Problems to Ch. 1 (solutions)

1.1 The charging energy required to charge a capacitor with capacitance C by the charge of a single electron is $E_C = e^2/(2C)$. Let us assume a simple parallel-plate model for a capacitor, $C = \epsilon A/d$, where $\epsilon = \epsilon_r\epsilon_0$, typical $\epsilon_r = 10$ and $\epsilon_0 = 8.85$ pF/m, A is the area of the plates and d their separation. We can also set $d = 2$ nm, a typical thickness of the oxide layer ('tunnel contact') formed between two metals. Assuming a square plate, $A = w^2$, estimate a width w of the junction which would correspond to the charging energy E_C being equal to 1 K. Estimate also the corresponding capacitances. How should these scales change so that charging effects would be observable at room temperature, i.e., $E_C/k_B \approx 300$ K? Remember that $e = 1.6 \times 10^{-19}$ C and $k_B = 1.38 \times 10^{-23}$ J/K.

- Solution:

  Charging energy: $E_C = e^2/(2C)$. Assumptions: $C = \epsilon_0\epsilon_r A/d$ and $A = w^2$. Then:

$$E_C = \frac{de^2}{2\epsilon_0\epsilon_r w^2} = 0.010\,\mathrm{K} \times k_B \left[\frac{1\mu m}{w}\right]^2 , \tag{1}$$

$$C = 89\,\mathrm{fF} \times \left[\frac{w}{1\mu m}\right]^2 . \tag{2}$$

  Hence, $E_C = 1\,\mathrm{K}$ corresponds to $w \approx 0.1\,\mu$m and $C = 0.89\,\mathrm{fF}$. Also, $E_C = 300\,\mathrm{K}$ equals $w = 0.1\,\mu\mathrm{m}/\sqrt{300} \approx 5.8\,\mathrm{nm}$, $C = 0.89\,\mathrm{fF}/300 \approx 3.0 \times 10^{-18}\,\mathrm{F}$.

1.2 At the end of Example 1.1 the density of states and the transmission probability through the tunnel barrier are assumed to be energy independent. To mimic the finite width Δ of the conduction band, assume that the density of states has a sharp cutoff at $|E - \mu_L| = \Delta/2$, i.e., it is constant for energies $|E - \mu_L| < \Delta/2$ and vanishes for $|E - \mu_L| > \Delta/2$. Show that the correction to the linear tunnel barrier conductance is proportional to $\exp[-\Delta/(2k_BT)]$ when $\Delta \gg k_BT$, i.e., in this limit the corrections to the linearity are exponentially small. In other words, show that

$$1/R_T - \frac{dI}{dV}|_{V=0} \overset{\Delta \gg k_BT}{\propto} \exp[-\Delta/(2k_BT)]$$

for the current defined in eqn (1.12) and R_T defined below eqn (1.15).

The solution for this exercise has not been keyed in yet.

1.3 Show that the heat current through a tunnel barrier in the linear response regime obeys a Wiedemann–Franz law, i.e., $\dot{Q}/\Delta T \propto T/R_T$. Find also the prefactor of this expression. Hint: Assume a vanishing voltage and that the temperatures $T_{L/R}$ of the left/right reservoirs are $T_{L/R} = T \pm \Delta T/2$. Finally, take the linear order in ΔT.

- Solution:

  The heat current through a tunnel barrier is (for $V = 0$)

$$\dot{Q} = \underbrace{cAtN_L(0)N_R(0)}_{[e^2R_T]^{-1}} \int_{-\infty}^{\infty} \mathrm{d}E E[f_L(E) - f_R(E)] .$$

  Here, expand in ΔT (let $k_B \equiv 1$ to simplify notation...)

$$\begin{aligned} f_{L/R}(E) &= [e^{E/(T\pm\frac{1}{2}\Delta T)} + 1]^{-1} \\ &= \underbrace{[e^{E/T} + 1]^{-1}}_{f_0(E)} + \Delta T \frac{\mathrm{d}f_0(E)}{\mathrm{d}(E/T)} \frac{\mathrm{d}}{\mathrm{d}(\Delta T)} \left.\frac{E}{T \pm \frac{1}{2}\Delta T}\right|_{\Delta T=0} + \mathcal{O}((\Delta T)^2) \\ &= f_0(E) \mp \frac{\Delta T}{2T} \frac{\mathrm{d}f_0(E)}{\mathrm{d}E} E + \mathcal{O}((\Delta T)^2) , \end{aligned}$$

so that

$$\dot{Q} \approx \frac{-\Delta T}{e^2 R_T T} \int_{-\infty}^{\infty} \mathrm{d}E E^2 \frac{\mathrm{d}f_0(E)}{\mathrm{d}E} = \frac{-\Delta T}{e^2 R_T} T(-1) \underbrace{\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{x^2 e^x}{(e^x+1)^2}}_{=\pi^2/3}$$

$$= \frac{\pi^2}{3} \left(\frac{k_B}{e}\right)^2 \frac{T}{R_T} \Delta T \, .$$

The linearized thermal conductance is then

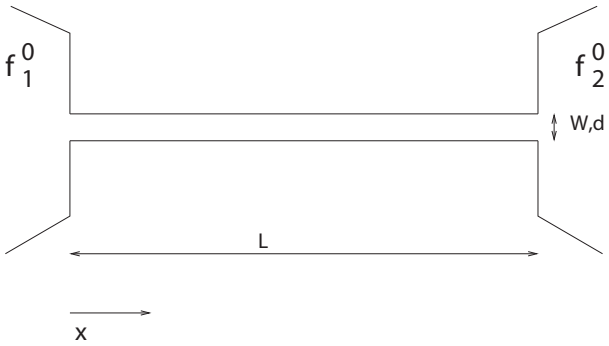
$$G_{th} = \frac{\mathrm{d}\dot{Q}}{\mathrm{d}(\Delta T)} = \frac{\pi^2}{3} \left(\frac{k_B}{e}\right)^2 \frac{T}{R_T} = \mathcal{L}_0 \frac{T}{R_T} \, ,$$

and is indeed proportional to T/R_T. Here, $\mathcal{L}_0$ is the Lorenz number for an ideal electron gas.

1.4 Question on a scientific paper. In nanoelectronics measurements, the most typically measured observable is either the current as a function of voltage or the (linear) conductance as a function of some control parameter, such as the gate voltage or a magnetic field. Sometimes the voltage is measured as a function of current. Consult the papers Smit *et al.*, *Nature* **419**, 906 (2002), Kuemmeth *et al.*, *Nature* **452**, 448 (2008) and Heersche *et al.*, *Nature* **446**, 56 (2007), and determine what the measured observables and the control parameters were in the reported experiments. Note that the size of the typically measured nanoelectronic structures ranges from a few nanometres to some micrometres. On the other hand, the measurement equipment is on our everyday scale (from some tens of cm to metres). Argue why and in which case the latter can show some information about the former.

The solution for this exercise has not been keyed in yet.

Problems to Ch. 2 (solutions)



2.1 Show that the diffusion constant (see eqn (2.21) for the three-dimensional case) in two- and one-dimensional cases equals $D = v^2\tau/2$ and $D = v^2\tau$, respectively.

- Solution: Repeating the derivation from eqn (2.17) to eqn (2.22), we notice that the diffusion constant is the angular average of $(\mathbf{v}(\hat{p})\cdot\hat{u}_z)^2\tau$. In the two-dimensional case we hence go to the polar coordinates and write $\mathbf{v}\cdot\hat{u}_z = v\cos(\theta)$, where θ is the angle between the speed $\mathbf{v}$ and the reference direction $\hat{u}_z$. Then the angular average is

$$D_2 = v^2\tau\int_0^{2\pi}\frac{d\theta}{2\pi}\cos^2\theta = v^2\tau/2. \tag{3}$$

In the 1d case, there is no other direction than the reference direction $\hat{u}_z$, but we can separate left- and right-going particles (those with $\mathbf{v}\cdot\hat{u}_z > 0$ and < 0, respectively). However, as the diffusion constant concerns only the square of $\mathbf{v}\cdot u_z$, these give the same contribution. Then

$$D_1 = \sum_{L,R}\frac{1}{2}v^2\tau = v^2\tau. \tag{4}$$

With these changes we can write the diffusion equation also in 2d and 1d systems.

2.2 Phonons in thin films may not always be strongly coupled to the phonons in the substrate. The thermal link between these two phonon systems is described by the Kapitza thermal resistance, which between two bulky phonon systems (dimensions $\gg$ phonon wavelength) is proportional to T^{-3}, i.e., $R_K = r_k T^{-3}$ with a system-dependent coefficient r_k. Assume an electron system in the mesoscopic film is heated with a small constant power P, resulting in an electron temperature $T_e = T_b + \Delta T_e$ and film phonon temperature $T_f = T_b + \Delta T_f$, $\Delta T_e, \Delta T_f \ll T_b$ slightly higher than the temperature T_b of the substrate. Linearize eqn (2.39) and find ΔT_e and ΔT_f.

- Solution: The power dissipated from the electrons (e) to the film phonons (f) is given by eqn (2.39)

$$P_{e-f} = \Sigma\Omega\left(T_e^5 - T_f^5\right). \tag{5}$$

Since $T_{e,f} = T_b + \Delta T_{e,f}$, where $\Delta T_{e,f} \ll T_b$, we can linearize this equation so that

$$P_{e-f} \approx 5\Sigma\Omega T_b^4\left(\Delta T_e - \Delta T_f\right). \tag{6}$$

The coupling between the film and substrate phonons (b for bath) is described by thermal resistance R_{th}, which is defined such that to linear order in the temperature difference ΔT

$$P_{f-b} = \frac{\Delta T}{R_{th}}. \tag{7}$$

Here $R_{th} = R_K = r_K T^{-3}$ so that $T_f - T_b = \Delta T_f = r_K P_{f-b} T^{-3} \approx r_K P_{f-b} T_b^{-3}$.

We are looking for a steady-state solution where the substrate is held at a constant temperature, e.g., by means of a cryostat. In this case $P = P_{e-f} = P_{f-b}$. This condition gives us two equations for two

unknowns, which are easily solved. As a result we obtain

$$\Delta T_f = \frac{r_K P}{T_b^3}, \tag{8}$$

$$\Delta T_e = \frac{P}{T_b^3}\left(r_K + \frac{1}{5\Sigma\Omega T_b}\right). \tag{9}$$

The Kapitza thermal resistance is strictly valid only for the coupling between bulky phonon systems. For metals the typical phonon wavelength is given by $\lambda = \frac{\hbar v_S}{k_B T}$, where v_s is the sound velocity, typically of the order of $3\ldots5$ km/s. At 1 K we then have $\lambda \approx 100$ nm and at 100 mK $\lambda \approx 1\ \mu$m. This is a lot larger than the thickness of the film, but Kapitza resistance still explains experimental results quite well.

2.3 Assume a conductor with an unknown type of scattering described via the relaxation time τ. Starting from eqns (2.4), (2.8) and (2.10), show that in linear response the conductance is still given by the Drude form. Hint: Write the distribution function as $f(\mathbf{r},\mathbf{p}) = f_0(\mathbf{p}) + g(\mathbf{p})$, where $f_0(\mathbf{p})$ is the Fermi function and to first approximation the correction $g(\mathbf{p})$ is independent of position. Note that around the Fermi level the momentum and the velocity are connected via the usual relation $\mathbf{p} \approx m^*\mathbf{v}$. Compute $g(\mathbf{p})$ from (2.4), and insert it in eqn (2.8) to obtain a formula for the current.

- Solution: With a time and position independent distribution function, the Boltzmann equation (2.4) reads

$$e\mathbf{E}\cdot\partial_{\mathbf{p}}[f_0(\mathbf{p}) + g(\mathbf{p})] = -\frac{1}{\tau}g(\mathbf{p}). \tag{10}$$

Since we assume $g(\mathbf{p})$ to be small, we can also drop it from the left hand side of the equation. We can thus write

$$g(\mathbf{p}) = -e\mathbf{E}\cdot\partial_{\mathbf{p}} f_0(\mathbf{p}). \tag{11}$$

Let us fix the coordinate axes so that $\mathbf{E} = E_0\hat{u}_z$. In this case $\mathbf{E}\cdot\partial_{\mathbf{p}} = E_0\partial_{p_z} = E_0[-p\sin(\theta)\partial_\theta + \cos(\theta)\partial_p]$, where the latter is written in spherical coordinates. For a spherical Fermi surface $f_0(\mathbf{p}) = f_0(E_p)$ is independent of the angle, and we can disregard the first term. Inserting the full distribution function into eqn (2.8) then yields the current density

$$\begin{aligned}\mathbf{j}_C &= -e\int \frac{d^3p}{4\pi^2\hbar^3}\mathbf{v}(\mathbf{p})g(\mathbf{p}) = e^2E_0\int dE N(E)\tau\int_0^{2\pi}\frac{d\varphi}{2\pi}\int_{-1}^{1}\frac{d(\cos\theta)}{2}v(E)\cos(\theta)[\cos(\theta)\partial_p]f_0(E_p)\hat{u}_z \\ &= \frac{e^2E_0}{3}\int dE N(E)v^2(E)\tau\partial_E f_0(E)\hat{u}_z.\end{aligned} \tag{12}$$

where we have taken into account the fact that the isotropic zeroth order term $f_0(\mathbf{p})$ does not yield a contribution to the average current and that the velocity components in other directions than $\hat{u}_z$ lead to vanishing currents (the integral over the phases averages to zero). In the last equation we use the fact that $v = \partial E_p/\partial p$. Now the Drude result comes from assuming that the density of states, velocity and the scattering time are energy independent in the region of energies where $\partial_E f_0(E)$ is non-zero — in practice, a few k_BT around the chemical potential. In that case, we get

$$\mathbf{j}_C = \frac{e^2N_F v_F^2\tau}{3}\mathbf{E} \equiv \sigma_D\mathbf{E}. \tag{13}$$

2.4 Using the Drude formula with a general scattering time, show how the conductance depends on temperature when $\tau_{e-ph}(T)$ given by eqn (2.40) becomes shorter than the elastic scattering time $\tau_{\rm el}$. Often the relaxation rates $1/\tau$ for different types of scattering process add up. This is called the *Matthiessen's rule*.

According to the Drude formula,

$$\sigma_D = \frac{e^2N_Fv_F^2}{3}\tau_{\rm tot}, \quad \text{with} \tau_{\rm tot}^{-1} = \tau_{\rm el}^{-1} + \tau_{\rm e-ph}^{-1}. \tag{14}$$