{A \cdot t} \qquad y = e^{-A \cdot t + C_2} = e^{C_2} \cdot e^{-A \cdot t} = c_2 \cdot e^{-A \cdot t}$$

The pathlines are $x = c_1 \cdot e^{A \cdot t}$ $\qquad y = c_2 \cdot e^{-A \cdot t}$

Eliminating t $\quad t = \frac{1}{A} \cdot \ln\left(\frac{x}{c_1}\right) = -\frac{1}{A} \cdot \ln\left(\frac{y}{c_2}\right)$ $\qquad \ln\left(x^{\frac{1}{A}} \cdot y^{\frac{1}{A}}\right) = \text{const}$ or $\ln\left(x^A \cdot y^A\right) = \text{const}$

so $x^A \cdot y^A = \text{const}$ or $x \cdot y = 4$ for given data

For streamlines $\frac{v}{u} = \frac{dy}{dx} = -\frac{A \cdot y}{A \cdot x} = \frac{y}{x}$

So, separating variables $\frac{dy}{y} = -\frac{dx}{x}$

Integrating $\ln(y) = -\ln(x) + c$

The solution is $\ln(x \cdot y) = c$ $\quad$ or $\quad x \cdot y = \text{const}$ $\quad$ or $\quad x \cdot y = 4$ $\quad$ for given data

The streamline passing through (2,2) and the pathline that started at (2,2) coincide because the flow is steady!

Problem 2.15

[Difficulty: 2]

2.15 A flow field is given by $\vec{V} = Ax\hat{i} + 2Ay\hat{j}$, where $A = 2\ \mathrm{s}^{-1}$. Verify that the parametric equations for particle motion are given by $x_p = c_1 e^{At}$ and $y_p = c_2 e^{2At}$. Obtain the equation for the pathline of the particle located at the point $(x, y) = (2,2)$ at the instant $t = 0$. Compare this pathline with the streamline through the same point.

Given: Velocity field

Find: Proof that the parametric equations for particle motion are $x_p = c_1 \cdot e^{A\cdot t}$ and $y_p = c_2 \cdot e^{2\cdot A\cdot t}$; pathline that was at (2,2) at t = 0; compare to streamline through same point, and explain why they are similar or not.

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $\quad u_p = \frac{dx}{dt} = A\cdot x \qquad v_p = \frac{dy}{dt} = 2\cdot A\cdot y$

So, separating variables $\quad \frac{dx}{x} = A\cdot dt \qquad \frac{dy}{y} = 2\cdot A\cdot dt$

Integrating $\quad \ln(x) = A\cdot t + C_1 \qquad \ln(y) = 2\cdot A\cdot t + C_2$

$$x = e^{A\cdot t + C_1} = e^{C_1}\cdot e^{A\cdot t} = c_1\cdot e^{A\cdot t} \qquad y = e^{2\cdot A\cdot t + C_2} = e^{C_2}\cdot e^{2\cdot A\cdot t} = c_2\cdot e^{2\cdot A\cdot t}$$

The pathlines are $\quad x = c_1\cdot e^{A\cdot t} \qquad y = c_2\cdot e^{2\cdot A\cdot t}$

Eliminating t $\quad y = c_2\cdot e^{2\cdot A\cdot t} = c_2\cdot\left(\frac{x}{c_1}\right)^2$ $\quad$ so $\quad y = c\cdot x^2$ $\quad$ or $\quad y = \frac{1}{2}\cdot x^2$ for given data

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{2\cdot A\cdot y}{A\cdot x} = \frac{2\cdot y}{x}$

So, separating variables $\quad \frac{dy}{y} = \frac{2\cdot dx}{x}$ $\quad$ Integrating $\quad \ln(y) = 2\cdot\ln(x) + c$

The solution is $\quad \ln\left(\frac{y}{x^2}\right) = c$

or $\quad y = C\cdot x^2$ $\quad$ or $\quad y = \frac{1}{2}\cdot x^2$ for given data

The streamline passing through (2,2) and the pathline that started at (2,2) coincide because the flow is steady!

Problem 2.16

[Difficulty: 2]

2.16 A velocity field is given by $\vec{V} = ayt\hat{i} - bx\hat{j}$, where $a = 1\ s^{-2}$ and $b = 4\ s^{-1}$. Find the equation of the streamlines at any time t. Plot several streamlines at $t = 0$ s, $t = 1$ s, and $t = 20$ s.

Given: Velocity field

Find: Equation of streamlines; Plot streamlines

Solution:

Streamlines are given by $\frac{v}{u} = \frac{dy}{dx} = \frac{-b \cdot x}{a \cdot y \cdot t}$

So, separating variables $a \cdot t \cdot y \cdot dy = -b \cdot x \cdot dx$

Integrating $\frac{1}{2} \cdot a \cdot t \cdot y^2 = -\frac{1}{2} \cdot b \cdot x^2 + C$

The solution is $y = \sqrt{C - \frac{b \cdot x^2}{a \cdot t}}$

For $t = 0$ s $\quad x = c$

For $t = 1$ s $\quad y = \sqrt{C - 4 \cdot x^2}$

For $t = 20$ s $\quad y = \sqrt{C - \frac{x^2}{5}}$

t = 0

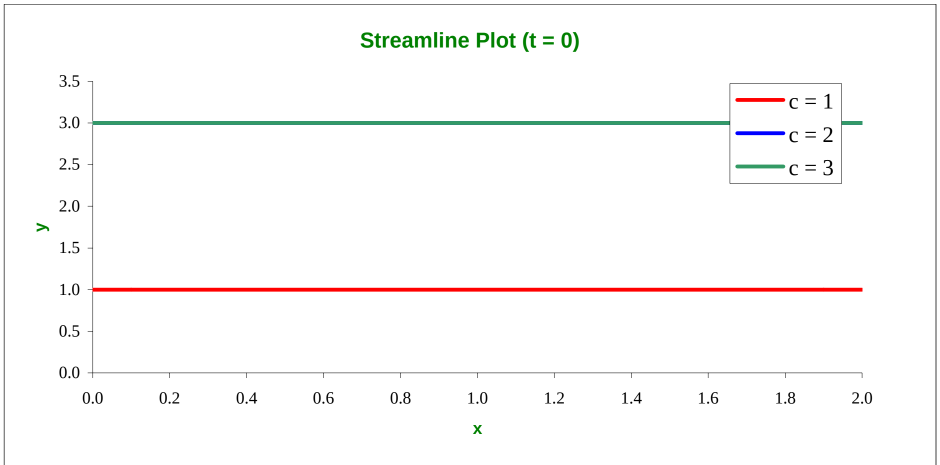
	C = 1	C = 2	C = 3
x	**y**	**y**	**y**
0.00	1.00	2.00	3.00
0.10	1.00	2.00	3.00
0.20	1.00	2.00	3.00
0.30	1.00	2.00	3.00
0.40	1.00	2.00	3.00
0.50	1.00	2.00	3.00
0.60	1.00	2.00	3.00
0.70	1.00	2.00	3.00
0.80	1.00	2.00	3.00
0.90	1.00	2.00	3.00
1.00	1.00	2.00	3.00
1.10	1.00	2.00	3.00
1.20	1.00	2.00	3.00
1.30	1.00	2.00	3.00
1.40	1.00	2.00	3.00
1.50	1.00	2.00	3.00
1.60	1.00	2.00	3.00
1.70	1.00	2.00	3.00
1.80	1.00	2.00	3.00
1.90	1.00	2.00	3.00
2.00	1.00	2.00	3.00

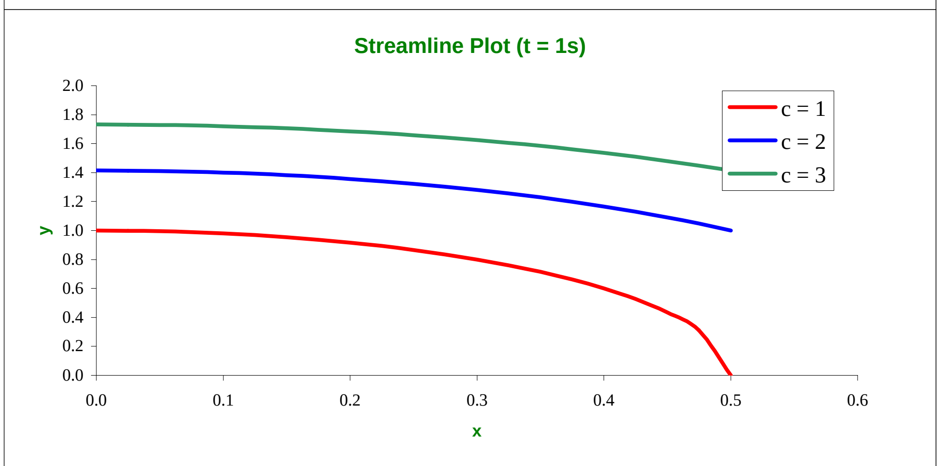
t = 1 s

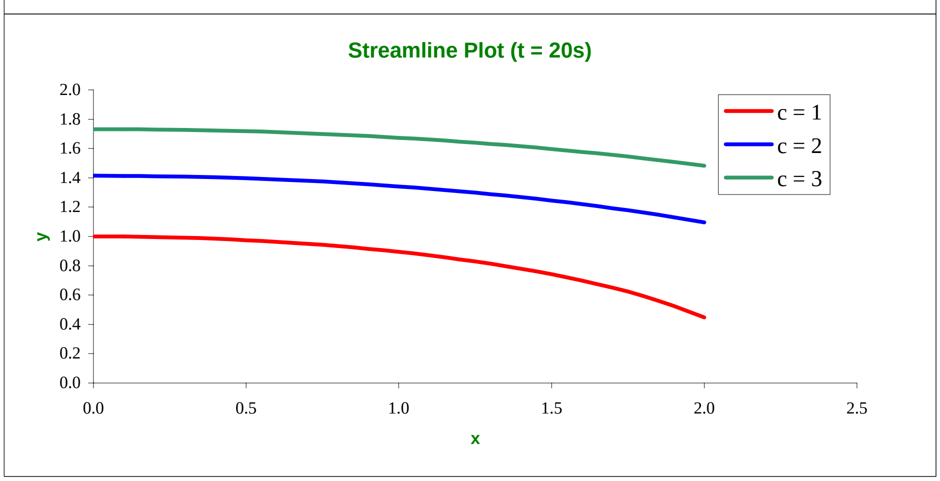
	C = 1	C = 2	C = 3
x	**y**	**y**	**y**
0.000	1.00	1.41	1.73
0.025	1.00	1.41	1.73
0.050	0.99	1.41	1.73
0.075	0.99	1.41	1.73
0.100	0.98	1.40	1.72
0.125	0.97	1.39	1.71
0.150	0.95	1.38	1.71
0.175	0.94	1.37	1.70
0.200	0.92	1.36	1.69
0.225	0.89	1.34	1.67
0.250	0.87	1.32	1.66
0.275	0.84	1.30	1.64
0.300	0.80	1.28	1.62
0.325	0.76	1.26	1.61
0.350	0.71	1.23	1.58
0.375	0.66	1.20	1.56
0.400	0.60	1.17	1.54
0.425	0.53	1.13	1.51
0.450	0.44	1.09	1.48
0.475	0.31	1.05	1.45
0.500	0.00	1.00	1.41

t = 20 s

	C = 1	C = 2	C = 3
x	**y**	**y**	**y**
0.00	1.00	1.41	1.73
0.10	1.00	1.41	1.73
0.20	1.00	1.41	1.73
0.30	0.99	1.41	1.73
0.40	0.98	1.40	1.72
0.50	0.97	1.40	1.72
0.60	0.96	1.39	1.71
0.70	0.95	1.38	1.70
0.80	0.93	1.37	1.69
0.90	0.92	1.36	1.68
1.00	0.89	1.34	1.67
1.10	0.87	1.33	1.66
1.20	0.84	1.31	1.65
1.30	0.81	1.29	1.63
1.40	0.78	1.27	1.61
1.50	0.74	1.24	1.60
1.60	0.70	1.22	1.58
1.70	0.65	1.19	1.56
1.80	0.59	1.16	1.53
1.90	0.53	1.13	1.51
2.00	0.45	1.10	1.48


Streamline Plot (t = 0)
c = 1
c = 2
c = 3
y
x
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0


Streamline Plot (t = 1s)
c = 1
c = 2
c = 3
y
x
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6


Streamline Plot (t = 20s)
c = 1
c = 2
c = 3
y
x
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
2.5

Problem 2.17

[Difficulty: 4]

2.17 Verify that $x_p = -a\sin(\omega t)$, $y_p = a\cos(\omega t)$ is the equation for the pathlines of particles for the flow field of Problem 2.12. Find the frequency of motion ω as a function of the amplitude of motion, a, and K. Verify that $x_p = -a\sin(\omega t)$, $y_p = a\cos(\omega t)$ is also the equation for the pathlines of particles for the flow field of Problem 2.11, except that ω is now a function of M. Plot typical pathlines for both flow fields and discuss the difference.

Given: Pathlines of particles

Find: Conditions that make them satisfy Problem 2.10 flow field; Also Problem 2.11 flow field; Plot pathlines

Solution:

The given pathlines are $x_p = -a\cdot\sin(\omega\cdot t)$ $\quad y_p = a\cdot\cos(\omega\cdot t)$

The velocity field of Problem 2.12 is $u = -\dfrac{K\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)}$ $\quad v = \dfrac{K\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)}$

If the pathlines are correct we should be able to substitute x_p and y_p into the velocity field to find the velocity as a function of time:

$$u = -\frac{K\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = -\frac{K\cdot a\cdot\cos(\omega\cdot t)}{2\cdot\pi\cdot\left(a^2\cdot\sin(\omega\cdot t)^2 + a^2\cdot\cos(\omega\cdot t)^2\right)} = -\frac{K\cdot\cos(\omega\cdot t)}{2\cdot\pi\cdot a} \qquad (1)$$

$$v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = -\frac{K\cdot(-a\cdot\sin(\omega\cdot t))}{2\cdot\pi\cdot\left(a^2\cdot\sin(\omega\cdot t)^2 + a^2\cdot\cos(\omega\cdot t)^2\right)} = -\frac{K\cdot\sin(\omega\cdot t)}{2\cdot\pi\cdot a} \qquad (2)$$

We should also be able to find the velocity field as a function of time from the pathline equations (Eq. 2.9):

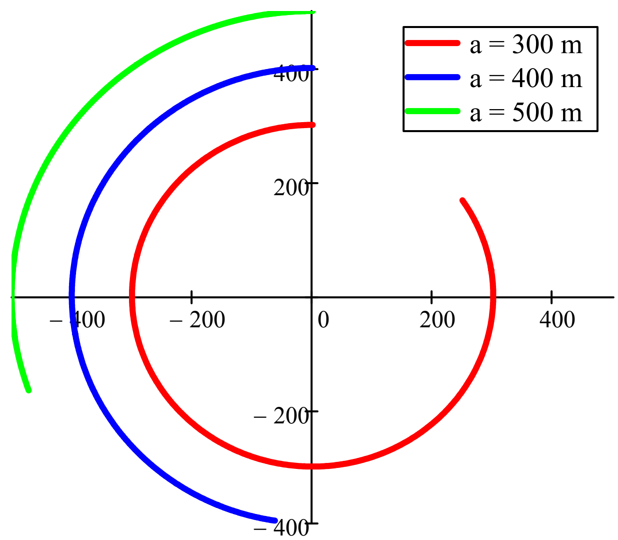
$$\frac{dx_p}{dt} = u \qquad\qquad \frac{dx_p}{dt} = v \qquad (2.9)$$

$$u = \frac{dx_p}{dt} = -a\cdot\omega\cdot\cos(\omega\cdot t) \qquad\qquad v = \frac{dy_p}{dt} = -a\cdot\omega\cdot\sin(\omega\cdot t) \qquad (3)$$

Comparing Eqs. 1, 2 and 3 $\quad u = -a\cdot\omega\cdot\cos(\omega\cdot t) = -\dfrac{K\cdot\cos(\omega\cdot t)}{2\cdot\pi\cdot a}$ $\quad v = -a\cdot\omega\cdot\sin(\omega\cdot t) = -\dfrac{K\cdot\sin(\omega\cdot t)}{2\cdot\pi\cdot a}$

Hence we see that $a\cdot\omega = \dfrac{K}{2\cdot\pi\cdot a}$ or $\omega = \dfrac{K}{2\cdot\pi\cdot a^2}$ for the pathlines to be correct.

The pathlines are



To plot this in Excel, compute x_p and y_p for t ranging from 0 to 60 s, with ω given by the above formula. Plot y_p versus x_p. Note that outer particles travel much slower!

This is the free vortex flow discussed in Example 5.6

The velocity field of Problem 2.11 is $u = -\frac{M \cdot y}{2 \cdot \pi}$ $\quad v = \frac{M \cdot x}{2 \cdot \pi}$

If the pathlines are correct we should be able to substitute x_p and y_p into the velocity field to find the velocity as a function of time:

$$u = -\frac{M \cdot y}{2 \cdot \pi} = -\frac{M \cdot (a \cdot \cos(\omega \cdot t))}{2 \cdot \pi} = -\frac{M \cdot a \cdot \cos(\omega \cdot t)}{2 \cdot \pi} \qquad (4)$$

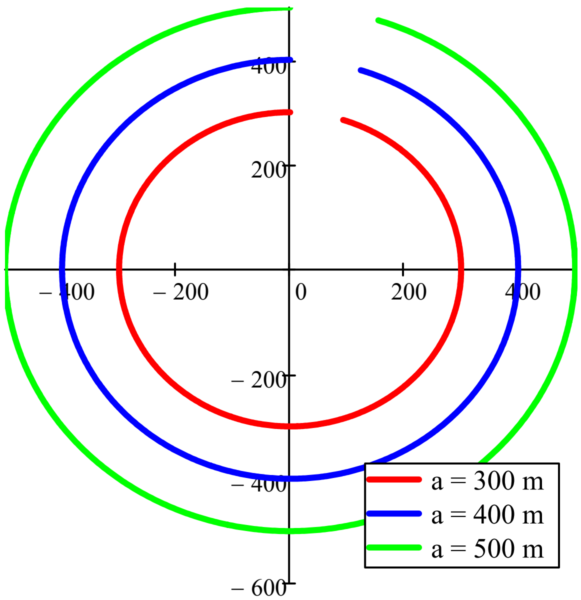
$$v = \frac{M \cdot x}{2 \cdot \pi} = \frac{M \cdot (-a \cdot \sin(\omega \cdot t))}{2 \cdot \pi} = -\frac{M \cdot a \cdot \sin(\omega \cdot t)}{2 \cdot \pi} \qquad (5)$$

Recall that $u = \frac{dx_p}{dt} = -a \cdot \omega \cdot \cos(\omega \cdot t)$ $\quad v = \frac{dy_p}{dt} = -a \cdot \omega \cdot \sin(\omega \cdot t)$ $\quad (3)$

Comparing Eqs. 1, 4 and 5 $\quad u = -a \cdot \omega \cdot \cos(\omega \cdot t) = -\frac{M \cdot a \cdot \cos(\omega \cdot t)}{2 \cdot \pi}$ $\quad v = -a \cdot \omega \cdot \sin(\omega \cdot t) = -\frac{M \cdot a \cdot \sin(\omega \cdot t)}{2 \cdot \pi}$

Hence we see that $\omega = \frac{M}{2 \cdot \pi}$ for the pathlines to be correct.

The pathlines



To plot this in Excel, compute x_p and y_p for t ranging from 0 to 75 s, with ω given by the above formula. Plot y_p versus x_p. Note that outer particles travel faster!

This is the forced vortex flow discussed in Example 5.6

Note that this is rigid body rotation!

Problem 2.18

[Difficulty: 2]

2.18 Air flows downward toward an infinitely wide horizontal flat plate. The velocity field is given by $\vec{V} = (ax\hat{i} - ay\hat{j})(2 + \cos \omega t)$, where $a = 5\ s^{-1}$, $\omega = 2\pi\ s^{-1}$, x and y (measured in meters) are horizontal and vertically upward, respectively, and t is in s. Obtain an algebraic equation for a streamline at $t = 0$. Plot the streamline that passes through point $(x, y) = (3, 3)$ at this instant. Will the streamline change with time? Explain briefly. Show the velocity vector on your plot at the same point and time. Is the velocity vector tangent to the streamline? Explain.

Given: Time-varying velocity field

Find: Streamlines at t = 0 s; Streamline through (3,3); velocity vector; will streamlines change with time

Solution:

For streamlines $$\frac{v}{u} = \frac{dy}{dx} = -\frac{a\cdot y\cdot(2+\cos(\omega\cdot t))}{a\cdot x\cdot(2+\cos(\omega\cdot t))} = -\frac{y}{x}$$

At t = 0 (actually all times!) $$\frac{dy}{dx} = -\frac{y}{x}$$

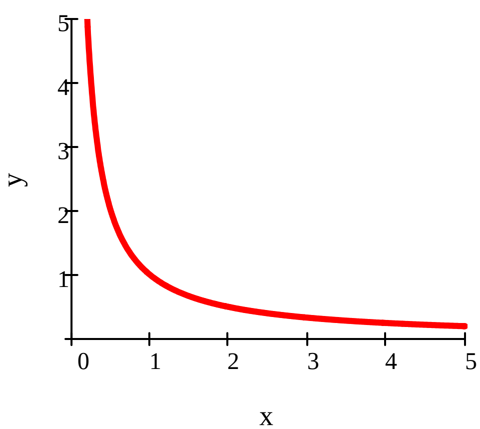
So, separating variables $$\frac{dy}{y} = -\frac{dx}{x}$$

Integrating $$\ln(y) = -\ln(x) + c$$

The solution is $$y = \frac{C}{x}$$ which is the equation of a hyperbola.

For the streamline through point (3,3) $C = \frac{3}{3}$ $\quad C = 1$ and $y = \frac{1}{x}$

The streamlines will not change with time since dy/dx does not change with time.



At t = 0 $$u = a\cdot x\cdot(2+\cos(\omega\cdot t)) = 5\cdot\frac{1}{s}\times 3\cdot m\times 3$$

$$u = 45\cdot\frac{m}{s}$$

$$v = -a\cdot y\cdot(2+\cos(\omega\cdot t)) = 5\cdot\frac{1}{s}\times 3\cdot m\times 3$$

$$v = -45\cdot\frac{m}{s}$$

The velocity vector is tangent to the curve;

Tangent of curve at (3,3) is $\frac{dy}{dx} = -\frac{y}{x} = -1$

Direction of velocity at (3,3) is $\frac{v}{u} = -1$

This curve can be plotted in Excel.

Problem 2.19

[Difficulty: 3]

2.19 Consider the flow described by the velocity field $\vec{V} = A(1 + Bt)\hat{i} + Cty\hat{j}$, with $A = 1$ m/s, $B = 1$ s^{-1}, and $C = 1$ s^{-2}. Coordinates are measured in meters. Plot the pathline traced out by the particle that passes through the point (1, 1) at time $t = 0$. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot of pathline traced out by particle that passes through point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 2s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = A\cdot(1 + B\cdot t)$ $\quad A = 1\cdot\frac{m}{s}$ $\quad B = 1\cdot\frac{1}{s}$ $\quad v_p = \frac{dy}{dt} = C\cdot t\cdot y$ $\quad C = 1\cdot\frac{1}{s^2}$

So, separating variables $dx = A\cdot(1 + B\cdot t)\cdot dt$ $\qquad \frac{dy}{y} = C\cdot t\cdot dt$

Integrating $x = A\cdot\left(t + B\cdot\frac{t^2}{2}\right) + C_1$ $\qquad \ln(y) = \frac{1}{2}\cdot C\cdot t^2 + C_2$

$$y = e^{\frac{1}{2}\cdot C\cdot t^2 + C_2} = e^{C_2}\cdot e^{\frac{1}{2}\cdot C\cdot t^2} = c_2\cdot e^{\frac{1}{2}\cdot C\cdot t^2}$$

The pathlines are $x = A\cdot\left(t + B\cdot\frac{t^2}{2}\right) + C_1$ $\qquad y = c_2\cdot e^{\frac{1}{2}\cdot C\cdot t^2}$

Using given data $x = A\cdot\left(t + B\cdot\frac{t^2}{2}\right) + 1$ $\qquad y = e^{\frac{1}{2}\cdot C\cdot t^2}$

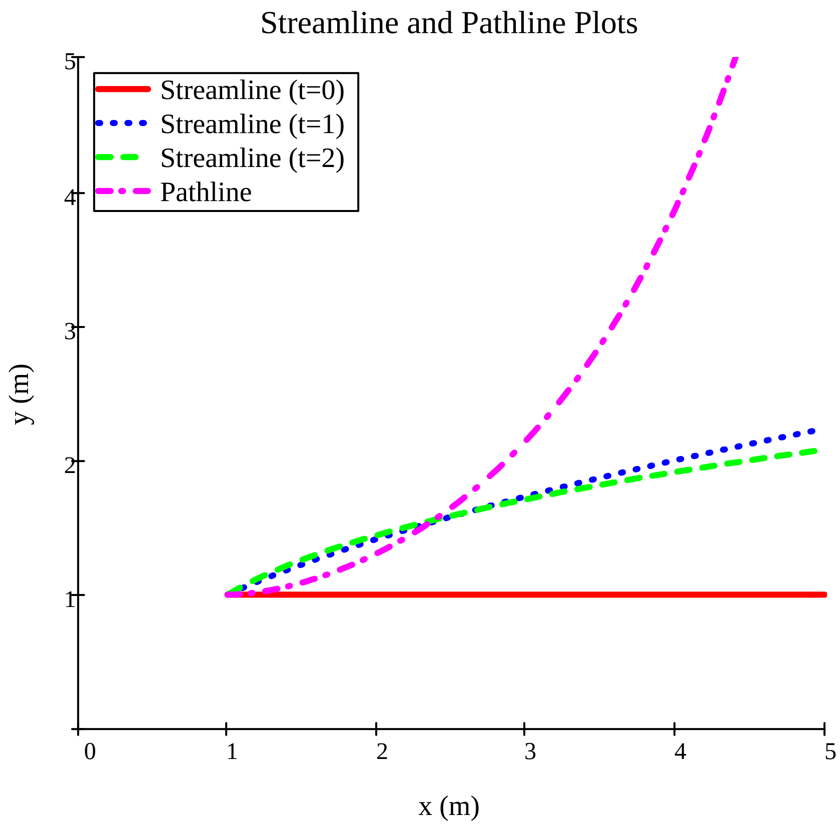
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{C\cdot y\cdot t}{A\cdot(1 + B\cdot t)}$

So, separating variables $(1 + B\cdot t)\cdot\frac{dy}{y} = \frac{C}{A}\cdot t\cdot dx$ $\quad$ which we can integrate for any given t (t is treated as a constant)

Integrating $(1 + B\cdot t)\cdot\ln(y) = \frac{C}{A}\cdot t\cdot x + c$

The solution is $y^{1+B\cdot t} = \frac{C}{A}\cdot t\cdot x + const$ $\qquad y = \left(\frac{C}{A}\cdot t\cdot x + const\right)^{\frac{1}{(1+B\cdot t)}}$

For particles at (1,1) at t = 0, 1, and 2s, using A, B, and C data: $y = 1$ $\quad y = x^{\frac{1}{2}}$ $\quad y = (2 \cdot x - 1)^{\frac{1}{3}}$



Problem 2.20

[Difficulty: 3]

2.20 Consider the flow described by the velocity field $\vec{V} = Bx(1 + At)\hat{i} + Cy\hat{j}$, with $A = 0.5\ s^{-1}$ and $B = C = 1\ s^{-1}$. Coordinates are measured in meters. Plot the pathline traced out by the particle that passes through the point (1, 1) at time $t = 0$. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot of pathline traced out by particle that passes through point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 2s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = B \cdot x \cdot (1 + A \cdot t)$ $\quad A = 0.5 \cdot \frac{1}{s}$ $\quad B = 1 \cdot \frac{1}{s}$ $\quad v_p = \frac{dy}{dt} = C \cdot y$ $\quad C = 1 \cdot \frac{1}{s}$

So, separating variables $\frac{dx}{x} = B \cdot (1 + A \cdot t) \cdot dt$ $\qquad \frac{dy}{y} = C \cdot dt$

Integrating $\ln(x) = B \cdot \left(t + A \cdot \frac{t^2}{2}\right) + C_1$ $\qquad \ln(y) = C \cdot t + C_2$

$$x = e^{B \cdot \left(t + A \cdot \frac{t^2}{2}\right) + C_1} = e^{C_1} \cdot e^{B \cdot \left(t + A \cdot \frac{t^2}{2}\right)} = c_1 \cdot e^{B \cdot \left(t + A \cdot \frac{t^2}{2}\right)} \qquad y = e^{C \cdot t + C_2} = e^{C_2} \cdot e^{C \cdot t} = c_2 \cdot e^{C \cdot t}$$

The pathlines are $x = c_1 \cdot e^{B \cdot \left(t + A \cdot \frac{t^2}{2}\right)}$ $\qquad y = c_2 \cdot e^{C \cdot t}$

Using given data $x = e^{B \cdot \left(t + A \cdot \frac{t^2}{2}\right)}$ $\qquad y = e^{C \cdot t}$

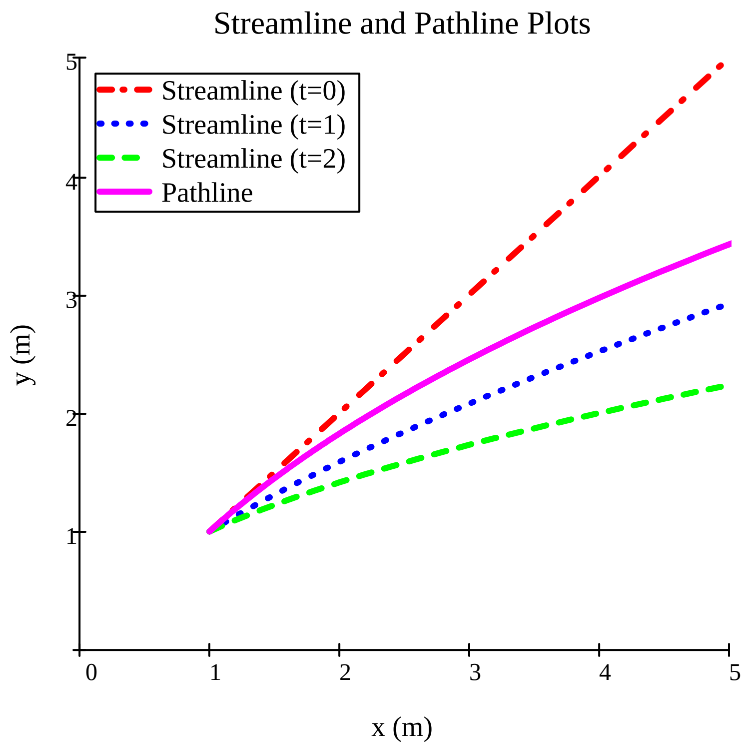
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{C \cdot y}{B \cdot x \cdot (1 + A \cdot t)}$

So, separating variables $(1 + A \cdot t) \cdot \frac{dy}{y} = \frac{C}{B} \cdot \frac{dx}{x}$ $\quad$ which we can integrate for any given t (t is treated as a constant)

Integrating $(1 + A \cdot t) \cdot \ln(y) = \frac{C}{B} \cdot \ln(x) + c$

The solution is $y^{1+A\cdot t} = \text{const}\cdot x^{\frac{C}{B}}$ or $y = \text{const}\cdot x^{\frac{C}{B(1+A\cdot t)}}$

For particles at (1,1) at t = 0, 1, and 2s $y = x^{\frac{C}{B}}$ $y = x^{\frac{C}{(1+A)B}}$ $y = x^{\frac{C}{(1+2\cdot A)B}}$



Problem 2.21

[Difficulty: 3]

2.21 Consider the flow field given in Eulerian description by the expression $\vec{V} = A\hat{i} - Bt\hat{j}$, where $A = 2$ m/s, $B = 2$ m/s^2, and the coordinates are measured in meters. Derive the Lagrangian position functions for the fluid particle that was located at the point $(x, y) = (1, 1)$ at the instant $t = 0$. Obtain an algebraic expression for the pathline followed by this particle. Plot the pathline and compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Eulerian Velocity field

Find: Lagrangian position function that was at point (1,1) at t = 0; expression for pathline; plot pathline and compare to streamlines through same point at the instants t = 0, 1 and 2s

Solution:

Governing equations: For pathlines (Lagrangian description) $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = A$ $A = 2\ \frac{m}{s}$ $v_p = \frac{dy}{dt} = -B\cdot t$ $B = 2\ \frac{m}{s^2}$

So, separating variables $dx = A\cdot dt$ $dy = -B\cdot t\cdot dt$

Integrating $x = A\cdot t + x_0$ $x_0 = 1\ m$ $y = -B\cdot\frac{t^2}{2} + y_0$ $y_0 = 1\ m$

The Lagrangian description is $x(t) = A\cdot t + x_0$ $y(t) = -B\cdot\frac{t^2}{2} + y_0$

Using given data $x(t) = 2\cdot t + 1$ $y(t) = 1 - t^2$

The pathlines are given by combining the equations $t = \frac{x - x_0}{A}$ $y = -B\cdot\frac{t^2}{2} + y_0 = -B\cdot\frac{\left(x - x_0\right)^2}{2\cdot A^2} + y_0$

Hence $y(x) = y_0 - B\cdot\frac{\left(x - x_0\right)^2}{2\cdot A^2}$ or, using given data $y(x) = 1 - \frac{(x - 1)^2}{4}$

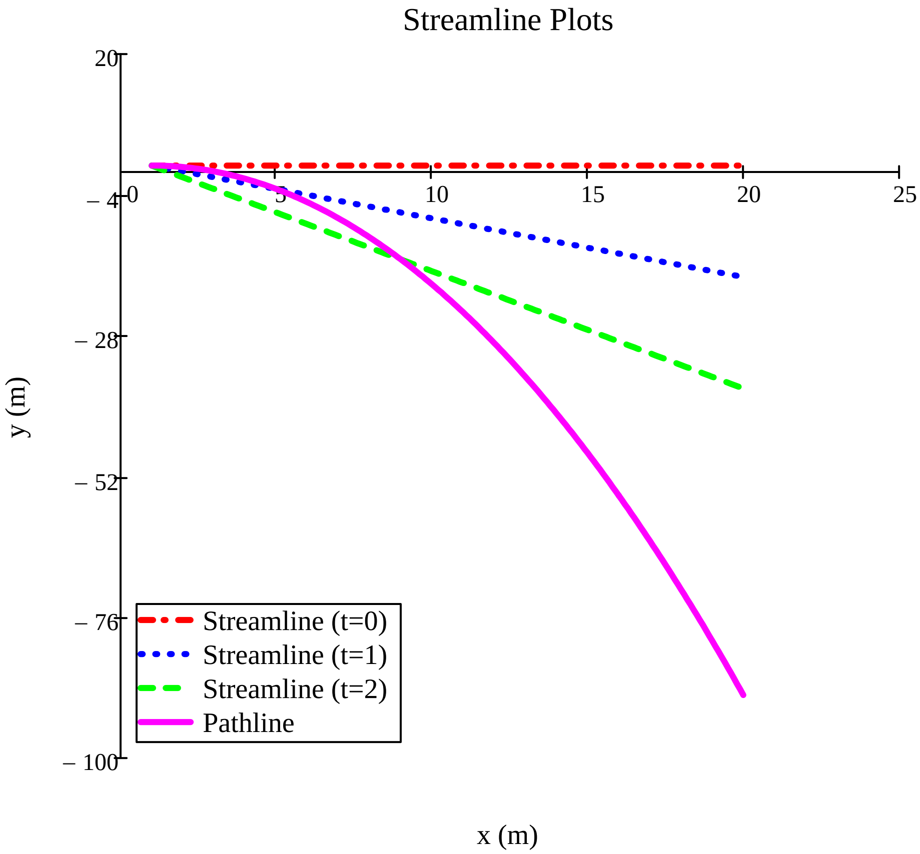
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{-B\cdot t}{A}$

So, separating variables $dy = -\frac{B\cdot t}{A}\cdot dx$ which we can integrate for any given t (t is treated as a constant)

The solution is $y = -\frac{B \cdot t}{A} \cdot x + c$ and for the one through (1,1) $1 = -\frac{B \cdot t}{A} \cdot 1 + c$ $c = 1 + \frac{B \cdot t}{A}$

$$y = -\frac{B \cdot t}{A} \cdot (x - 1) + 1 \qquad y = 1 - t \cdot (x - 1)$$

$$x = 1, 1.1 .. 20$$



Problem 2.22

[Difficulty: 3]

2.22 Consider the velocity field $V = ax\hat{i} + by(1 + ct)\hat{j}$, where $a = b = 2\ s^{-1}$ and $c = 0.4\ s^{-1}$. Coordinates are measured in meters. For the particle that passes through the point $(x, y) = (1, 1)$ at the instant $t = 0$, plot the pathline during the interval from $t = 0$ to 1.5 s. Compare this pathline with the streamlines plotted through the same point at the instants $t = 0$, 1, and 1.5 s.

Given: Velocity field

Find: Plot of pathline of particle for t = 0 to 1.5 s that was at point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 1.5 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = ax$ $a = 2\ \frac{1}{s}$ $v_p = \frac{dy}{dt} = b \cdot y \cdot (1 + c \cdot t)$ $b = 2\ \frac{1}{s^2}$ $c = 0.4\ \frac{1}{s}$

So, separating variables $\frac{dx}{x} = a \cdot dt$ $dy = b \cdot y \cdot (1 + c \cdot t) \cdot dt$ $\frac{dy}{y} = b \cdot (1 + c \cdot t) \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a \cdot t$ $x_0 = 1\ m$ $\ln\left(\frac{y}{y_0}\right) = b \cdot \left(t + \frac{1}{2} \cdot c \cdot t^2\right)$ $y_0 = 1\ m$

Hence $x(t) = x_0 \cdot e^{a \cdot t}$ $y(t) = e^{b \cdot \left(t + \frac{1}{2} \cdot c \cdot t^2\right)}$

Using given data $x(t) = e^{2 \cdot t}$ $y(t) = e^{2 \cdot t + 0.4 \cdot t^2}$

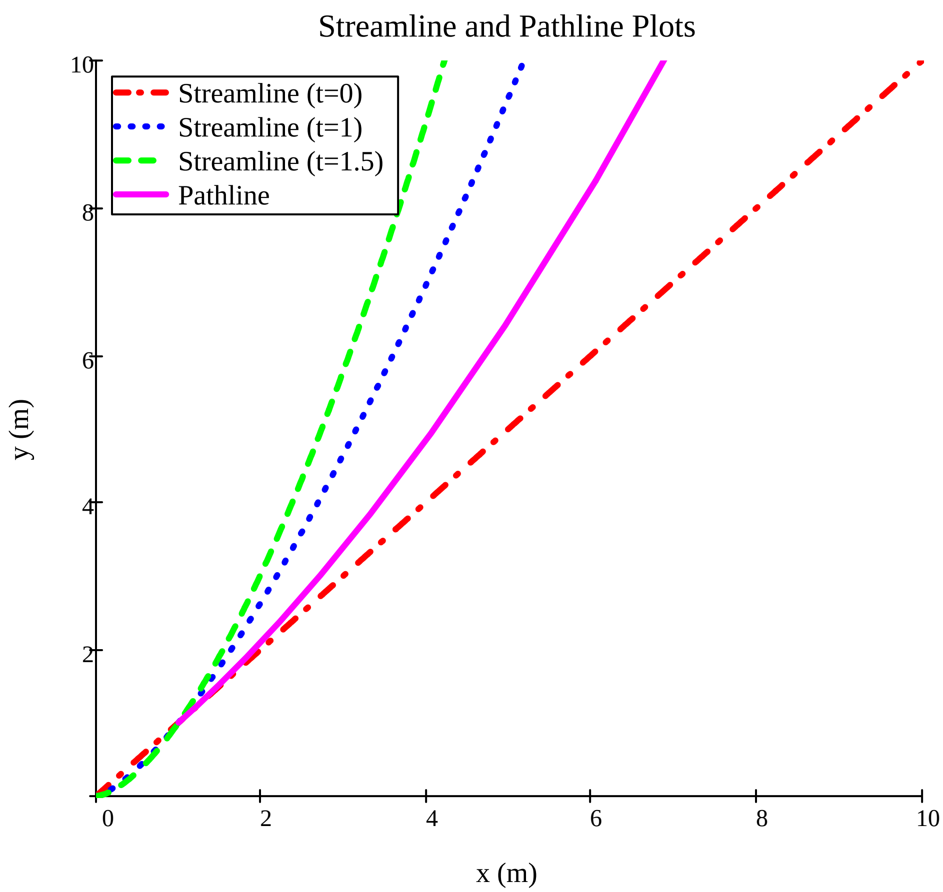
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot y \cdot (1 + c \cdot t)}{a \cdot x}$

So, separating variables $\frac{dy}{y} = \frac{b \cdot (1 + c \cdot t)}{a \cdot x} \cdot dx$ which we can integrate for any given t (t is treated as a constant)

Hence $\ln\left(\frac{y}{y_0}\right) = \frac{b}{a} \cdot (1 + c \cdot t) \cdot \ln\left(\frac{x}{x_0}\right)$

The solution is $y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot (1 + c \cdot t)}$

For $t = 0$ $\quad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a}\cdot(1+c\cdot t)} = x$ $\quad t = 1$ $\quad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a}\cdot(1+c\cdot t)} = x^{1.4}$ $\quad t = 1.5$ $\quad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a}\cdot(1+c\cdot t)} = x^{1.6}$



Problem 2.23

[Difficulty: 3]

2.23 Consider the flow field given in Eulerian description by the expression $\vec{V} = ax\hat{i} + byt\hat{j}$, where $a = 0.2\ s^{-1}$, $b = 0.04\ s^{-2}$, and the coordinates are measured in meters. Derive the Lagrangian position functions for the fluid particle that was located at the point $(x, y) = (1, 1)$ at the instant $t = 0$. Obtain an algebraic expression for the pathline followed by this particle. Plot the pathline and compare with the streamlines plotted through the same point at the instants $t = 0$, 10, and 20 s.

Given: Velocity field

Find: Plot of pathline of particle for t = 0 to 1.5 s that was at point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 1.5 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = a \cdot x$ $\quad a = \frac{1}{5}\frac{1}{s}$ $\quad v_p = \frac{dy}{dt} = b \cdot y \cdot t$ $\quad b = \frac{1}{25}\frac{1}{s^2}$

So, separating variables $\frac{dx}{x} = a \cdot dt$ $\quad dy = b \cdot y \cdot t \cdot dt$ $\quad \frac{dy}{y} = b \cdot t \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a \cdot t$ $\quad x_0 = 1\ m$ $\quad \ln\left(\frac{y}{y_0}\right) = b \cdot \frac{1}{2} \cdot t^2$ $\quad y_0 = 1\ m$

Hence $x(t) = x_0 \cdot e^{a \cdot t}$ $\quad y(t) = y_0 \cdot e^{\frac{1}{2} \cdot b \cdot t^2}$

Using given data $x(t) = e^{\frac{t}{5}}$ $\quad y(t) = e^{\frac{t^2}{50}}$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot y \cdot t}{a \cdot x}$

So, separating variables $\frac{dy}{y} = \frac{b \cdot t}{a \cdot x} \cdot dx$ $\quad$ which we can integrate for any given t (t is treated as a constant)

Hence $\ln\left(\frac{y}{y_0}\right) = \frac{b}{a} \cdot t \cdot \ln\left(\frac{x}{x_0}\right)$

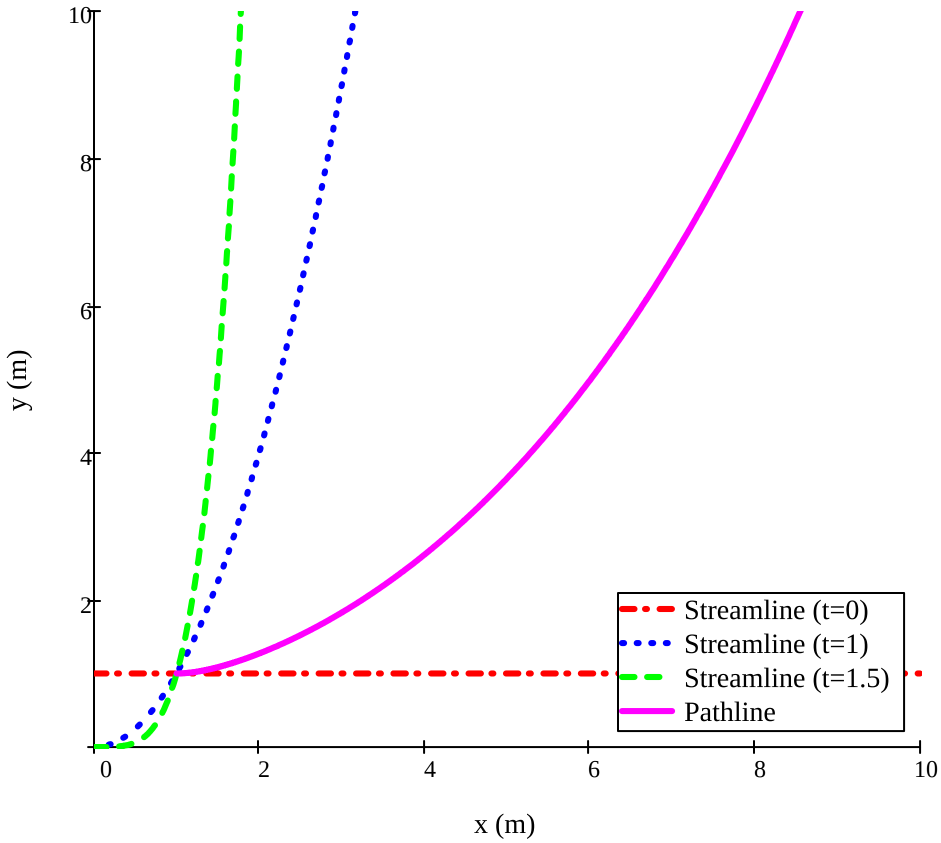
The solution is $y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t}$ $\quad \frac{b}{a} = 0.2$ $\quad x_0 = 1$ $\quad y_0 = 1$

For $t = 0$ $\qquad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t} = 1$

$t = 5$ $\qquad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t} = x \qquad \frac{b}{a} \cdot t = 1$

$t = 10$ $\qquad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t} = x^2 \qquad \frac{b}{a} \cdot t = 2$

Streamline and Pathline Plots



Problem 2.24

[Difficulty: 3]

2.24 A velocity field is given by $\vec{V} = axt\hat{i} - by\hat{j}$, where $a = 0.1\ \text{s}^{-2}$ and $b = 1\ \text{s}^{-1}$. For the particle that passes through the point $(x, y) = (1, 1)$ at instant $t = 0$ s, plot the pathline during the interval from $t = 0$ to $t = 3$ s. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot pathlines and streamlines

Solution:

Pathlines are given by $\frac{dx}{dt} = u = a\cdot x\cdot t$ $\quad \frac{dy}{dt} = v = -b\cdot y$

So, separating variables $\frac{dx}{x} = a\cdot t\cdot dt$ $\quad \frac{dy}{y} = -b\cdot dt$

Integrating $\ln(x) = \frac{1}{2}\cdot a\cdot t^2 + c_1$ $\quad \ln(y) = -b\cdot t + c_2$

For initial position (x_0, y_0) $\quad x = x_0\cdot e^{\frac{a}{2}\cdot t^2}$ $\quad y = y_0\cdot e^{-b\cdot t}$

Using the given data, and IC $(x_0, y_0) = (1,1)$ at $t = 0$

$$x = e^{0.05\cdot t^2} \qquad y = e^{-t}$$

Streamlines are given by $\frac{v}{u} = \frac{dy}{dx} = \frac{-b\cdot y}{a\cdot x\cdot t}$

So, separating variables $\frac{dy}{y} = -\frac{b}{a\cdot t}\cdot\frac{dx}{x}$ $\quad$ Integrating $\quad \ln(y) = -\frac{b}{a\cdot t}\cdot\ln(x) + C$

The solution is $y = C\cdot x^{-\frac{b}{a\cdot t}}$

For streamline at (1,1) at $t = 0$ s $\quad x = c$

For streamline at (1,1) at $t = 1$ s $\quad y = x^{-10}$

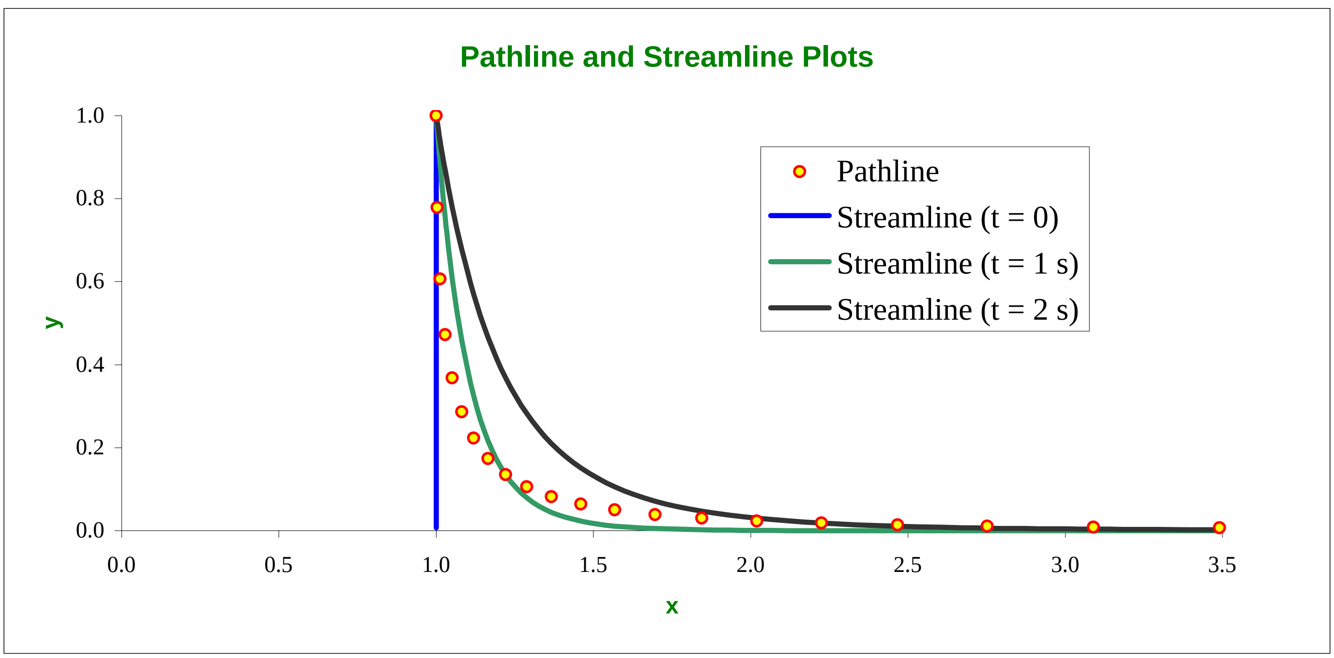
For streamline at (1,1) at $t = 2$ s $\quad y = x^{-5}$

Pathline

t	x	y
0.00	1.00	1.00
0.25	1.00	0.78
0.50	1.01	0.61
0.75	1.03	0.47
1.00	1.05	0.37
1.25	1.08	0.29
1.50	1.12	0.22
1.75	1.17	0.17
2.00	1.22	0.14
2.25	1.29	0.11
2.50	1.37	0.08
2.75	1.46	0.06
3.00	1.57	0.05
3.25	1.70	0.04
3.50	1.85	0.03
3.75	2.02	0.02
4.00	2.23	0.02
4.25	2.47	0.01
4.50	2.75	0.01
4.75	3.09	0.01
5.00	3.49	0.01

Streamlines

t = 0		t = 1 s		t = 2 s	
x	y	x	y	x	y
1.00	1.00	1.00	1.00	1.00	1.00
1.00	0.78	1.00	0.97	1.00	0.98
1.00	0.61	1.01	0.88	1.01	0.94
1.00	0.47	1.03	0.75	1.03	0.87
1.00	0.37	1.05	0.61	1.05	0.78
1.00	0.29	1.08	0.46	1.08	0.68
1.00	0.22	1.12	0.32	1.12	0.57
1.00	0.17	1.17	0.22	1.17	0.47
1.00	0.14	1.22	0.14	1.22	0.37
1.00	0.11	1.29	0.08	1.29	0.28
1.00	0.08	1.37	0.04	1.37	0.21
1.00	0.06	1.46	0.02	1.46	0.15
1.00	0.05	1.57	0.01	1.57	0.11
1.00	0.04	1.70	0.01	1.70	0.07
1.00	0.03	1.85	0.00	1.85	0.05
1.00	0.02	2.02	0.00	2.02	0.03
1.00	0.02	2.23	0.00	2.23	0.02
1.00	0.01	2.47	0.00	2.47	0.01
1.00	0.01	2.75	0.00	2.75	0.01
1.00	0.01	3.09	0.00	3.09	0.00
1.00	0.01	3.49	0.00	3.49	0.00



Problem 2.25

[Difficulty: 3]

2.25 Consider the flow field $\vec{V} = axt\hat{i} + b\hat{j}$, where $a = 0.1\ \text{s}^{-2}$ and $b = 4$ m/s. Coordinates are measured in meters. For the particle that passes through the point $(x, y) = (3, 1)$ at the instant $t = 0$, plot the pathline during the interval from $t = 0$ to 3 s. Compare this pathline with the streamlines plotted through the same point at the instants $t = 1$, 2, and 3 s.

Given: Flow field

Find: Pathline for particle starting at (3,1); Streamlines through same point at t = 1, 2, and 3 s

Solution:

For particle paths $\frac{dx}{dt} = u = a \cdot x \cdot t$ and $\frac{dy}{dt} = v = b$

Separating variables and integrating $\frac{dx}{x} = a \cdot t \cdot dt$ or $\ln(x) = \frac{1}{2} \cdot a \cdot t^2 + c_1$

$dy = b \cdot dt$ or $y = b \cdot t + c_2$

Using initial condition (x,y) = (3,1) and the given values for a and b

$c_1 = \ln(3 \cdot m)$ and $c_2 = 1 \cdot m$

The pathline is then $x = 3 \cdot e^{0.05 \cdot t^2}$ and $y = 4 \cdot t + 1$

For streamlines (at any time t) $\frac{v}{u} = \frac{dy}{dx} = \frac{b}{a \cdot x \cdot t}$

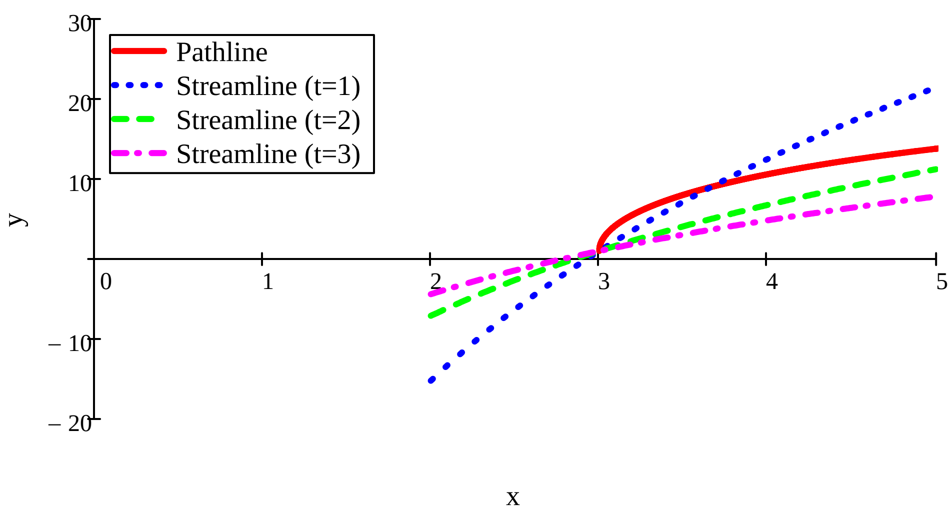
So, separating variables $dy = \frac{b}{a \cdot t} \cdot \frac{dx}{x}$

Integrating $y = \frac{b}{a \cdot t} \cdot \ln(x) + c$

We are interested in instantaneous streamlines at various times that always pass through point (3,1). Using a and b values:

$$c = y - \frac{b}{a \cdot t} \cdot \ln(x) = 1 - \frac{4}{0.1 \cdot t} \cdot \ln(3)$$

The streamline equation is $y = 1 + \frac{40}{t} \cdot \ln\left(\frac{x}{3}\right)$



These curves can be plotted in Excel.

Problem 2.26

[Difficulty: 4]

2.26 Consider the garden hose of Fig. 2.5. Suppose the velocity field is given by $\vec{V} = u_0\hat{i} + v_0\sin[\omega(t - x/u_0)]\hat{j}$, where the x direction is horizontal and the origin is at the mean position of the hose, $u_0 = 10$ m/s, $v_0 = 2$ m/s, and $\omega = 5$ cycle/s. Find and plot on one graph the instantaneous streamlines that pass through the origin at $t = 0$ s, 0.05 s, 0.1 s, and 0.15 s. Also find and plot on one graph the pathlines of particles that left the origin at the same four times.

Given: Velocity field

Find: Plot streamlines that are at origin at various times and pathlines that left origin at these times

Solution:

For streamlines
$$\frac{v}{u} = \frac{dy}{dx} = \frac{v_0 \cdot \sin\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right]}{u_0}$$

So, separating variables (t=const)
$$dy = \frac{v_0 \cdot \sin\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right]}{u_0} \cdot dx$$

Integrating
$$y = \frac{v_0 \cdot \cos\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right]}{\omega} + c$$

Using condition y = 0 when x = 0
$$y = \frac{v_0 \cdot \left[\cos\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right] - \cos(\omega \cdot t)\right]}{\omega}$$
This gives streamlines y(x) at each time t

For particle paths, first find x(t)
$$\frac{dx}{dt} = u = u_0$$

Separating variables and integrating
$$dx = u_0 \cdot dt \qquad \text{or} \qquad x = u_0 \cdot t + c_1$$

Using initial condition x = 0 at t = τ
$$c_1 = -u_0 \cdot \tau \qquad x = u_0 \cdot (t - \tau)$$

For y(t) we have
$$\frac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right] \qquad \text{so} \qquad \frac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left[t - \frac{u_0 \cdot (t - \tau)}{u_0}\right]\right]$$

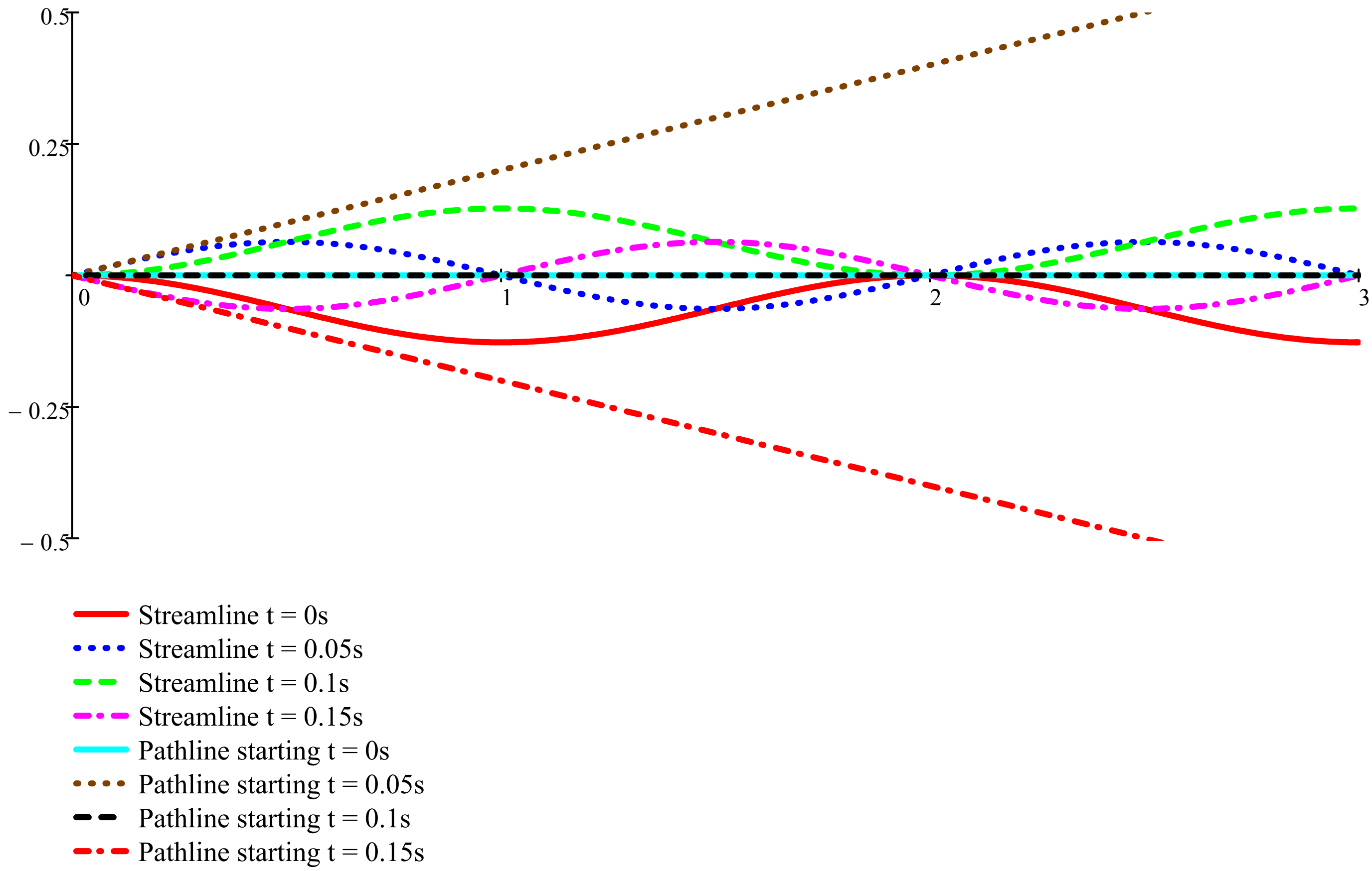
and
$$\frac{dy}{dt} = v = v_0 \cdot \sin(\omega \cdot \tau)$$

Separating variables and integrating
$$dy = v_0 \cdot \sin(\omega \cdot \tau) \cdot dt \qquad y = v_0 \cdot \sin(\omega \cdot \tau) \cdot t + c_2$$

Using initial condition y = 0 at t = τ
$$c_2 = -v_0 \cdot \sin(\omega \cdot \tau) \cdot \tau \qquad y = v_0 \cdot \sin(\omega \cdot \tau) \cdot (t - \tau)$$

The pathline is then

$$x(t,\tau) = u_0 \cdot (t - \tau) \qquad y(t,\tau) = v_0 \cdot \sin(\omega \cdot \tau) \cdot (t - \tau)$$
These terms give the path of a particle (x(t),y(t)) that started at t = τ.



The streamlines are sinusoids; the pathlines are straight (once a water particle is fired it travels in a straight line).

These curves can be plotted in *Excel*.

Problem 2.27

[Difficulty: 5]

2.27 Using the data of Problem 2.26, find and plot the streakline shape produced after the first second of flow.

Given: Velocity field

Find: Plot streakline for first second of flow

Solution:

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \qquad \text{and} \qquad y_p(t) = y(t, x_0, y_0, t_0)$$

where x_0, y_0 is the position of the particle at $t = t_0$, and re-interprete the results as streaklines

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \qquad \text{and} \qquad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For particle paths, first find x(t) $\quad \frac{dx}{dt} = u = u_0$

Separating variables and integrating $\quad dx = u_0 \cdot dt \quad$ or $\quad x = x_0 + u_0 \cdot (t - t_0)$

For y(t) we have $\quad \frac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right] \quad$ so $\quad \frac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left[t - \frac{x_0 + u_0 \cdot (t - t_0)}{u_0}\right]\right]$

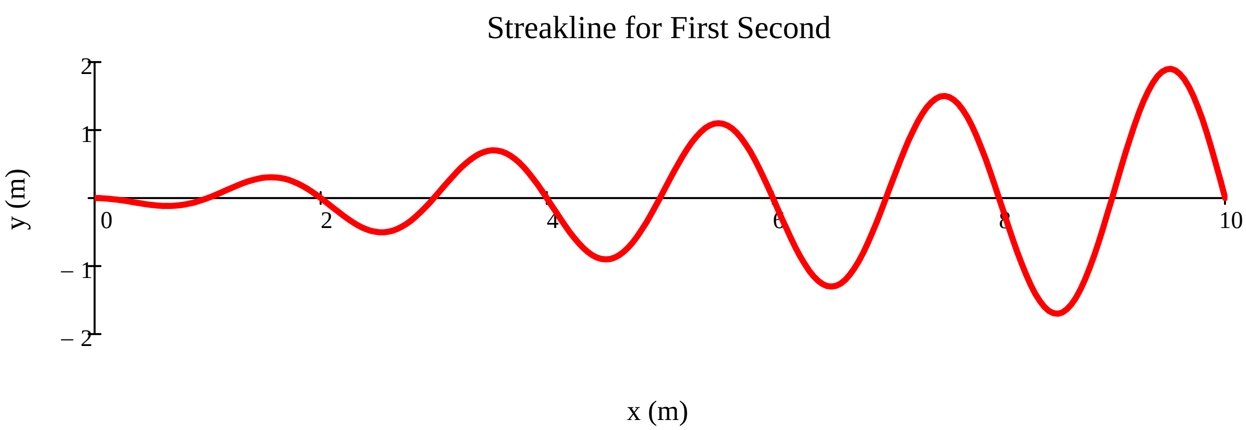
and $\quad \frac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \frac{x_0}{u_0}\right)\right]$

Separating variables and integrating $\quad dy = v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \frac{x_0}{u_0}\right)\right] \cdot dt \qquad y = y_0 + v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \frac{x_0}{u_0}\right)\right] \cdot (t - t_0)$

The streakline is then $\quad x_{st}(t_0) = x_0 + u_0 (t - t_0) \qquad y_{st}(t_0) = y_0 + v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \frac{x_0}{u_0}\right)\right] \cdot (t - t_0)$

With $\quad x_0 = y_0 = 0$

$$x_{st}(t_0) = u_0 \cdot (t - t_0) \qquad y_{st}(t_0) = v_0 \cdot \sin[\omega \cdot (t_0)] \cdot (t - t_0)$$



This curve can be plotted in *Excel*. For t = 1, t_0 ranges from 0 to t.

Problem 2.28

[Difficulty: 4]

2.28 Consider the velocity field of Problem 2.20. Plot the streakline formed by particles that passed through the point (1, 1) during the interval from $t = 0$ to $t = 3$ s. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot of streakline for t = 0 to 3 s at point (1,1); compare to streamlines through same point at the instants t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines $u_p = \frac{dx}{dt} = B\cdot x\cdot(1 + A\cdot t)$ $A = 0.5\ \frac{1}{s}$ $B = 1\ \frac{1}{s}$ $v_p = \frac{dy}{dt} = C\cdot y$ $C = 1\ \frac{1}{s}$

So, separating variables $\frac{dx}{x} = B\cdot(1 + A\cdot t)\cdot dt$ $\frac{dy}{y} = C\cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = B\cdot\left(t - t_0 + A\cdot\frac{t^2 - t_0^{\,2}}{2}\right)$ $\ln\left(\frac{y}{y_0}\right) = C\cdot(t - t_0)$

$$x = x_0\cdot e^{B\cdot\left(t-t_0+A\cdot\frac{t^2-t_0^{\,2}}{2}\right)} \qquad y = y_0\cdot e^{C\cdot(t-t_0)}$$

The pathlines are

$$x_p(t) = x_0\cdot e^{B\cdot\left(t-t_0+A\cdot\frac{t^2-t_0^{\,2}}{2}\right)} \qquad y_p(t) = y_0\cdot e^{C\cdot(t-t_0)}$$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The streaklines are then

$$x_{st}(t_0) = x_0\cdot e^{B\cdot\left(t-t_0+A\cdot\frac{t^2-t_0^{\,2}}{2}\right)} \qquad y_{st}(t_0) = y_0\cdot e^{C\cdot(t-t_0)}$$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

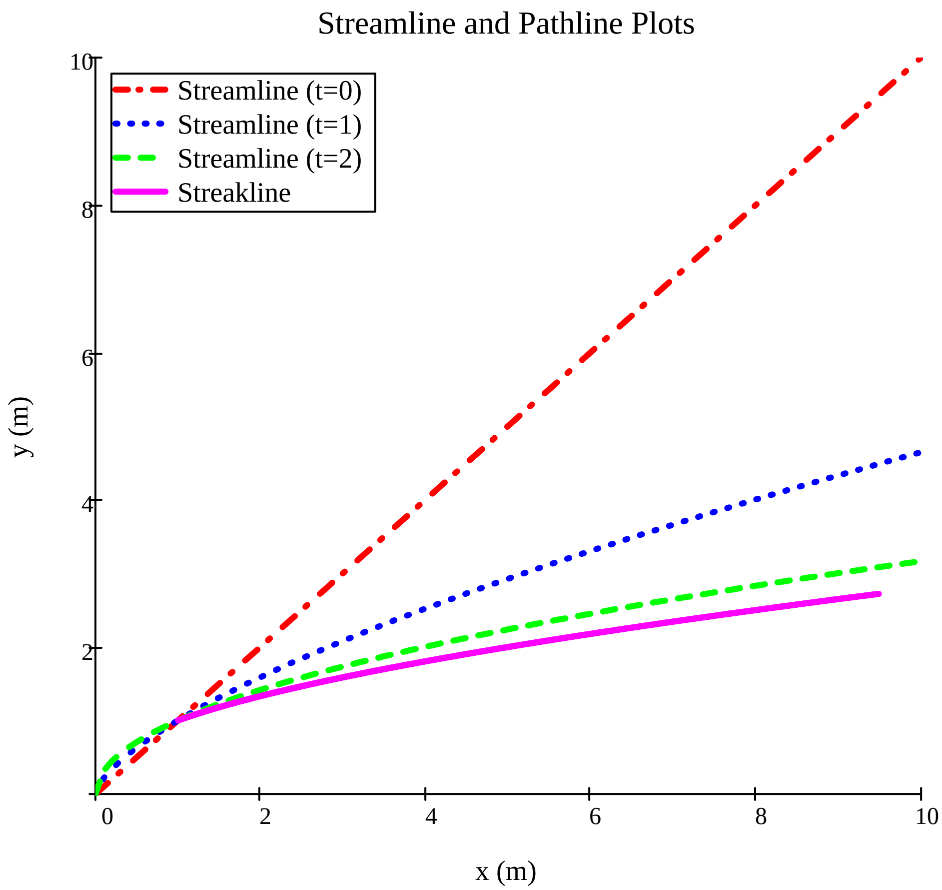
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{C \cdot y}{B \cdot x \cdot (1 + A \cdot t)}$

So, separating variables $(1 + A \cdot t) \cdot \frac{dy}{y} = \frac{C}{B} \cdot \frac{dx}{x}$ which we can integrate for any given t (t is treated as a constant)

Integrating $(1 + A \cdot t) \cdot \ln(y) = \frac{C}{B} \cdot \ln(x) + const$

The solution is $y^{1+A \cdot t} = const \cdot x^{\frac{C}{B}}$

For particles at (1,1) at t = 0, 1, and 2s $y = x$ $y = x^{\frac{2}{3}}$ $y = x^{\frac{1}{2}}$



Problem 2.29

[Difficulty: 4]

2.29 Streaklines are traced out by neutrally buoyant marker fluid injected into a flow field from a fixed point in space. A particle of the marker fluid that is at point (x, y) at time t must have passed through the injection point (x_0, y_0) at some earlier instant $t = \tau$. The time history of a marker particle may be found by solving the pathline equations for the initial conditions that $x = x_0$, $y = y_0$ when $t = \tau$. The present locations of particles on the streakline are obtained by setting τ equal to values in the range $0 \le \tau \le t$. Consider the flow field $\vec{V} = ax(1 + bt)\hat{i} + cy\hat{j}$, where $a = c = 1\ \text{s}^{-1}$ and $b = 0.2\ \text{s}^{-1}$. Coordinates are measured in meters. Plot the streakline that passes through the initial point $(x_0, y_0) = (1, 1)$, during the interval from $t = 0$ to $t = 3$ s. Compare with the streamline plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot of streakline for t = 0 to 3 s at point (1,1); compare to streamlines through same point at the instants t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines $u_p = \frac{dx}{dt} = a\cdot x\cdot(1 + b\cdot t)$ $\quad a = 1\ \frac{1}{s}$ $\quad b = \frac{1}{5}\ \frac{1}{s}$ $\quad v_p = \frac{dy}{dt} = c\cdot y$ $\quad c = 1\ \frac{1}{s}$

So, separating variables $\frac{dx}{x} = a\cdot(1 + b\cdot t)\cdot dt$ $\qquad \frac{dy}{y} = c\cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a\cdot\left(t - t_0 + b\cdot\frac{t^2 - t_0^{\,2}}{2}\right)$ $\qquad \ln\left(\frac{y}{y_0}\right) = c\cdot(t - t_0)$

$$x = x_0\cdot e^{a\cdot\left(t - t_0 + b\cdot\frac{t^2 - t_0^{\,2}}{2}\right)} \qquad y = y_0\cdot e^{c\cdot(t - t_0)}$$

The pathlines are $x_p(t) = x_0 \cdot e^{a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^{\,2}}{2}\right)}$ $\quad y_p(t) = y_0 \cdot e^{c \cdot (t - t_0)}$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The streaklines are then $x_{st}(t_0) = x_0 \cdot e^{a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^{\,2}}{2}\right)}$ $\quad y_{st}(t_0) = y_0 \cdot e^{c \cdot (t - t_0)}$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

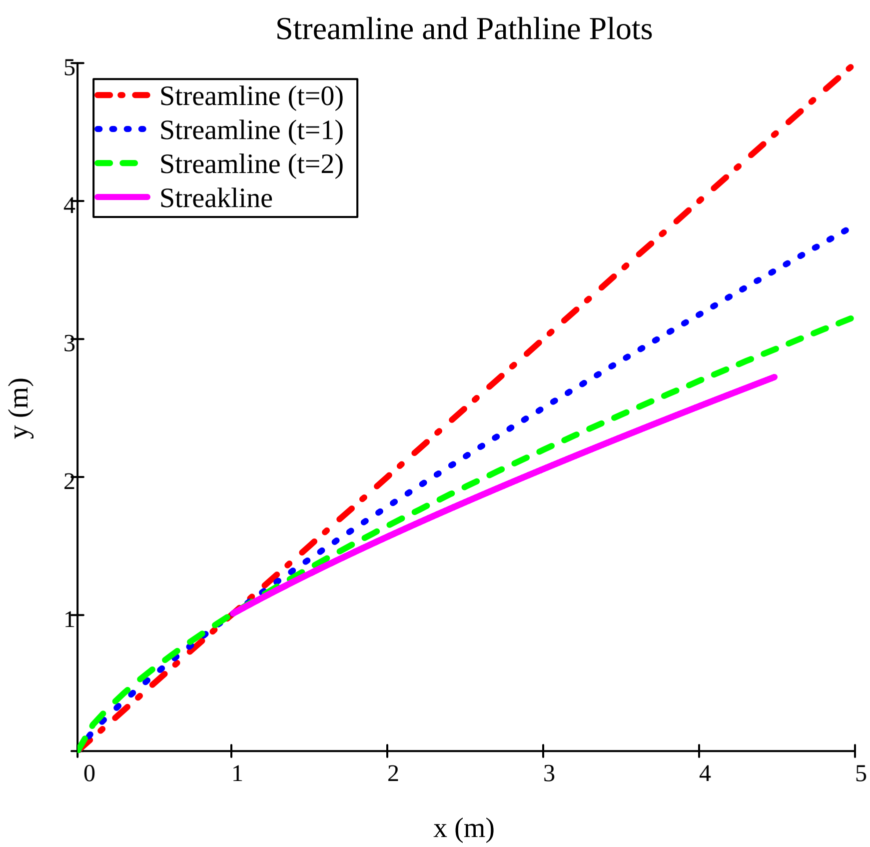
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{c \cdot y}{a \cdot x \cdot (1 + b \cdot t)}$

So, separating variables $(1 + b \cdot t) \cdot \frac{dy}{y} = \frac{c}{a} \cdot \frac{dx}{x}$ which we can integrate for any given t (t is treated as a constant)

Integrating $(1 + b \cdot t) \cdot \ln(y) = \frac{c}{a} \cdot \ln(x) + \text{const}$

The solution is $y^{1+b \cdot t} = \text{const} \cdot x^{\frac{c}{a}}$

For particles at (1,1) at t = 0, 1, and 2s $\quad y = x \quad y = x^{\frac{2}{3}} \quad y = x^{\frac{1}{2}}$



Problem 2.30

[Difficulty: 4]

2.30 Consider the flow field $\vec{V} = axt\hat{i} + b\hat{j}$, where $a = 1/4\ \text{s}^{-2}$ and $b = 1/3$ m/s. Coordinates are measured in meters. For the particle that passes through the point $(x, y) = (1, 2)$ at the instant $t = 0$, plot the pathline during the time interval from $t = 0$ to 3 s. Compare this pathline with the streakline through the same point at the instant $t = 3$ s.

Given: Velocity field

Find: Plot of pathline for t = 0 to 3 s for particle that started at point (1,2) at t = 0; compare to streakline through same point at the instant t = 3

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines $u_p = \frac{dx}{dt} = a \cdot x \cdot t$ $\quad a = \frac{1}{4} \frac{1}{s^2}$ $\quad b = \frac{1}{3} \frac{m}{s}$ $\quad v_p = \frac{dy}{dt} = b$

So, separating variables $\frac{dx}{x} = a \cdot t \cdot dt$ $\quad dy = b \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = \frac{a}{2} \cdot \left(t^2 - t_0^{\ 2}\right)$ $\quad y - y_0 = b \cdot (t - t_0)$

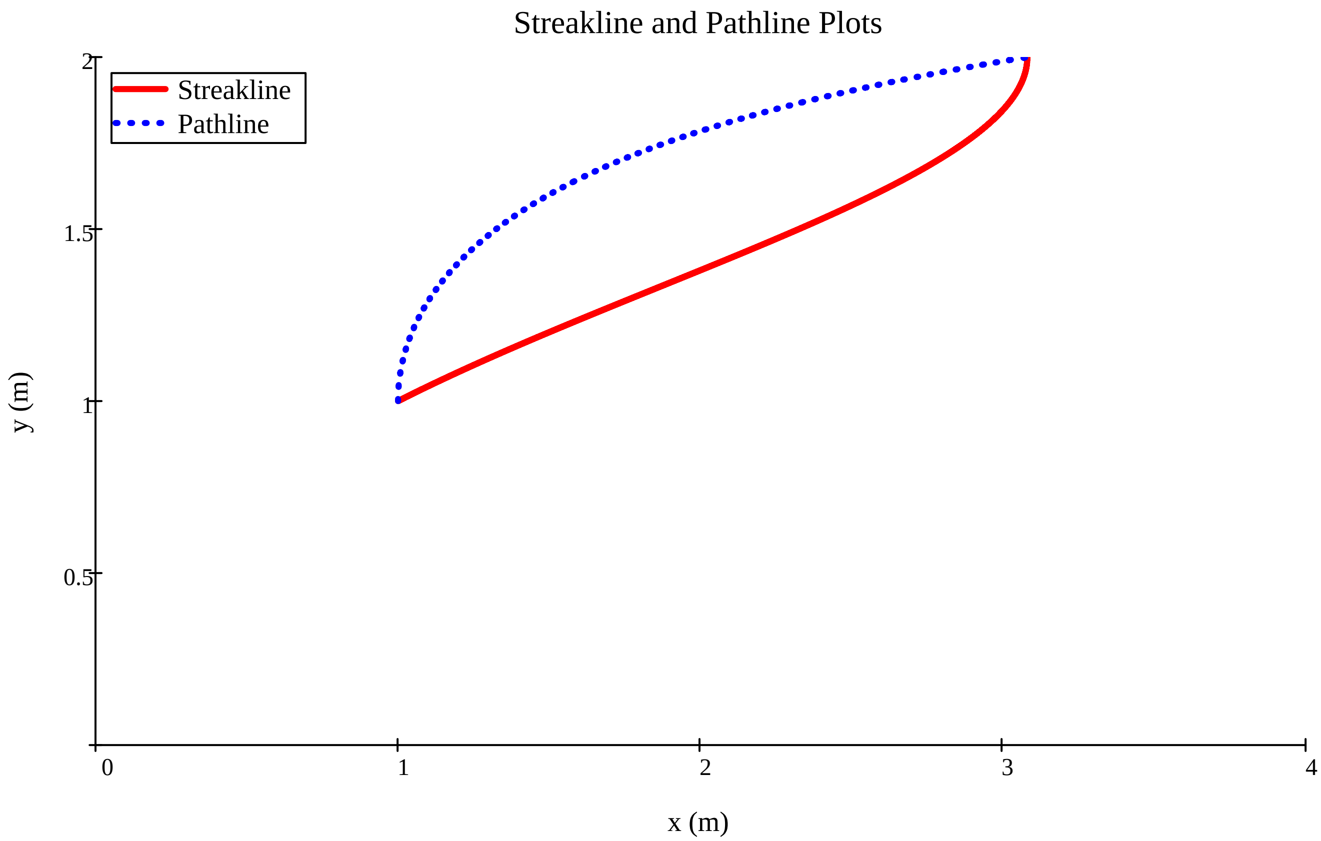
$$x = x_0 \cdot e^{\frac{a}{2} \cdot \left(t^2 - t_0^{\ 2}\right)} \qquad y = y_0 + b \cdot (t - t_0)$$

The pathlines are $x_p(t) = x_0 \cdot e^{\frac{a}{2} \cdot \left(t^2 - t_0^{\ 2}\right)}$ $\quad y_p(t) = y_0 + b \cdot (t - t_0)$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The pathlines are then $x_{st}(t_0) = x_0 \cdot e^{\frac{a}{2} \cdot \left(t^2 - t_0^{\ 2}\right)}$ $\quad y_{st}(t_0) = y_0 + b \cdot (t - t_0)$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)


Streakline and Pathline Plots
Streakline
Pathline
y (m)
x (m)
0
0.5
1
1.5
2
1
2
3
4

Problem 2.31

[Difficulty: 4]

2.31 A flow is described by velocity field $\vec{V} = ay^2\hat{i} + b\hat{j}$, where $a = 1\ m^{-1}s^{-1}$ and $b = 2$ m/s. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (6, 6). At $t = 1$ s, what are the coordinates of the particle that passed through point (1, 4) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point $(-3, 0)$ 2 s earlier? Show that pathlines, streamlines, and streaklines for this flow coincide.

Given: 2D velocity field

Find: Streamlines passing through (6,6); Coordinates of particle starting at (1,4); that pathlines, streamlines and streaklines coincide

Solution:

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{b}{a\cdot y^2} \quad$ or $\quad \int a\cdot y^2\, dy = \int b\, dx$

Integrating $\quad \frac{a\cdot y^3}{3} = b\cdot x + c$

For the streamline through point (6,6) $\quad c = 60$ and $\quad y^3 = 6\cdot x + 180$

For particle that passed through (1,4) at t = 0 $\quad u = \frac{dx}{dt} = a\cdot y^2 \quad \int 1\, dx = x - x_0 = \int a\cdot y^2\, dt \quad$ We need y(t)

$v = \frac{dy}{dt} = b \quad \int 1\, dy = \int b\, dt \quad y = y_0 + b\cdot t = y_0 + 2\cdot t$

Then $\quad x - x_0 = \int_0^t a\cdot(y_0 + b\cdot t)^2\, dt \quad x = x_0 + a\cdot\left(y_0^2\cdot t + b\cdot y_0\cdot t^2 + \frac{b^2\cdot t^3}{3}\right)$

Hence, with $x_0 = 1 \quad y_0 = 4 \quad x = 1 + 16\cdot t + 8\cdot t^2 + \frac{4}{3}\cdot t^3 \quad$ At t = 1 s $\quad x = 26.3\cdot m$

$y = 4 + 2\cdot t \quad y = 6\cdot m$

For particle that passed through (-3,0) at t = 1 $\quad \int 1\, dy = \int b\, dt \quad y = y_0 + b\cdot(t - t_0)$

$$x - x_0 = \int_{t_0}^t a\cdot(y_0 + b\cdot t)^2\, dt \qquad x = x_0 + a\cdot\left[y_0^2\cdot(t - t_0) + b\cdot y_0\cdot(t^2 - t_0^2) + \frac{b^2}{3}\cdot(t^3 - t_0^3)\right]$$

Hence, with $x_0 = -3$, $y_0 = 0$ at $t_0 = 1 \quad x = -3 + \frac{4}{3}\cdot(t^3 - 1) = \frac{1}{3}\cdot(4\cdot t^3 - 13) \quad y = 2\cdot(t - 1)$

Evaluating at t = 3 $\quad x = 31.7\cdot m \quad y = 4\cdot m$

This is a steady flow, so pathlines, streamlines and streaklines always coincide

Problem 2.32

[Difficulty: 3]

2.32 Tiny hydrogen bubbles are being used as tracers to visualize a flow. All the bubbles are generated at the origin ($x = 0$, $y = 0$). The velocity field is unsteady and obeys the equations:

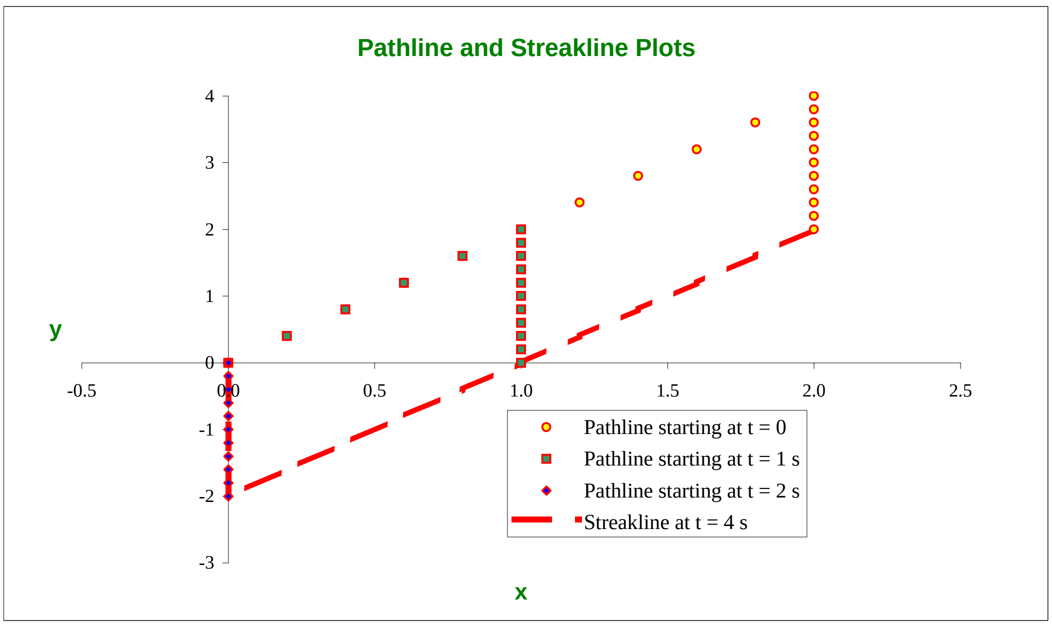
$$u = 1\ \text{m/s} \qquad v = 2\ \text{m/s} \qquad 0 \le t < 2\ \text{s}$$
$$u = 0 \qquad v = -1\ \text{m/s} \qquad 0 \le t \le 4\ \text{s}$$

Plot the pathlines of bubbles that leave the origin at $t = 0$, 1, 2, 3, and 4 s. Mark the locations of these five bubbles at $t = 4$ s. Use a dashed line to indicate the position of a streakline at $t = 4$ s.

Solution The particle starting at t = 3 s follows the particle starting at t = 2 s;
The particle starting at t = 4 s doesn't move!

Pathlines:

t	Starting at t = 0		Starting at t = 1 s		Starting at t = 2 s		Streakline at t = 4 s	
	x	y	x	y	x	y	x	y
0.00	0.00	0.00					2.00	2.00
0.20	0.20	0.40					1.80	1.60
0.40	0.40	0.80					1.60	1.20
0.60	0.60	1.20					1.40	0.80
0.80	0.80	1.60					1.20	0.40
1.00	1.00	2.00	0.00	0.00			1.00	0.00
1.20	1.20	2.40	0.20	0.40			0.80	-0.40
1.40	1.40	2.80	0.40	0.80			0.60	-0.80
1.60	1.60	3.20	0.60	1.20			0.40	-1.20
1.80	1.80	3.60	0.80	1.60			0.20	-1.60
2.00	2.00	4.00	1.00	2.00	0.00	0.00	0.00	-2.00
2.20	2.00	3.80	1.00	1.80	0.00	-0.20	0.00	-1.80
2.40	2.00	3.60	1.00	1.60	0.00	-0.40	0.00	-1.60
2.60	2.00	3.40	1.00	1.40	0.00	-0.60	0.00	-1.40
2.80	2.00	3.20	1.00	1.20	0.00	-0.80	0.00	-1.20
3.00	2.00	3.00	1.00	1.00	0.00	-1.00	0.00	-1.00
3.20	2.00	2.80	1.00	0.80	0.00	-1.20	0.00	-0.80
3.40	2.00	2.60	1.00	0.60	0.00	-1.40	0.00	-0.60
3.60	2.00	2.40	1.00	0.40	0.00	-1.60	0.00	-0.40
3.80	2.00	2.20	1.00	0.20	0.00	-1.80	0.00	-0.20
4.00	2.00	2.00	1.00	0.00	0.00	-2.00	0.00	0.00



Problem 2.33

[Difficulty: 3]

2.33 A flow is described by velocity field $\vec{V} = ax\hat{i} + b\hat{j}$, where $a = 1/5\ \text{s}^{-1}$ and $b = 1$ m/s. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (1, 1). At $t = 5$ s, what are the coordinates of the particle that initially (at $t = 0$) passed through point (1, 1)? What are its coordinates at $t = 10$ s? Plot the streamline and the initial, 5 s, and 10 s positions of the particle. What conclusions can you draw about the pathline, streamline, and streakline for this flow?

Given: Velocity field

Find: Equation for streamline through point (1.1); coordinates of particle at t = 5 s and t = 10 s that was at (1,1) at t = 0; compare pathline, streamline, streakline

Solution:

Governing equations: For streamlines $\frac{v}{u} = \frac{dy}{dx}$ For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$

Assumption: 2D flow

Given data $a = \frac{1}{5}\cdot\frac{1}{s}$ $b = 1\cdot\frac{m}{s}$ $x_0 = 1$ $y_0 = 1$ $t_0 = 0$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b}{a\cdot x}$

So, separating variables $\frac{a}{b}\cdot dy = \frac{dx}{x}$

Integrating $\frac{a}{b}\cdot\left(y - y_0\right) = \ln\left(\frac{x}{x_0}\right)$

The solution is then $y = y_0 + \frac{b}{a}\cdot\ln\left(\frac{x}{x_0}\right) = 5\cdot\ln(x) + 1$

Hence for pathlines $u_p = \frac{dx}{dt} = a\cdot x$ $v_p = \frac{dy}{dt} = b$

Hence $\frac{dx}{x} = a\cdot dt$ $dy = b\cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a\cdot\left(t - t_0\right)$ $y - y_0 = b\cdot\left(t - t_0\right)$

The pathlines are $x = x_0\cdot e^{a\cdot\left(t-t_0\right)}$ $y = y_0 + b\cdot\left(t - t_0\right)$ or $y = y_0 + \frac{b}{a}\cdot\ln\left(\frac{x}{x_0}\right)$

For a particle that was at $x_0 = 1$ m, $y_0 = 1$ m at $t_0 = 0$ s, at time $t = 1$ s we find the position is

$$x = x_0 \cdot e^{a \cdot (t - t_0)} = e^{\frac{1}{5}} \text{ m} \qquad y = y_0 + b \cdot (t - t_0) = 2 \text{ m}$$

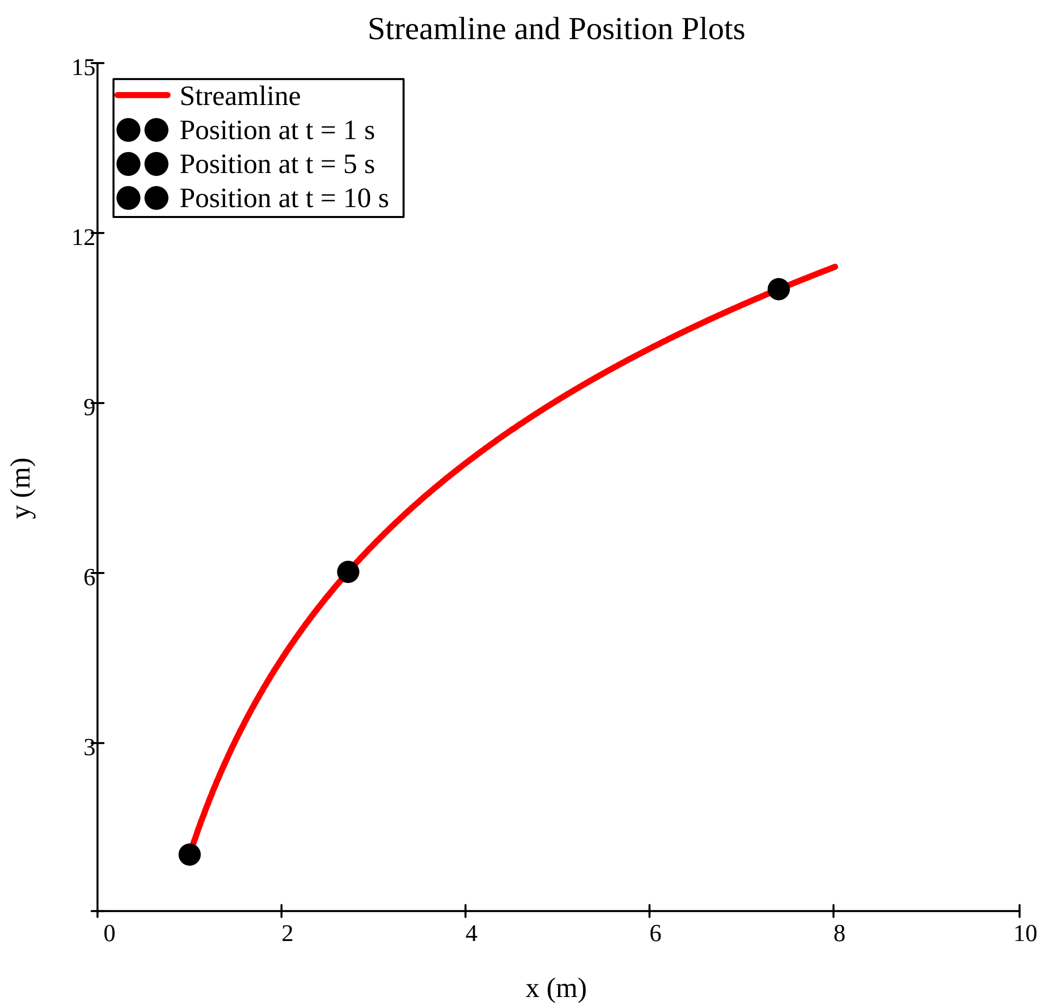
For a particle that was at $x_0 = 1$ m, $y_0 = 1$ m at $t_0 = 0$ s, at time $t = 5$ s we find the position is

$$x = x_0 \cdot e^{a \cdot (t - t_0)} = e \text{ m} \qquad y = y_0 + b \cdot (t - t_0) = 6 \text{ m}$$

For a particle that was at $x_0 = 1$ m, $y_0 = 1$ at $t_0 = 0$ s, at time $t = 10$ s we find the position is

$$x = x_0 \cdot e^{a \cdot (t - t_0)} = e^2 \text{ m} \qquad y = y_0 + b \cdot (t - t_0) = 11 \text{ m}$$

For this steady flow streamlines, streaklines and pathlines coincide



Problem 2.34

[Difficulty: 3]

2.34 A flow is described by velocity field $\vec{V} = a\hat{i} + bx\hat{j}$, where $a = 2$ m/s and $b = 1\ \text{s}^{-1}$. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (2, 5). At $t = 2$ s, what are the coordinates of the particle that passed through point (0, 4) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point (1, 4.25) 2 s earlier? What conclusions can you draw about the pathline, streamline, and streakline for this flow?

Given: Velocity field

Find: Equation for streamline through point (2.5); coordinates of particle at t = 2 s that was at (0,4) at t = 0; coordinates of particle at t = 3 s that was at (1,4.25) at t = 1 s; compare pathline, streamline, streakline

Solution:

Governing equations: For streamlines $\frac{v}{u} = \frac{dy}{dx}$ For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$

Assumption: 2D flow

Given data $a = 2\,\frac{m}{s}$ $b = 1\,\frac{1}{s}$ $x_0 = 2$ $y_0 = 5$ $x = 1$ $x = x$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot x}{a}$

So, separating variables $\frac{a}{b} \cdot dy = x \cdot dx$

Integrating $\frac{a}{b} \cdot (y - y_0) = \frac{1}{2} \cdot \left(x^2 - x_0^{\,2}\right)$

The solution is then $y = y_0 + \frac{b}{2 \cdot a} \cdot \left(x^2 - x_0^{\,2}\right) = \frac{x^2}{4} + 4$

Hence for pathlines $u_p = \frac{dx}{dt} = a$ $v_p = \frac{dy}{dt} = b \cdot x$

Hence $dx = a \cdot dt$ $dy = b \cdot x \cdot dt$

Integrating $x - x_0 = a \cdot (t - t_0)$ $dy = b \cdot \left[x_0 + a \cdot (t - t_0)\right] \cdot dt$

$$y - y_0 = b \cdot \left[x_0 \cdot (t - t_0) + \frac{a}{2} \cdot \left(\left(t^2 - t_0^{\,2}\right)\right) - a \cdot t_0 \cdot (t - t_0)\right]$$

The pathlines are $x = x_0 + a \cdot (t - t_0)$ $y = y_0 + b \cdot \left[x_0 \cdot (t - t_0) + \frac{a}{2} \cdot \left(\left(t^2 - t_0^{\,2}\right)\right) - a \cdot t_0 \cdot (t - t_0)\right]$

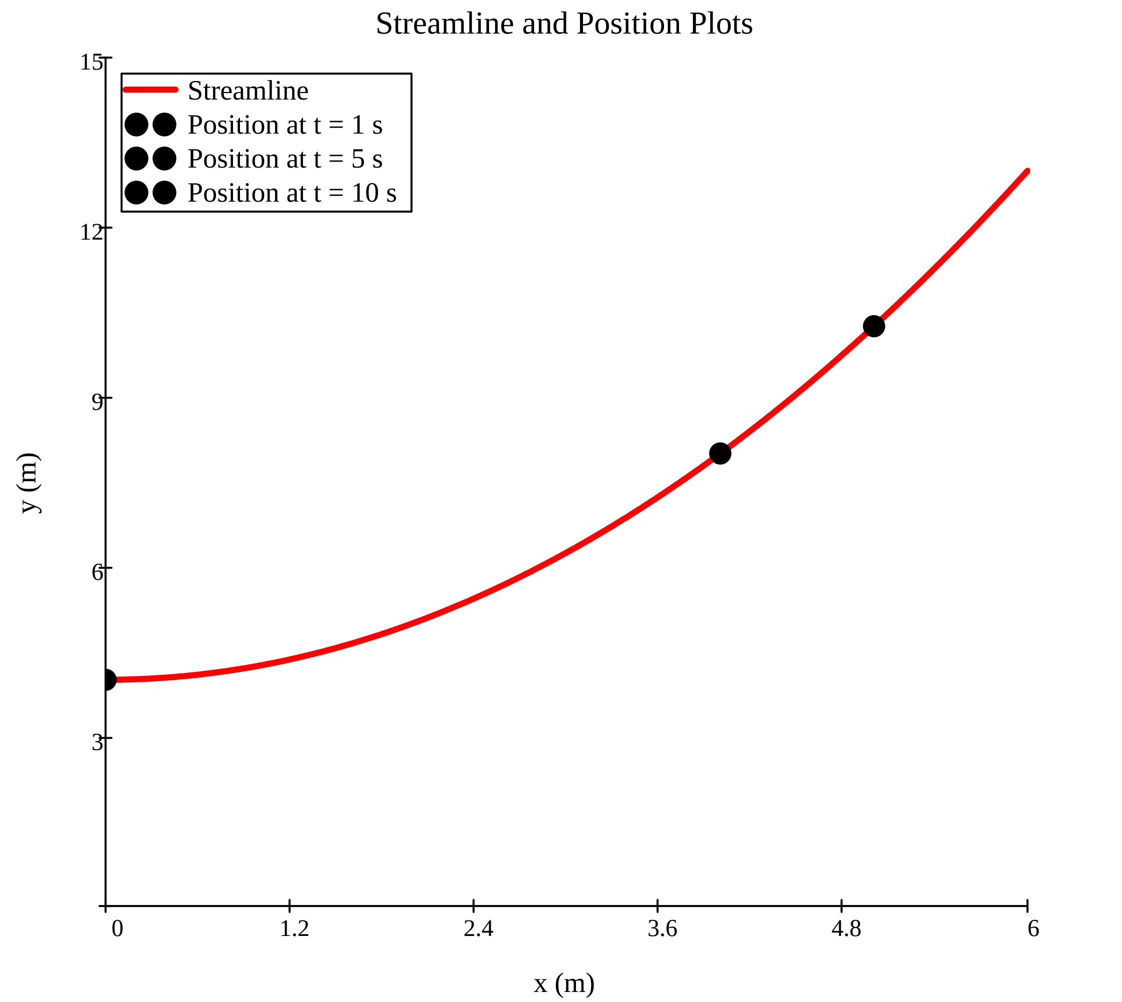
For a particle that was at $x_0 = 0\,m$, $y_0 = 4\,m$ at $t_0 = 0s$, at time $t = 2\,s$ we find the position is

$$x = x_0 + a\cdot(t - t_0) = 4\,m \qquad y = y_0 + b\cdot\left[x_0\cdot(t - t_0) + \frac{a}{2}\cdot\left(\left(t^2 - t_0^{\,2}\right)\right) - a\cdot t_0\cdot(t - t_0)\right] = \ m$$

For a particle that was at $x_0 = 1\,m$, $y_0 = 4.25\,m$ at $t_0 = 1\,s$, at time $t = 3\,s$ we find the position is

$$x = x_0 + a\cdot(t - t_0) = 5\,m \qquad y = y_0 + b\cdot\left[x_0\cdot(t - t_0) + \frac{a}{2}\cdot\left(\left(t^2 - t_0^{\,2}\right)\right) - a\cdot t_0\cdot(t - t_0)\right] = 10.\,m$$

For this steady flow streamlines, streaklines and pathlines coincide; the particles refered to are the same particle!



Problem 2.35

[Difficulty: 4]

2.35 A flow is described by velocity field $\vec{V} = ay\hat{i} + bt\hat{j}$, where $a = 0.2\ \text{s}^{-1}$ and $b = 0.4\ \text{m/s}^2$. At $t = 2$ s, what are the coordinates of the particle that passed through point (1, 2) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point (1, 2) at $t = 2$ s? Plot the pathline and streakline through point (1, 2), and plot the streamlines through the same point at the instants $t = 0$, 1, 2, and 3 s.

Given: Velocity field

Find: Coordinates of particle at t = 2 s that was at (1,2) at t = 0; coordinates of particle at t = 3 s that was at (1,2) at t = 2 s; plot pathline and streakline through point (1,2) and compare with streamlines through same point at t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

Given data $a = 0.2\ \frac{1}{s}$ $b = 0.4\ \frac{m}{s^2}$

Hence for pathlines $u_p = \frac{dx}{dt} = a \cdot y$ $v_p = \frac{dy}{dt} = b \cdot t$

Hence $dx = a \cdot y \cdot dt$ $dy = b \cdot t \cdot dt$ $y - y_0 = \frac{b}{2} \cdot \left(t^2 - t_0^2\right)$

For x $dx = \left[a \cdot y_0 + a \cdot \frac{b}{2} \cdot \left(t^2 - t_0^2\right)\right] \cdot dt$

Integrating $x - x_0 = a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0)\right]$

The pathlines are $x(t) = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0)\right]$ $y(t) = y_0 + \frac{b}{2} \cdot \left(t^2 - t_0^2\right)$

These give the position (x,y) at any time t of a particle that was at (x_0,y_0) at time t_0

Note that streaklines are obtained using the logic of the Governing equations, above

The streaklines are $x(t_0) = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0) \right]$ $y(t_0) = y_0 + \frac{b}{2} \cdot (t^2 - t_0^2)$

These gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For a particle that was at $x_0 = 1\,m$, $y_0 = 2\,m$ at $t_0 = 0s$, at time $t = 2\,s$ we find the position is (from pathline equations)

$$x = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0) \right] = 1.9\,m \qquad y = y_0 + \frac{b}{2} \cdot (t^2 - t_0^2) = 2.8\,m$$

For a particle that was at $x_0 = 1\,m$, $y_0 = 2\,m$ at $t_0 = 2\,s$, at time $t = 3\,s$ we find the position is

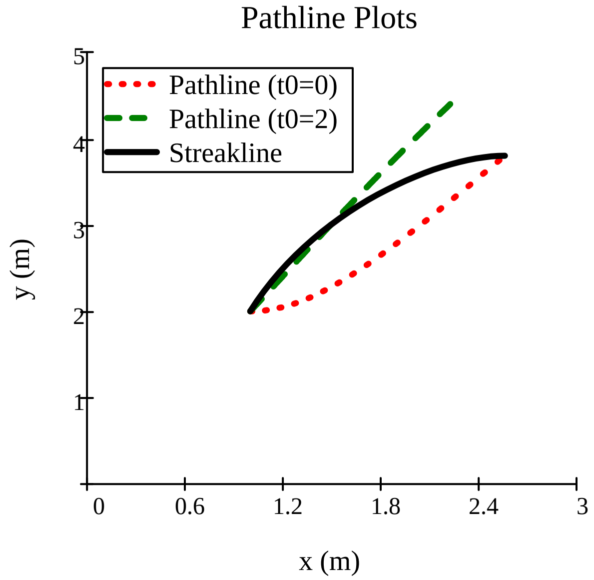
$$x = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0) \right] = 1.4\,m \qquad y = y_0 + \frac{b}{2} \cdot (t^2 - t_0^2) = 3.0 \quad m$$

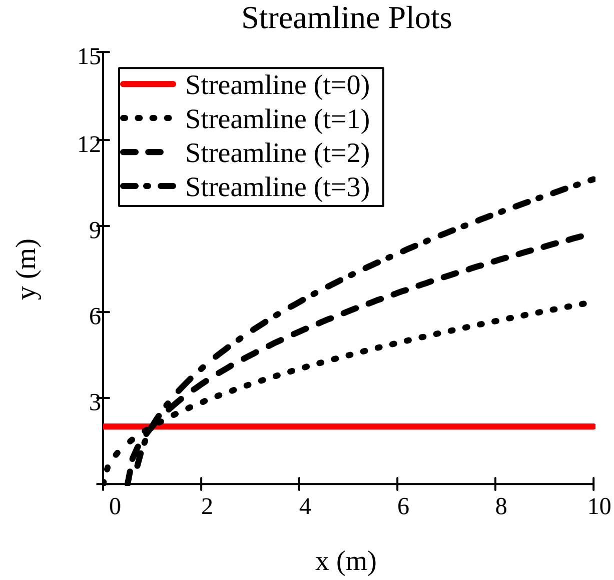
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot t}{a \cdot y}$

So, separating variables $y \cdot dy = \frac{b}{a} \cdot t \cdot dx$ where we treat t as a constant

Integrating $\frac{y^2 - y_0^2}{2} = \frac{b \cdot t}{a} \cdot (x - x_0)$ and we have $x_0 = 1 \ m$ $y_0 = 2 \ m$

The streamlines are then $y = \sqrt{y_0^2 + \frac{2 \cdot b \cdot t}{a} \cdot (x - x_0)} = \sqrt{4 \cdot t \cdot (x - 1) + 4}$


Pathline Plots
Pathline (t0=0)
Pathline (t0=2)
Streakline
y (m)
x (m)
0
0.6
1.2
1.8
2.4
3
1
2
3
4
5


Streamline Plots
Streamline (t=0)
Streamline (t=1)
Streamline (t=2)
Streamline (t=3)
y (m)
x (m)
0
2
4
6
8
10
3
6
9
12
15