1 Quadratic functions

Exercise 1B

4 **a**

$$\begin{aligned} y &= x^2 - 6x + 11 \\ &= (x-3)^2 - 9 + 11 \\ &= (x-3)^2 + 2 \end{aligned}$$

b Minimum value of y is 2.

5 **a** Minimum at (3, 6) ⇒ $y = a(x - 3)^2 + 6$
So $a = -3$, $b = 6$

b Curve passes through (1, 14), so substituting $x = 1$ and $y = 14$ into $y = a(x - 3)^2 + 6$:

$$\begin{aligned} 14 &= a(1-3)^2 + 6 \\ 8 &= 4a \\ a &= 2 \end{aligned}$$

6 **a**

$$\begin{aligned} 2x^2 + 4x - 1 &= 2[x^2 + 2x] - 1 \\ &= 2[(x+1)^2 - 1] - 1 \\ &= 2(x+1)^2 - 2 - 1 \\ &= 2(x+1)^2 - 3 \end{aligned}$$

b Line of symmetry is $x = -1$

c

$$2x^2 + 4x - 1 = 0$$
$$2(x+1)^2 - 3 = 0$$
$$2(x+1)^2 = 3$$
$$(x+1)^2 = \frac{3}{2}$$
$$x + 1 = \pm\sqrt{\frac{3}{2}}$$
$$x = -1 \pm \sqrt{\frac{3}{2}}$$

Exercise 1C

4 **a** $2x^2 + 5x - 12 = (2x - 3)(x + 4)$

b The graph crosses the x-axis where $y = 0$, i.e. where $2x^2 + 5x - 12 = 0$:

$$2x^2 + 5x - 12 = 0$$
$$(2x - 3)(x + 4) = 0$$
$$2x - 3 = 0 \text{ or } x = 4 = 0$$
$$x = \frac{3}{2} \text{ or } x = -4$$

So the intersections with the x-axis are at $\left(\frac{3}{2}, 0\right)$ and $(-4,0)$.

5 Roots at -5 and 2

$$\Rightarrow y = a(x-2)(x+5)$$
$$= ax^2 + 3ax - 10a$$

So $c = -10a$ and $b = 3a$.

y-intercept at 3 $\Rightarrow c = 3$

$$\therefore 3 = -10a$$
$$a = -\frac{3}{10}$$

and $b = 3\left(-\frac{3}{10}\right) = -\frac{9}{10}$

Exercise 1D

5 $3x^2 = 4x + 1 \Leftrightarrow 3x^2 - 4x - 1 = 0$

Using the quadratic formula with $a = 3$, $b = -4$ and $c = -1$:

$$x = \frac{-(-4)\pm\sqrt{(-4)^2 - 4\times 3\times(-1)}}{2\times 3}$$
$$= \frac{4\pm\sqrt{28}}{6}$$
$$= \frac{2\pm\sqrt{7}}{3}$$

6 The discriminant with $a = 4$, $b = 1$ and $c = \frac{1}{16}$ is

$$\Delta = b^2 - 4ac$$
$$= 1^2 - 4\times 4\times\frac{1}{16}$$
$$= 0$$

So there is only one root, i.e. the vertex lies on the x-axis.

7 Equal roots when discriminant is zero:

$\Delta = b^2 - 4ac = 0$

$(-4)^2 - 4 \times m \times 2m = 0$

$16 - 8m^2 = 0$

$m^2 = 2$

$m = \pm\sqrt{2}$

8 Tangent to the *x*-axis implies equal roots, so discriminant is zero:

$\Delta = b^2 - 4ac = 0$

$(2k+1)^2 - 4 \times (-3) \times (-4k) = 0$

$4k^2 + 4k + 1 - 48k = 0$

$4k^2 - 44k + 1 = 0$

$$\begin{aligned} k &= \frac{44 \pm \sqrt{44^2 - 4\times4\times1}}{2\times4} \\ &= \frac{44 \pm \sqrt{1920}}{8} \\ &= \frac{11}{2} \pm \sqrt{30} \end{aligned}$$

9 No real solutions when discriminant $\Delta < 0$:

$b^2 - 4ac < 0$

$(-6)^2 - 4 \times 1 \times 2k < 0$

$36 - 8k < 0$

$k > \frac{9}{2}$

10 For a quadratic to be non-negative (≥ 0) for all *x*, it must have at most one root, so $\Delta \leq 0$ and $a > 0$.

$b^2 - 4ac \leq 0$

$(-3)^2 - 4 \times 2 \times (2c - 1) \leq 0$

$9 - 16c + 8 \leq 0$

$c \geq \frac{17}{16}$

COMMENT

Note that $\Delta \leq 0$ is not sufficient in general for a quadratic to be non-negative. The condition $a > 0$ is also necessary to ensure that the quadratic has a positive shape (opening upward) rather than a negative shape (opening downward), so that the curve remains above the *x*-axis and never goes below it, as would be the case if $a < 0$. In this question a was given as positive (2), so we did not need to use this condition at all.

11 For a quadratic to be negative for all x, it must have no real roots, so $\Delta < 0$ and $a < 0$.

$b^2 - 4ac < 0$

$3^2 - 4 \times m \times (-4) < 0$

$9 + 16m < 0$

$m < -\frac{9}{16}$

COMMENT

The condition $a < 0$ ensures that the function is negative shaped and therefore remains below the x-axis. In this case $a = m$, and it followed from the condition on Δ that $a < 0$, as seen in the answer.

12 The two zeros of $ax^2 + bx + c$ are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$. The positive difference between these zeros is

$$\left|\frac{-b+\sqrt{b^2-4ac}}{2a} - \frac{-b-\sqrt{b^2-4ac}}{2a}\right| = \left|\frac{2\sqrt{b^2-4ac}}{2a}\right|$$
$$= \left|\frac{\sqrt{b^2-4ac}}{a}\right|$$

So, in this case,

$\frac{\sqrt{k^2-12}}{1} = \sqrt{69}$

$k^2 - 12 = 69$

$k^2 = 81$

$k = \pm 9$

COMMENT

Note that modulus signs were used in the general expression for the positive distance, as a could be negative. Here $a = 1$ and so the modulus was not required in the specific case in this question.

Exercise 1E

3

$$y = x^2 - 4 \quad \dots(1)$$
$$y = 8 - x \quad \dots(2)$$

Substituting (1) into (2):

$x^2 - 4 = 8 - x$

$x^2 + x - 12 = 0$

$(x - 3)(x + 4) = 0$

$x = 3$ or $x = -4$

Substituting into (2):

$$x = 3: \quad y = 8 - 3 = 5$$
$$x = -4: \quad y = 8 - (-4) = 12$$

So the points of intersection are (3, 5) and (−4, 12).

4

$$y = 2x^2 - 3x + 2 \quad \dots(1)$$
$$3x + 2y = 5 \quad \dots(2)$$

Substituting (1) into (2):

$$3x + 2\left(2x^2 - 3x + 2\right) = 5$$
$$4x^2 - 3x - 1 = 0$$
$$(4x + 1)(x - 1) = 0$$
$$x = -\tfrac{1}{4} \text{ or } x = 1$$

Substituting into (1):

$$x = -\tfrac{1}{4}: \quad y = 2\left(-\tfrac{1}{4}\right)^2 - 3\left(-\tfrac{1}{4}\right) + 2 = \tfrac{23}{8}$$
$$x = 1: \quad y = 2 \times 1^2 - 3 \times 1 + 2 = 1$$

So the solutions are $\left(-\frac{1}{4}, \frac{23}{8}\right)$ and (1, 1).

5

a

$$x^2 - 6x + y^2 - 2y - 8 = 0 \quad \dots(1)$$
$$y = x - 8 \quad \dots(2)$$

Substituting (2) into (1):

$x^2 - 6x + (x - 8)^2 - 2(x - 8) - 8 = 0$

$2x^2 - 24x + 72 = 0$

$x^2 - 12x + 36 = 0$

as required.

b

$x^2 - 12x + 36 = 0$

$(x - 6)^2 = 0$

$x = 6$

There is only one point of intersection, which means that the line is tangent to the circle.

6 $y = mx + 3 \quad \ldots(1)$

$y = 3x^2 - x + 5 \quad \ldots(2)$

Substituting (1) into (2):

$mx + 3 = 3x^2 - x + 5$

$3x^2 - x - mx + 2 = 0$

$3x^2 - (m + 1)x + 2 = 0$

Only one intersection means that this quadratic has a single root, so $\Delta = 0$:

$b^2 - 4ac = 0$

$[-(m+1)]^2 - 4 \times 3 \times 2 = 0$

$(m+1)^2 = 24$

$m + 1 = \pm\sqrt{24}$

$m = -1 \pm \sqrt{24} = -1 \pm 2\sqrt{6}$

Exercise 1F

1 Let one number be x and the other be y.

Sum of x and y is 8: $x + y = 8 \quad \ldots(1)$

Product is 9.75: $xy = 9.75 \quad \ldots(2)$

From (1): $y = 8 - x \quad \ldots(3)$

Substituting (3) into (2):

$x(8 - x) = 9.75$

$x^2 - 8x + \frac{39}{4} = 0$

$4x^2 - 32x + 39 = 0$

$(2x - 3)(2x - 13) = 0$

$x = \frac{3}{2}$ or $x = \frac{13}{2}$

The two numbers are 1.5 and 6.5.

COMMENT

It is often wise to convert decimals (such as 9.75) into fractions to make subsequent manipulation easier. The quadratic could also have been solved with a GDC, of course.

2 The length is x; let the width be y. Perimeter of 12:

$2(x + y) = 12$

$x + y = 6$

$y = 6 - x$

$$\begin{aligned}\text{Area } A &= xy \\ &= x(6-x) \\ &= 6x - x^2\end{aligned}$$

Completing the square:

$$\begin{aligned}A &= -\left[x^2 - 6x\right] \\ &= -(x-3)^2 + 9\end{aligned}$$

So the maximum area is 9 cm^2 (and occurs when $x = y = 3$, i.e. when the rectangle is a square with side 3 cm).

COMMENT

Note that the negative sign makes the quadratic negative shaped, which results in a maximum rather than minimum turning point.

3 **a** New fencing required is 200 m, so

$2x - 10 + y = 200$

$\Rightarrow y = 210 - 2x$

$$\begin{aligned}\text{Area } A &= xy \\ &= x(210-2x) \\ &= 210x - 2x^2\end{aligned}$$

b Completing the square:

$$\begin{aligned}A &= -2\left[x^2 - 105x\right] \\ &= -2\left[\left(x - \frac{105}{2}\right)^2 - \left(\frac{105}{2}\right)^2\right] \\ &= 2\left(x - \frac{105}{2}\right)^2 - \frac{105^2}{2}\end{aligned}$$

∴ maximum area when

$x = \frac{105}{2} = 52.5\text{m}$, for which

$y = 210 - 2 \times \frac{105}{2} = 105\text{m}$

4 **a** Ball is at ground level when $h = 0$.

$8t - 4.9t^2 = 0$

$t(8 - 4.9t) = 0$

$t = 0$ or $t = \frac{8}{4.9} = 1.63\,(3\text{SF})$

So the ball returns to the ground after 1.63 s.

b By symmetry of a quadratic, maximum height (vertex) is halfway between the roots, $t = 0$ and $t = \frac{8}{4.9}$.

$$\begin{aligned} t_{\max} &= \frac{0+\frac{8}{4.9}}{2} = \frac{4}{4.9} \\ \therefore h_{\max} &= 8\left(\frac{4}{4.9}\right) - 4.9\left(\frac{4}{4.9}\right)^2 \\ &= \frac{32-16}{4.9} = \frac{16}{4.9} = 3.27\text{ m}\,(3\text{SF}) \end{aligned}$$

COMMENT

This could also have been solved using a GDC or by completing the square.

5 **a** Perimeter of 60:

$2x + \frac{1}{2}\pi y = 60$

$\Rightarrow 2x = 60 - \frac{\pi y}{2}$

$\Rightarrow x = 30 - \frac{\pi y}{4}$

$$\begin{aligned} A &= xy + \frac{1}{2}\pi\left(\frac{y}{2}\right)^2 \\ &= \left(30 - \frac{\pi y}{4}\right)y + \frac{\pi y^2}{8} \\ &= 30y - \frac{\pi y^2}{4} + \frac{\pi y^2}{8} \\ &= 30y - \frac{1}{8}\pi y^2 \end{aligned}$$

b Finding the roots of the area expression:

$\left(30y - \frac{1}{8}\pi y^2\right) = 0$

$y\left(30 - \frac{1}{8}\pi y\right) = 0$

$y = 0$ or $y = \frac{240}{\pi}$

By the symmetry of a quadratic, the maximum area (vertex) is halfway between the roots:

$$y = \frac{0+\frac{240}{\pi}}{2} = \frac{120}{\pi}$$

When $y = \frac{120}{\pi}$,

$$\begin{aligned} x &= 30 - \frac{\pi y}{4} \\ &= 30 - \frac{\pi}{4}\left(\frac{120}{\pi}\right) \\ &= 30 - 30 \\ &= 0 \end{aligned}$$

COMMENT

This could also have been solved using a GDC or by completing the square.

c If $A = 200$,

$$30y - \frac{1}{8}\pi y^2 = 200$$

$$\frac{\pi y^2}{8} - 30y + 200 = 0$$

Using the quadratic formula with $a = \frac{\pi}{8}$, $b = -30$ and $c = 200$:

$$\begin{aligned} y &= \frac{30 \pm \sqrt{30^2 - 4 \times \frac{\pi}{8} \times 200}}{\frac{\pi}{4}} \\ &= 7.38 \text{ or } 69.0 \text{ (3SF)} \end{aligned}$$

When $y = 7.38,\ x = 30 - \frac{\pi \times 7.38}{4} = 24.2$

When $y = 69.0,\ x = 30 - \frac{\pi \times 69.0}{4} = -24.2$

(therefore reject as $x < 0$)

So $x = 24.2$ m and $y = 7.38$ m.

6 Total profit = $n(200 - 4n) = 4n(50 - n)$

Finding the roots of the total profit function:

$4n(50 - n) = 0$

$n = 0$ or $n = 50$

By the symmetry of a quadratic, the maximum lies halfway between the roots, i.e. at $n = 25$.

COMMENT
This could also have been solved by completing the square.

Mixed examination practice 1

Short questions

1 **a** $x^2 + 5x - 14 = (x + 7)(x - 2)$

b $x^2 + 5x - 14 = 0$

$(x + 7)(x - 2) = 0$

$x = -7$ or $x = 2$

2 **a** Positive quadratic, so the vertex is a minimum point.

b Minimum at (3, 7) ⇒ $y = (x - 3)^2 + 7$ So $a = 3$, $b = 7$

3 Maximum y-value is 48 ⇒ c ⇒ 48. Passes through (−2, 0) and (6, 0) means that its roots are $x = -2$ and $x = 6$. The line of symmetry is midway between the roots, i.e. at $x = 2$, so $b = 2$.

Substituting $x = -2$ and $y = 0$ into $y = a(x - 2)^2 + 48$:

$0 = a(-2 - 2)^2 + 48$

$0 = 16a + 48$

$a = -3$

So $a = -3$, $b = 2$ and $c = 48$.

4 Roots at $x = k$ and $x = k + 4$ ⇒ line of symmetry is $x = k + 2$ (midway between the roots).

So the x-coordinate of the turning point is $k + 2$.

5 **a** Roots at $-\frac{1}{2}$ and 2, so

$f(x) = \left(x + \frac{1}{2}\right)(x - 2)$

i.e. $p = -\frac{1}{2}, q = 2$

b Line of symmetry is midway between the roots: $x = \frac{2 + \left(-\frac{1}{2}\right)}{2} = \frac{3}{4}$

∴ x-coordinate of C is $\frac{3}{4}$

6
- Negative quadratic ⇒ a is negative
- Negative y-intercept ⇒ c is negative

- Single (repeated) root ⇒ $b^2 - 4ac = 0$
- Line of symmetry $x = -\frac{b}{2a}$ is positive ⇒ b is positive (as a is negative)

TABLE 1MS.6

Expression	Positive	Negative	Zero
a		✓	
c		✓	
$b^2 - 4ac$			✓
b	✓		

7 **a**

$$\begin{aligned} x^2 - 10x + 35 &= (x-5)^2 - 25 + 35 \\ &= (x-5)^2 + 10 \end{aligned}$$

b From (a), the minimum value of $x^2 - 10x + 35$ is 10.

Hence the maximum value of $\frac{1}{(x^2-10x+35)^3}$ is $\frac{1}{10^3} = \frac{1}{1000}$

8 Equal roots ⇒ $\Delta = 0$

$b^2 - 4ac = 0$

$(k + 1)^2 - 4 \times 2k \times 1 = 0$

$k^2 - 6k + 1 = 0$

$k = \frac{6\pm\sqrt{32}}{2} = 3 \pm 2\sqrt{2}$

9 No real roots ⇒ $\Delta < 0$

$b^2 - 4ac < 0$

$6^2 - 4 \times 2 \times k < 0$

$36 - 8k < 0$

$k > \frac{9}{2}$

10 Only one zero ⇒ $\Delta = 0$

$b^2 - 4ac = 0$

$[-(k+1)]^2 - 4 \times 1 \times 3 = 0$

$(k+1)^2 - 12 = 0$

$k + 1 = \pm 2\sqrt{3}$

$k = -1 \pm 2\sqrt{3}$

11 **a** Roots of $x^2 - kx + (k - 1) = 0$ are

$$\frac{k \pm \sqrt{k^2 - 4(k-1)}}{2} = \frac{k \pm \sqrt{k^2 - 4k + 4}}{2} = \frac{k \pm \sqrt{(k-2)^2}}{2} = \frac{k \pm (k-2)}{2} = k - 1 \text{ or } 1$$

$\therefore \alpha = k - 1, \beta = 1$

b $\alpha^2 + \beta^2 = 17$

$(k - 1)^2 + 1 = 17$

$k^2 - 2k + 2 = 17$

$k^2 - 2k - 15 = 0$

$(k - 5)(k + 3) = 0$

$k = 5$ or $k = -3$

Long questions

1 **a** **i** Square perimeter = $4x$

ii Circle perimeter = $2\pi y$

b $4x + 2\pi y = 8 \Rightarrow x = 2 - \frac{\pi}{2}y$

c
$$\begin{aligned} A &= \text{area of square} + \text{area of circle} \\ &= x^2 + \pi y^2 \\ &= \left(2 - \frac{\pi y}{2}\right)^2 + \pi y^2 \\ &= 4 - 2\pi y + \frac{\pi^2 y^2}{4} + \pi y^2 \\ &= \frac{\pi}{4}(\pi + 4)y^2 - 2\pi y + 4 \end{aligned}$$

d Completing the square:

$$\begin{aligned} A &= \frac{\pi}{4}(\pi+4)\left[y^2 - \frac{8}{\pi+4}y + \frac{16}{\pi(\pi+4)}\right] \\ &= \frac{\pi}{4}(\pi+4)\left[\left(y - \frac{4}{\pi+4}\right)^2 - \left(\frac{4}{\pi+4}\right)^2 + \frac{16}{\pi(\pi+4)}\right] \end{aligned}$$

So the minimum area occurs when $y = \frac{4}{\pi+4}$ Percentage of wire in circle

$= \frac{\text{length of wire in circle}}{\text{total length of wire}} \times 100\,\%$

$= \frac{2\pi y}{8} \times 100\,\%$

$= \frac{2\pi\left(\frac{4}{\pi+4}\right)}{8} \times 100\,\%$

$= 44.0\,\%\,(3\text{SF})$

COMMENT

Note that it isn't necessary to simplify the constant in the expression for *A* after completing the square, as the question asks only for the value of *y* where the area is minimised and not for the actual value of that minimum.

2 **a** Car A has position ($20t - 50$, 0) and Car B has position (0, $15t - 30$).

$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= [0 - (20t - 50)]^2 + [(15t - 30) - 0]^2 \\ &= (20t - 50)^2 + (15t - 30)^2 \\ &= 400t^2 - 2000t + 2500 + 225t^2 - 900t + 900 \\ &= 625t^2 - 2900t + 3400 \end{aligned}$$

b Completing the square:

$$\begin{aligned} d^2 &= 625\left[t^2 - \frac{116}{25}t\right] + 3400 \\ &= 625\left[\left(t - \frac{58}{25}\right)^2 - \left(\frac{58}{25}\right)^2\right] + 3400 \\ &= 625\left(t - \frac{58}{25}\right)^2 - 58^2 + 3400 \\ &= 625\left(t - \frac{58}{25}\right)^2 + 36 \end{aligned}$$

So $d^2 \geq 36$ and, since $d > 0$, it follows that the minimum value of d is 6 km.

3 **a** Vertex on the x-axis ⇒ has only one root, so $\Delta = 0$.

$b^2 - 4ac = 0$

$36 - 4k = 0$

$k = 9$

b Equation of first graph is

$y = x^2 - 6x + 9 = (x - 3)^2$

So vertex is at (3, 0).

Second graph has vertex at (−2, 5), so its equation is $y = a\,(x + 2)^2 + 5$

It passes through (3, 0); substituting into the equation gives

$0 = a(3+2)^2 + 5$

$25a = -5$

$a = -\frac{1}{5}$

$$\begin{aligned} \therefore y &= -\tfrac{1}{5}(x+2)^2 + 5 \\ &= -\tfrac{1}{5}\left(x^2 + 4x + 4\right) + 5 \\ &= -\tfrac{1}{5}x^2 - \tfrac{4}{5}x + \tfrac{21}{5} \end{aligned}$$

c For intersection of $y = x^2 - 6x + 9$ and $y = -\frac{1}{5}x^2 - \frac{4}{5}x + \frac{21}{5}$:

$x^2 - 6x + 9 = -\frac{1}{5}x^2 - \frac{4}{5}x + \frac{21}{5}$

$5x^2 - 30x + 45 = -x^2 - 4x + 21$

$6x^2 - 26x + 24 = 0$

$3x^2 - 13x + 12 = 0$

$(3x - 4)(x - 3) = 0$

$x = \frac{4}{3}$ or $x = 3$

$x = 3$ is the point of intersection at the vertex (3, 0) of the first graph.

To find the y-coordinate of the other point, substitute $x = \frac{4}{3}$ into $y = (x - 3)^2$:

$y = \left(\frac{4}{3} - 3\right)^2 = \frac{25}{9}$

So the other point of intersection is $\left(\frac{4}{3}, \frac{25}{9}\right)$.