Solutions of Chapter 2 Problems

Problem 2.1

a) Consider the binary hypothesis testing problem

$$H_0 \; : \; X = 0$$
$$H_1 \; : \; X = 1 \, ,$$

where the a-priori probabilities of the two hypotheses are given by

$$\pi_0 = P[X=0] = 1/4 \quad , \quad \pi_1 = P[X=1] = 3/4 \, .$$

Since $Y = X + V$, under H_0 we have $Y = V$, whereas under H_1 $Y = 1 + V$, so that the probability mass distributions of Y under H_0 and H_1 are as sketched below.

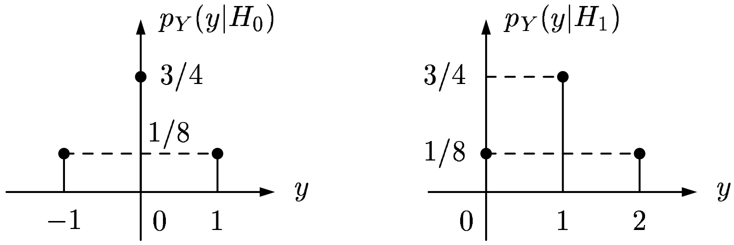

Figure 1: Probability mass distribution of Y under H_0 and H_1.

The costs corresponding to the minimum probability of error risk function are given by $C_{ij} = 1 - \delta_{ij}$.

The likelihood ratio function is given by

$$L(y) = \frac{p_Y(y|H_1)}{p_Y(y|H_0)} = \begin{cases} \infty & y = 2 \\ 6 & y = 1 \\ 1/6 & y = 0 \\ 0 & y = -1 \, , \end{cases}$$

and the LRT takes the form

$$L(y) \underset{H_0}{\overset{H_1}{\gtrless}} \tau = \frac{(C_{10} - C_{00})\pi_0}{(C_{01} - C_{11})\pi_1} = \frac{1}{3} \, .$$

The optimum decision rule can therefore be expressed as

$$\hat{X} = \begin{cases} 1 & \text{for} \quad y \geq 1 \\ 0 & \text{for} \quad y \leq 0 \, . \end{cases}$$

b) The probability of a correct decision is

$$\begin{aligned} P[\hat{X} = X] &= P[\hat{X} = X = 1] + P[\hat{X} = X = 0] \\ &= P[\hat{X} = 1|H_1]\pi_1 + P[\hat{X} = 0|H_0]\pi_0 \\ &= P[Y \geq 1|H_1]\pi_1 + P[Y \leq 0|H_0]\pi_0 \\ &= (\frac{1}{8} + \frac{3}{4})\frac{1}{4} + (\frac{1}{8} + \frac{3}{4})\frac{3}{4} = \frac{7}{8}\,. \end{aligned}$$

Problem 2.2

a) Since $Y = AX + V$ where X takes values 0 and 1 with probabilities $\pi_0 = 1/3$ and $\pi_1 = 2/3$, respectively, the hypothesis testing problem takes the Gaussian form

$$\begin{aligned} H_0 &: Y = V \sim N(0,1) \\ H_1 &: Y = A + V \sim N(1,4)\,. \end{aligned}$$

So the LRT is given by

$$\begin{aligned} L(y) &= \frac{f_Y(y|H_1)}{f_Y(y|H_0)} \\ &= \frac{1}{2}\frac{\exp(-(y-1)^2/8)}{\exp(-y^2/2)} \underset{H_0}{\overset{H_1}{\gtrless}} \tau\,, \end{aligned} \tag{2.1}$$

where since we seek to minimize the probability of error

$$\tau = \frac{\pi_0}{\pi_1} = \frac{1}{2}\,. \tag{2.2}$$

Substituting (2.2) inside (2.1), we obtain the test

$$\exp(-(y-1)^2/8) \underset{H_0}{\overset{H_1}{\gtrless}} \exp(-y^2/2)\,.$$

Equivalently, since the logarithm function preserves ordering, we have

$$-(Y-1)^2/8 \underset{H_0}{\overset{H_1}{\gtrless}} -Y^2/2\,,$$

and after some simple algebraic manipulations, we get

$$q(Y) \triangleq 3Y^2 + 2Y - 1 \underset{H_0}{\overset{H_1}{\gtrless}} 0\,.$$

The quadratic equation $q(y) = 0$ has for roots -1 and $1/3$ so the Bayesian test is given by

$$\delta_B(Y) = \begin{cases} 0 & \text{for } -1 < Y < 1/3 \\ 1 & \text{otherwise}\,. \end{cases}$$

Problem 2.3

a) Under H_1, since $Y = X + V$ with X and V independent, the probability density of Y is the convolution of the densities of X and V. This gives

$$\begin{aligned} f_Y(y|H_1) &= f_X(y) * f_V(y) \\ &= ab\Big[\int_0^y \exp(-a(y-v))\exp(-bv)dv\Big]u(y) \\ &= ab\exp(-ay)u(y)\int_0^y \exp(-(b-a)v)dv \\ &= \frac{ab}{b-a}\exp(-ay)u(y)[1-\exp(-(b-a)y)] \\ &= \frac{ab}{b-a}[\exp(-ay)-\exp(-by)]u(y)\ , \end{aligned}$$

where $u(\cdot)$ denotes the unit step function.

The likelihood ratio test (LRT) takes the form

$$\begin{aligned} L(y) &= \frac{f_Y(y|H_1)}{f_Y(y|H_0)} \\ &= \frac{ab}{b-a}\frac{[\exp(-ay)-\exp(-by)]}{b\exp(-by)} \underset{H_0}{\overset{H_1}{\gtrless}} \tau\ , \end{aligned} \tag{2.3}$$

which yields after simplifications

$$\frac{a}{b-a}[\exp((b-a)y)-1] \underset{H_0}{\overset{H_1}{\gtrless}} \tau\ .$$

Assuming without loss of generality that $b > a$, this gives

$$\exp((b-a)y) \underset{H_0}{\overset{H_1}{\gtrless}} \big(\frac{b}{a}-1\big)\tau + 1$$

or equivalently

$$Y \underset{H_0}{\overset{H_1}{\gtrless}} \eta \triangleq \frac{1}{b-a}\log\big[\big(\frac{b}{a}-1\big)\tau+1\big]\ . \tag{2.4}$$

b) The cost function corresponding to the minimum probability of error criterion is given by $C_{ij} = 1 - \delta_{ij}$. If the two hypotheses are equally likely, $\pi_0 = \pi_1 = 1/2$, so that the threshold $\tau = 1$ Substituting in (2.4), we find that the threshold

$$\eta = \frac{1}{b-a}\log(b/a)\ .$$

Problem 2.4:

a) For the densities

$$f_Y(y|H_0) = \frac{1}{(2\pi)^{1/2}} \exp(-y^2/2)$$
$$f_Y(y|H_1) = \frac{1}{2} \exp(-|y|)$$

the likelihood ratio function takes the form

$$L(y) = \frac{f_Y(y|H_1)}{f_Y(y|H_0)} = \left(\pi/2\right)^{1/2} \exp\left(\frac{y^2}{2} - |y|\right) .$$

b) The function $L(y)$ is plotted as a function of y in Fig. 2 below.

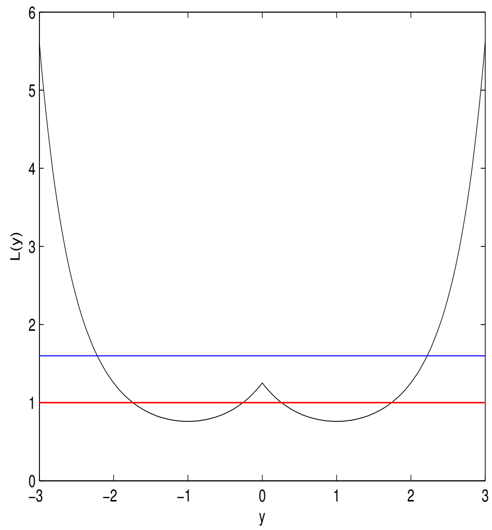

Figure 2: Plot of likelihood ratio function $L(y)$.

The decision rule

$$L(y) \underset{H_0}{\overset{H_1}{\gtrless}} \tau$$

is implemented by drawing a horizontal line at $L(y) = \tau$ and choosing H_1 for all values of y for which $L(y)$ is above the line. From this simple observation we deduce that if

$$\tau \leq \left(\pi/2\right)^{1/2} \exp(-1/2) ,$$

we always choose H_1. If

$$\left(\pi/2\right)^{1/2} \exp(-1/2) < \tau \leq \left(\pi/2\right)^{1/2} ,$$

there are two regions of identical width, centered about 1 and -1, for which H_0 is selected. If

$$\tau > \left(\pi/2\right)^{1/2}$$

there is only one region, centered about zero, where H_0 is selected. These last two cases are illustrated by red and blue horizontal lines in Fig. 2, which correspond respectively to the choices $\tau = 1$ and $\tau = 1.6$.

Problem 2.5

a) Consider the binary hypothesis testing problem

$$\begin{aligned} H_0 &: \quad f_{Y_k}(y_k|H_0) = \frac{\lambda}{2}\exp(-\lambda|y_k|) \\ H_1 &: \quad f_{Y_k}(y_k|H_1) = \frac{\lambda}{2}\exp(-\lambda|y_k - A|) \,, \end{aligned}$$

where the Y_k's are independent for $1 \leq k \leq N$. The likelihood ratio function can be expressed as

$$L(\mathbf{y}) = \prod_{k=1}^{N} L(y_k) \tag{2.5}$$

with

$$L(y_k) = \exp[\lambda(|y_k| - |y_k - A|)] \,.$$

Accordingly, by taking logarithms and dividing by N, the LR test can written as

$$S \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \frac{1}{N}\ln(\tau) \,,$$

where

$$S = \frac{1}{N}\sum_{k=1}^{N} \ln L(Y_k) \,,$$

with

$$\ln L(y) = c(y - A/2) = \begin{cases} -A & y \leq 0 \\ 2y - A & 0 \leq y \leq A \\ A & y \geq A \,. \end{cases}$$

b) When the two hypotheses are equally likely ($\pi_0 = \pi_1 = 1/2$) and we seek to minimize the probability of error, so the Bayesian costs are given by $C_{ij} = 1 - \delta_{ij}$, the Bayesian threshold $\tau = 1$, so $\eta = 0$.

c) The clipping function $c(\cdot)$ truncates the high and low values of centered observations $Y_k - A/2$ to prevent them from dominating the sum S.

Problem 2.6:

a) Since $f_Y(y|H_0)$ and $f_Y(y|H_1)$ are densities, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y|H_0)dy &= A_0 \int_{-a}^{a} (a - |y|)dy = A_0 a^2 = 1 \\ \int_{-\infty}^{\infty} f_Y(y|H_1)dy &= A_1 \int_{-b}^{b} (b - |y|)dy = A_1 b^2 = 1 \,, \end{aligned}$$

so that

$$A_0 = \frac{1}{a^2} \quad , \quad A_1 = \frac{1}{b^2} \, .$$

b) We have

$$\begin{aligned} L(y) &= \frac{f_Y(y|H_1)}{f_Y(y|H_0)} \\ &= \begin{cases} \infty & \text{for} \quad a \leq |y| \leq b \\ \dfrac{a^2}{b^2}\dfrac{b-|y|}{a-|y|} & \text{for} \quad 0 \leq |y| < a \, . \end{cases} \end{aligned}$$

Thus for $\tau < a/b$ we always choose H_1, whereas for $\tau > a/b$, the LRT takes the form

$$\frac{a^2}{b^2}\frac{(b-|y|)}{(a-|y|)} \underset{H_0}{\overset{H_1}{\gtrless}} \tau \, ,$$

or equivalently

$$|Y| \underset{H_0}{\overset{H_1}{\gtrless}} \eta \triangleq ba\frac{(b\tau - a)}{(b^2\tau - a^2)} \, .$$

c) We have

$$\begin{aligned} P_D(\eta) &= \int_\eta^\infty f_Y(y|H_1)dy + \int_{-\infty}^{-\eta} f_Y(y|H_1)dy \\ &= \frac{2}{b^2}\int_\eta^b (b-y)dy = \frac{(b-\eta)^2}{b^2} \, . \end{aligned}$$

Similarly

$$\begin{aligned} P_F(\eta) &= \int_\eta^\infty f_Y(y|H_0)dy + \int_{-\infty}^{-\eta} f_Y(y|H_0)dy \\ &= \frac{(a-\eta)^2}{a^2} \, . \end{aligned}$$

Since

$$\begin{aligned} \eta &= b\big(P_D^{1/2} - 1\big) \\ &= a\big(P_F^{1/2} - 1\big) \, , \end{aligned} \tag{2.6}$$

the ROC takes theform

$$P_D = \big[1 + \frac{a}{b}(P_F^{1/2} - 1)\big]^2 \, .$$

For $b = 2a$, this gives

$$P_D = \frac{(1 + P_F^{1/2})^2}{4} \, , \tag{2.7}$$

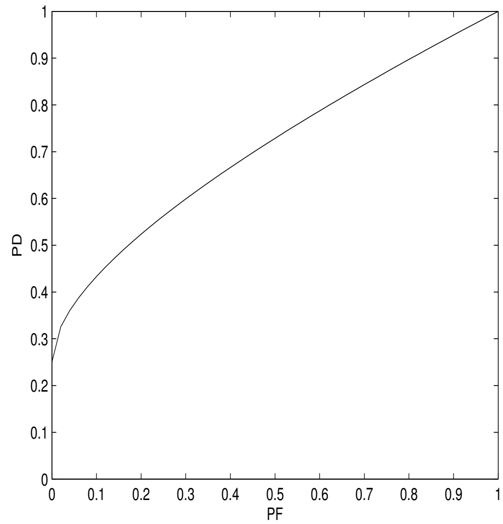

Figure 3: Plot of receiver operating characteristic (2.7).

which is sketched below.

Problem 2.7:

The minimum probability of error decision rule is the LRT

$$L(k) = \frac{P[K=k|H_1]}{P[K=k|H_0]} = \frac{q_1^k(1-q_1)^{n-k}}{q_0^k(1-q_0)^{n-k}} \underset{H_0}{\overset{H_1}{\gtrless}} \tau = \frac{\pi_0}{\pi_1} \,,$$

where we have used the costs $C_{ij} = 1 - \delta_{ij}$ with $i,\ j = 0,\ 1$. This rule can be expressed as

$$K = k \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

with

$$\eta = \frac{\ln\left(\frac{\pi_0(1-q_0)^n}{\pi_1(1-q_1)^n}\right)}{\ln\left(\frac{q_1(1-q_0)}{q_0(1-q_1)}\right)} \,.$$

b) Since the random variable K is integer valued, the ROC is obtained by connecting linearly the discrete points $(P_F(m), P_D(m))$ corresponding to the test

$$K \underset{H_0}{\overset{H_1}{\gtrless}} m \tag{2.8}$$

for $-1 \leq m \leq n$. The probabilities of false alarm and of detection for test (2.8) can be expressed as

$$\begin{aligned} P_F(m) &= \sum_{k=m+1}^{n} \binom{n}{k} q_0^k (1-q_0)^{n-k} \\ P_F(m) &= \sum_{k=m+1}^{n} \binom{n}{k} q_1^k (1-q_1)^{n-k} , \end{aligned}$$

where

$$\binom{n}{k} \triangleq \frac{n!}{k!(n-k)!} .$$

For $n = 2$ we only need to consider $-1 \leq m \leq 2$, and in this case $(P_F(m), P_D(m))$ are specified by Table 1 shown below.

m	$P_F(m)$	$P_D(m)$
-1	1	1
0	$2q_0(1-q_0)+q_0^2$	$2q_1(1-q_1)+q_1^2$
1	q_0^2	q_1^2
2	0	0

Table 1: Probabilities $(P_F(m), P_D(m))$ for $n = 2$ and $-1 \leq m \leq 2$.

The ROC is plotted in Fig. 4 below for $q_0 = 0.2$ and $q_1 = 0.6$. Note that the linear segments connecting the four discrete points $(P_F(m), P_D(m))$ with $-1 \leq m \leq 2$ are obtained by randomization.

Problem 2.8:

a) The Bayesian test can be expressed as

$$\begin{aligned} L(y) &= \frac{f_Y(y|H_1)}{f_Y(y|H_0)} \\ &= 2\frac{\exp(-2|y|)}{\exp(-|y|)} = 2\exp(-|y|) \underset{H_0}{\overset{H_1}{\gtrless}} \tau , \end{aligned}$$

or equivalently

$$|Y| \underset{H_1}{\overset{H_0}{\gtrless}} \eta \triangleq -\ln(\tau/2) .$$

b) The probability of false alarm can be expressed as

$$\begin{aligned} P_F(\tau) &= \int_{-\eta}^{\eta} f_Y(y|H_0)dy \\ &= 1 - 2\int_{\eta}^{\infty} f_Y(y|H_0)dy \\ &= 1 - \exp(-\eta) = 1 - \frac{\tau}{2} \end{aligned} \tag{2.9}$$

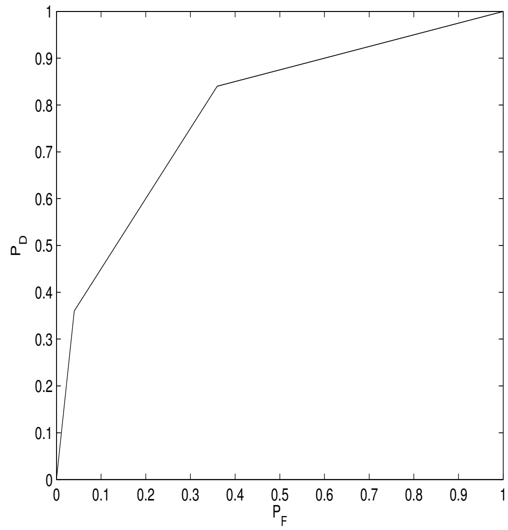

Figure 4: Plot of the ROC for $q_0 = 0.2$, $q_1 = 0.6$ and $n = 2$.

for $\tau \leq 2$ and $P_F(\tau) = 0$ for $\tau > 2$. Similarly

$$\begin{aligned} P_D(\tau) &= \int_{-\eta}^{\eta} f_Y(y|H_1)dy \\ &= 1 - 2\int_{\eta}^{\infty} f_Y(y|H_1)dy \\ &= 1 - \exp(-2\eta) = 1 - \frac{\tau^2}{4} \end{aligned} \tag{2.10}$$

for $\tau \leq 2$ and $P_D(\tau) = 0$ for $\tau > 2$.

c) Eliminating η from identities (2.9) and (2.10) yields the ROC equation

$$P_D = 1 - (1 - P_F)^2 \tag{2.11}$$

which is sketched below.

d) To design a Neyman-Pearson test with probability of false alarm less or equal to α, we must in fact select $P_F = \alpha$ which, after substitution in (2.9), yields

$$\tau = 2(1 - \alpha)\,.$$

Problem 2.9

a) For the Bayesian costs $C_{00} = C_{11} = 0$, $C_{10} = 1$, and $C_{01} = 2$, the minimax equation is given by

$$P_F = 2(1 - P_D)\,,$$

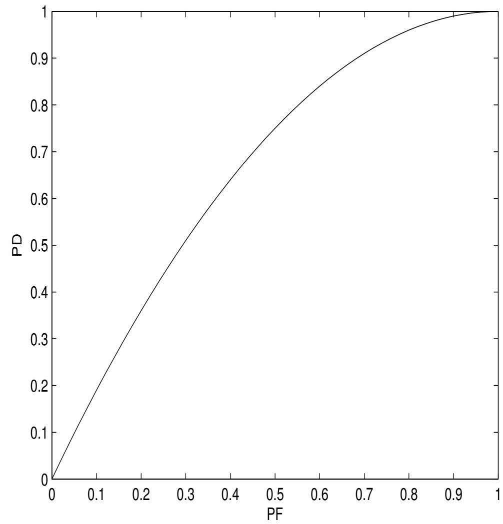

Figure 5: Plot of receiver operating characteristic (2.11).

or equivalently

$$P_D = 1 - \frac{P_F}{2} \,. \tag{2.12}$$

Eliminating P_D from equations (2.11) and (2.12) gives the quadratic equation

$$P_F^2 - \frac{5}{2} P_F + 1 = 0$$

whose roots are

$$P_F = \frac{1}{2}\Big[\frac{5}{2} \pm \big(\frac{25}{4} - 4\big)^{1/2}\Big] = \frac{1}{2}, \, 2 \,.$$

Since P_F must be less than 1, this gives $P_F = 1/2$, and backsubstituting inside (2.12), we obtain

$$P_D = 3/4 \,.$$

The slope of the ROC (2.11) is

$$\frac{dP_D}{dP_F} = 2(1 - P_F) \,.$$

Evaluating it at $P_F = 1/2$, we find that the threshold of the minimax test is

$$\tau_{\rm M} = 1 \,.$$

According to equation (2.96) of the book, we have therefore

$$\pi_{0\rm M} = [1 + 1/2]^{-1} = \frac{2}{3} \,.$$

This completes the specification of the minimax test.

b) For the given costs, the Bayesian risk of an arbitrary test δ is

$$R(\delta, \pi_0) = 2(1-\pi_0)(1-P_D(\delta)) + \pi_0 P_F(\delta) \tag{2.13}$$

for a fixed π_0, the test δ that minimizes the risk $R(\delta, \pi_0)$ is the LRT with threshold

$$\tau = \frac{\pi_0}{2(1-\pi_0)} .$$

Note that $\tau \leq 2$ as long as

$$\pi_0 \leq \frac{4}{5} .$$

For $\pi_0 \leq 4/5$, the probability of false alarm and of a miss are given by

$$\begin{aligned} P_F &= 1 - \frac{\tau}{2} = \frac{4-5\pi_0}{4(1-\pi_0)} \\ 1 - P_D &= \frac{\tau^2}{4} = \frac{\pi_0^2}{16(1-\pi_0)^2} . \end{aligned}$$

Substituting these values inside (2.13) yields

$$\begin{aligned} V(\pi_0) &= \frac{\pi_0^2}{8(1-\pi_0)} + \pi_0 \frac{(4-5\pi_0)}{4(1-\pi_0)} \\ &= \frac{8\pi_0 - 9\pi_0^2}{8(1-\pi_0)} \end{aligned} \tag{2.14}$$

for $0 \leq \pi_0 \leq 4/5$, and observing that $P_F = P_D = 0$ for $\tau > 2$, or equivalently $\pi_0 > 4/$, we find

$$V(\pi_0) = 2(1-\pi_0) \tag{2.15}$$

for $4/5 < \pi_0 \leq 1$. The function $V(\pi_0)$ defined by (2.14)-(2.15) is sktched below in Fig. 6. Its maximum is obtained by setting the derivative of (2.14) equal to zero. We have

$$\begin{aligned} \frac{dV}{d\pi_0} &= \frac{(8-18\pi_0)(1-\pi_0) + (8\pi_0 - 9\pi_0^2)}{8(1-\pi_0)^2} \\ &= \frac{9(1-\pi_0)^2 - 1}{8(1-\pi_0)^2} , \end{aligned}$$

which is zero whenever

$$3(1-\pi_0) = 1$$

or $\pi_{0\mathrm{M}} = 2/3$, which verifies the result obtained earlier in part a) of this problem.

Problem 2.10

a) For the probability densities

$$f_Y(y|H_0) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} , \end{cases}$$

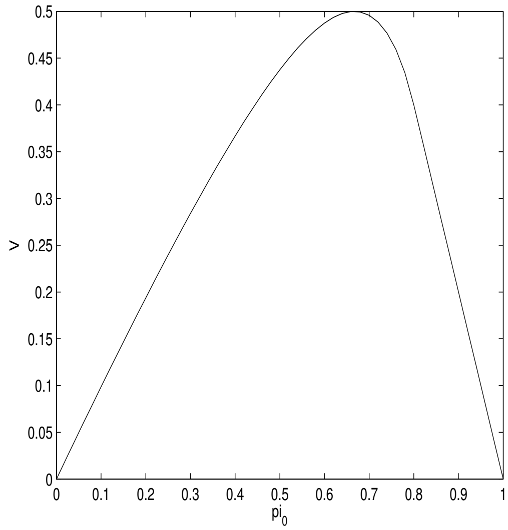

Figure 6: Plot of function $V(\pi_0)$.

and

$$f_Y(y|H_1) = \begin{cases} 4\left(\frac{1}{2} - \left|\frac{1}{2} - y\right|\right) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases},$$

the LRT can be expressed as

$$\begin{aligned} L(y) &= \frac{f_Y(y|H_1)}{f_Y(y|H_0)} \\ &= 4\Big(\frac{1}{2} - \Big|\frac{1}{2} - y\Big|\Big) \underset{H_0}{\overset{H_1}{\gtrless}} \tau \end{aligned} \tag{2.16}$$

for $0 \le y \le 1$. Since the maximum value of $L(y)$ is 2, 2 cases occur. When $\tau > 2$, we always select H_0. On the other hand when $\tau \le 2$, the test (2.16) can be rewritten as

$$\eta = \frac{2-\tau}{4} \underset{H_0}{\overset{H_1}{\gtrless}} \Big|\frac{1}{2} - y\Big| .$$

b) The probability of false alarm can be expressed as

$$\begin{aligned} P_F(\tau) &= \int_{1/2-\eta}^{1/2+\eta} f_Y(y|H_0)dy \\ &= 2\eta = 1 - \frac{\tau}{2} . \end{aligned} \tag{2.17}$$