Exercises

1.1 Numbers 101: The Very Basics

1. (a) The claim makes sense and is true.
   (b) The claim makes no sense; $\sqrt{8}$ isn't a subset.
   (c) The claim makes sense and is true.
   (d) The claim makes sense but is false; consider $a = 0$ and $b = \sqrt{2}$.
   (e) The claim makes sense and is true.
   (f) The claim makes sense but is false: consider $a = 0$.

2. (a) The claim is false; let $a = \sqrt{2}$.
   (b) The expression $\mathbb{Q}^2$ doesn't make sense.
   (c) The claim is true. Since $a^2 > 0$, some large n will work.
   (d) The claim is true; see Theorem **??**.
   (e) If $a \in \mathbb{Q}$ and $a \neq 0$, then $a\sqrt{2} \notin \mathbb{Q}$.

3. The number $1/a$ is an integer only if $a = \pm 1$. The number $1/a$ is rational for all nonzero integers a. The equation $1/a = a$ holds only if $a = \pm 1$.

4. (a) $1 \in S_1$ but $-1 \notin S_1$
   (b) $2 \in S_2$ but $1/2 \notin S_2$
   (c) $\sqrt{2} \in S_3$ but $1/\sqrt{2} = \sqrt{2}/2 \notin S_3$

5. (a) $1 \in S_1$ but $-1 \notin S_1$
   (b) $2 \in S_2$ but $1/2 \notin S_2$
   (c) $\sqrt{2} \in S_3$ but $1/\sqrt{2} = \sqrt{2}/2 \notin S_3$
   (d) $\pi \in S_4$ but $\pi^2 \notin S_4$

6. Theorem **??** is useful. Because $\mathbb{Q}$ is closed under addition, multiplication, and division (if denominators aren't zero), expressions like those in (a) and (b) are rational—unless, for (b), $p = q = 0$. Expressions involving square roots are different. If $p = q = 1$, for instance, then $\sqrt{p^2 + q^2} = \sqrt{2}$ is irrational; the same expression is rational if $p = 3$ and $q = 4$. The quantity $\sqrt{p^2 + 2pq + q^2}$ is always rational, since $\sqrt{p^2 + 2pq + q^2} = \sqrt{(p+q)^2} = \pm(p+q)$. (Note that $\sqrt{(p+q)^2} = p+q$ may be false.)

7. All of xy, $x+y$, $x-y$ and x/y can be either rational or irrational. Examples are easy to find.

8. The quantities in (a), (b), (c), (d) and (f) are all irrational; proofs are by contradiction. (E.g., if x/r were rational, then we could multiply by the rational number r. Then the product x is also rational, a contradiction.) $\sqrt{r}$ can go either way.

9. Assume toward contradiction that $\sqrt{3} = a/b$ for integers a and b, where a/b is in reduced form. Then squaring both sides gives $3b^2 = a^2$. This implies (essentially as in the proof of Theorem **??**) that 3 divides both a and b, which contradicts the assumption that a/b is in reduced form.

10. (a) Say $x^2 \notin \mathbb{Q}$. If $x \in \mathbb{Q}$, then (by Theorem **??**) x^2 is rational, too, which contradicts our assumption.

    (b) Another proof by contradiction. If $x = \sqrt{2} + \sqrt{3}$ is rational, then $x^2 = 5 + 2\sqrt{6}$ is rational, too. This implies, in turn, that $\sqrt{6}$ is rational, which is absurd.

    (c) Yet another proof by contradiction. Let's write $x = \sqrt{2} + \sqrt{3} + \sqrt{5}$, then we have $x - \sqrt{5} = \sqrt{2} + \sqrt{3}$, and suppose x is rational. Squaring both sides of the last equation and simplifying gives

$$x^2 - 2x\sqrt{5} = 2\sqrt{6},$$

    which is progress, since only *two* square roots remain. Squaring again gives

$$x^4 - 4x^3\sqrt{5} + 20x^2 = 24,$$

    which is even better, as only *one* square root is left. The last equation implies that

$$\sqrt{5} = \frac{x^4 + 20x^2 - 24}{4x^3}.$$

    Because x is rational, so is the right-hand side above, and thus so is the left. This absurdity completes the proof.

11. Parts (i) and (ii) follow from the fact that $1 < a/b < 2$. For part (iii), note that

$$\frac{a'^2}{b'^2} = \frac{(2b-a)^2}{(a-b)^2} = \frac{4b^2 - 4ab + a^2}{a^2 - 2ab + b^2},$$

    and substituting $a^2 = 2b^2$ shows that the last fraction is 2.

    This all shows that if $\sqrt{2} = a/b$ holds for *any* positive integers a and b, then we can find a new fraction a'/b' with $\sqrt{2} = a'/b'$ and $b' < b$, which is absurd.

12. (a) $\mathbb{Z}_2$ is not closed under addition: $1 + 1 = 2 \notin \mathbb{Z}_2$.

    (b) $\mathbb{Z}_2$ satisfies all the requirements in Theorem **??**.

13. Matrix addition in $M_{2\times 2}$ is commutative, but multiplication is not; examples are easy to find. Every matrix A in $M_{2\times 2}$ has an additive inverse $-A$, but multiplicative inverses exist only for some nonzero matrices (those with nonzero determinant); again, examples are easy to find. Distributivity does indeed hold in $M_{2\times 2}$.

14. If a and b are rational, then
$$\frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2+2b^2}.$$
This shows that elements of F have multiplicative inverses in F. The rest is easier.

15. Since $\ln\ln\ln n$ tends to infinity, it must exceed two for large n. The well-ordering property guarantees that a smallest such n_0 exists. (Using a calculator we can see that n_0 has about 703 decimal digits.)

16. (a) The answer is no. For instance, the set $\{1, 1/2, 1/3, \dots\}$ is a subset of $\mathbb{Q}$, but has no least element.
    (b) The set $R = \{1, 10, 100, 1000, \dots\}$ does have the well-ordering property; every nonempty subset includes a *smallest* power of 10.
    (c) The set $T = \{-3, -2, -1, \dots, 41, 42\}$ (like all finite sets of real numbers) *does* have the well-ordering property, since every nonempty subset of T is also finite, and hence has a least element.
    (d) If we trade "least" for "greatest" in the well-ordering property, the result no longer holds for $\mathbb{N}$, since $\mathbb{N}$ itself has no greatest element. The property *does* hold for the finite set T, and also for $\mathbb{Z} \setminus \mathbb{N} = \{\dots, -3, -2, -1, 0\}$.

1.2 Sets 101: Getting Started

1. (a) $D \subset I$; $D \in C$.
   (b) $B = \{m \in A \mid m \text{ has 31 days}\}$.
   (c) $A \times D$ is the set of ordered pairs (January, 2), (February, 2), …, (December, 2), (January, 3), (February, 3), …, (December, 3). There are 24 such pairs.
   (d) $A \setminus B = \{$February, April, June, September, November$\}$; $B \setminus A = \emptyset$; $A \cap C = \{$November$\}$; $B \cap A = B$; $D \cap I = D$; $D \cup I = I$.

2. (a) $S = \{0, -1\}$; T is the interval of numbers between $(-1-\sqrt{21})/2 \approx -2.791$ and $(-1+\sqrt{21})/2 \approx 1.791$.

(b) Decide whether each of the following statements is true or false, and explain: $S \subset \mathbb{N}$ is false because $-1 \notin \mathbb{N}$; $S \subset T$ is true; $T \cap \mathbb{Q} \neq \emptyset$ is true, since $0 \in T \cap \mathbb{Q}$; $-2.8 \in \mathbb{Q} \setminus T$ is true.

(c) The quadratic formula shows that $U = \left\{x \in \mathbb{R} \mid x^2 + x < 0\right\} = (-1, 0)$.

3. (a) $\mathbb{R} \setminus A = (-\infty, 1) \cup (3, \infty)$

(c) $\mathbb{R} \setminus A = (-\infty, 1]) \cup [2, 3] \cup [4, \infty)$

(e) $\mathbb{R} \setminus A = \{0\}$

4. (a) $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$

(b) $I = (-\infty, \infty)$ has empty complement; $I = (-\infty, 17)$ has closed complement $[17, \infty)$; $I = (0, 17)$ has complement $(-\infty, 0] \cup [17, \infty)$.

(c) $\mathbb{R} \setminus \mathbb{Z} = (0, 1) \cup (-1, 0) \cup (1, 2) \cup (-2, -1) \cup \ldots$.

5. To say that a is in $\mathbb{R} \setminus (\mathbb{R} \setminus A)$ means that a is *not* in $\mathbb{R} \setminus A$; this means, in turn that $a \in A$.

6. (a) Claim (i) is false. As one example, take $A = \mathbb{R}$ and $B = \emptyset$. Then $R \setminus (A \cup B) = \emptyset$ but $(R \setminus A) \cup (R \setminus B) = \mathbb{R}$.

(b) If $A = B$, then $A \cup B = A \cap B = A$, and both claims are clearly true.

(c) To prove that (ii) holds, suppose $x \in \mathbb{R} \setminus (A \cup B)$. Thus $x \notin A \cup B$, so $x \notin A$ and $x \notin B$; in other words, $x \in \mathbb{R} \setminus A$ and $x \in \mathbb{R} \setminus B$. This is just another way of saying that $x \in (\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B)$. The "vice versa" implication is similar.

7. We know $\mathbb{R} \setminus A_1 = (-\infty, 1] \cup [3, \infty)$ and $\mathbb{R} \setminus A_2 = (-\infty, 2] \cup [5, \infty)$. Also, $\mathbb{R} \setminus (A_1 \cap A_2) = (-\infty, 2] \cup [3, \infty)$ and $\mathbb{R} \setminus (A_1 \cup A_2) = (-\infty, 1] \cup [5, \infty)$.

It's easy to see that, as claimed, $\mathbb{R} \setminus (A_1 \cap A_2) = (-\infty, 2] \cup [3, \infty) = (-\infty, 1] \cup [3, \infty) \cup (-\infty, 2] \cup [5, \infty) = (\mathbb{R} \setminus A_1) \cup (\mathbb{R} \setminus A_2)$. Similarly, $\mathbb{R} \setminus (A_1 \cup A_2) = (-\infty, 1] \cup [5, \infty) = ((-\infty, 1] \cup [3, \infty)) \cap ((-\infty, 2] \cup [5, \infty) = (\mathbb{R} \setminus A_1) \cup (\mathbb{R} \setminus A_2)$

8. Here we have $A_1 \cup A_2 = (0, 1) \cup (2, \infty)$ and $A_1 \cap A_2 = \emptyset$. Thus $\mathbb{R} \setminus A_1 = (-\infty, 0] \cup [1, \infty)$ and $\mathbb{R} \setminus A_2 = (-\infty, 2]$. This implies that $(\mathbb{R} \setminus A_1) \cup (R \setminus A_2) = (-\infty, \infty)$ and $(\mathbb{R} \setminus A_1) \cap (R \setminus A_2) = (-\infty, 0) \cup [1, 2]$. Consistent with De Morgan, $\mathbb{R} \setminus (A_1 \cap A_2) = (-\infty, infty)$; $\mathbb{R} \setminus (A_1 \cup A_2) = (-\infty, 0] \cup [1, 2]$.

9. If $x \in T'$ then $x \notin T$. Since $T \supset S$, we have $x \notin S$, which means $x \in S'$, as desired.

10. Many possibilities exist for (a), (b), and (c). For (d) we could use $I = (0, \infty)$ and $J = (1, \infty)$; note that here, as in *all* possibilities for (d), one interval is contained in the other.

11. (a) $I = (-42, 0)$ and $J = (0, \infty)$ work.

    (b) $I = (-42, 0)$ and $J = [0, \infty)$ work.

    (c) The given conditions (draw a picture) mean that $a < c < 0 < b < d$, so $I \cup J = (a, d)$ which is indeed an open interval.

12. Suppose $a \in I \cup J$, $b \in I \cup J$, and $a < x < b$. We're done if we show that $x \in I \cup J$. This is trivial if $x = c$, so we assume $x \neq c$. Now if both $a \in I$ and $b \in I$, then $x \in I$ by Definition **??**, and we're done. Similarly, we're done if both $a \in J$ and $b \in J$. So let's assume $a \in I$ and $b \in J$. If $x < c$, then we have $a < x < c$ with a and c in I; by Definition **??**, $x \in I$, too. Similarly, if $x > c$, then we have $c < x < b$ and so $x \in J$. We're done.

13. It's easy for I and $\mathbb{R} \setminus I$ to be intervals. For instance, if $I = (-\infty, 0)$, then $\mathbb{R} \setminus I = [0, \infty)$ is another interval. I and $\mathbb{R} \setminus I$ cannot both be bounded intervals; two bounded intervals can't "add up" to the unbounded set $(-\infty, \infty)$.

14. No. Any finite set I of numbers contains a smallest number, say a, and a second smallest, say b. If I were an interval, it would also have to contain the average, $(a + b)/2$, which lies (illegally) between a and b.

15. No. Suppose a and b are rational numbers in I, with $a < b$. Consider $c = a + (b - a)/\sqrt{2}$. Note that $c \in \mathbb{R} \setminus \mathbb{Q}$ and that $a < c < b$. If I were an interval, we'd have $c \in I$, which is impossible.

16. (a) $(1, 2) \cup (3, \infty)$ is the union of two open intervals, and hence also open. The complement, $(-\infty, 2] \cup [2, 3]$, is therefore closed.

    (b) $\mathbb{R} \setminus \{a\} = (-\infty, a) \cup (a, \infty)$.

    (c) $(-\infty, a)$ is itself an open interval, and therefore open. The complement of $(-\infty, a]$ is the open interval (a, ∞), so $(-\infty, a]$ is closed.

    (d) If $I = (0, 1)$ were closed, then $\mathbb{R} \setminus I$ would be open. This is false, because $1 \in \mathbb{R} \setminus I$, but no open interval containing 1 is contained in $\mathbb{R} \setminus I$.

17. (a) The complement of $\{1, 2, 3\}$ consists of four open intervals.

    (b) $\mathbb{R} \setminus \mathbb{Z}$ is the union of all open intervals of the form $(n, n + 1)$, where $n \in \mathbb{Z}$.

    (c) If $\mathbb{Q}$ were open, we could find for each rational q an open interval I with $q \in I \subseteq$. But $I \subseteq \mathbb{Q}$ is impossible.