## Solution exercise 2.1

a) Want to show that the following expression satisfies Laplace equation and a bottom boundary condition.

$$\phi_T = A \cosh(k(z+h)) \cos ky \cdot \cos\omega t \qquad (1)$$

Laplace equation (2-dimensions)

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (2)$$

$$\frac{\partial^2 \phi_T}{\partial y^2} = -k^2 A\cosh(k(z+h)) \cos ky \cdot \cos\omega t \qquad (3)$$

$$\frac{\partial^2 \phi_T}{\partial z^2} = k^2 A\cosh(k(z+h)) \cos ky \cdot \cos\omega t \qquad (4)$$

Hence;

$$\frac{\partial^2 \phi_T}{\partial y^2} + \frac{\partial^2 \phi_T}{\partial z^2} = 0 \qquad (5)$$

Bottom boundary condition

$$\frac{\partial \phi}{\partial z}\Big|_{z=-h} = 0 \qquad (6)$$

$$\frac{\partial \phi_T}{\partial z} = kA\sinh(k(z+h)) \cos ky \cdot \cos\omega t \qquad (7)$$

$$\Rightarrow \frac{\partial \phi_T}{\partial z}\Big|_{z=-h} = 0 \qquad (8)$$

b) Boundary condition at the walls of the tank

$$\frac{\partial \phi}{\partial y}\Big|_{y=-b,b} = 0 \qquad (9)$$

$$\frac{\partial\phi_T}{\partial y} = -kA\cosh(k(z+h))\sin ky \cdot \cos\omega t$$

In order to satisfy the condition for y = -b or y = b, the following must hold

$$\sin(k \cdot b) = 0 \tag{11}$$

$$\Rightarrow k \cdot b = n \cdot \pi \qquad ; n = 1,2,\ldots\ldots \tag{12}$$

Hence ;

$$k = \frac{n\pi}{b} \qquad ; n = 1,2,\ldots. \tag{13}$$

c) Free surface condition:

$$-\omega^2\phi\,|_{z=0} + g\frac{\partial\phi}{\partial z}\,|_{z=0} = 0 \tag{14}$$

Inserting values for $\phi_T$ and $\frac{\partial\phi_T}{\partial z}$ we obtain

$$-\omega^2 A\cosh(k(z+h))\cos ky \cdot \cos\omega t + gkA\sinh(k(z+h))\cos ky \cdot \cos\omega t = 0 \tag{15}$$

Assuming A, cosky as well as $\cos\omega t$ to be non-zero, we get:

$$\omega^2 = kg\tanh kh \tag{16}$$

Use that $T = \frac{2\pi}{\omega}$ and insert values of k, we get:

$$T_N = 2\pi/\left(\frac{gn\pi}{b}\cdot\tanh\left(\frac{n\pi h}{b}\right)\right)^{1/2} \qquad ; n = 1,2,\ldots.. \tag{17}$$

Approximate formulation as $h/b \rightarrow 0$. Then the trigonometric function can be written as: (choose n = 1 in order to avoid large arguments)

$$\tanh \frac{n\pi h}{b} = \frac{\pi h}{b} \tag{18}$$

Further ;

$$T_N = 2\pi / \left( \frac{g\pi}{b} \cdot \frac{\pi h}{b} \right)^{1/2} = \frac{2b}{(gh)^{1/2}} \tag{19}$$

d) Fluid motion at the free surface as function of time. Free surface elevation

$$\zeta = -\frac{1}{g} \frac{\partial \phi_T}{\partial t} \Big|_{z=0} = \frac{\omega}{g} A \cosh kh \ \cos ky \cdot \sin \omega t \tag{20}$$

We can see that the free surface oscillates harmonically with an amplitude $\zeta_a$ given as

$$\zeta_a = \frac{\omega}{g} A \cosh kh \ \cos ky \tag{21}$$

We observe that the amplitude varies in space, and is maximum at the ends of the tank.

## Solution exercise 2.2

Velocity potential:

$$\phi = Ae^{kz}\left(\frac{1}{r}\right)^{1/2}\cos(\omega t - kr) \tag{1}$$

a) Laplace equation is not satisfied everywhere. This is for r = 0 where the velocity potential is not defined.

b) Propagation direction of waves. Have to investigate the argument of the trigonometric function, in time and space. Assume

$$\omega t - kr = \text{constant} = 0 \tag{2}$$

$$\Rightarrow \omega t = kr \tag{3}$$

Hence; increased time corresponds to increased radius. Then the waves are propagating outward to larger radius r.

c) Variation of wave amplitude in space. (z=0)

$$\zeta = -\frac{1}{g}\frac{\partial\phi}{\partial t}\Big|_{z=0} = \frac{\omega}{g}A\left(\frac{1}{r}\right)^{1/2}\sin(\omega t - kr) \tag{4}$$

Wave amplitude $\zeta_a$ is given as:

$$\zeta_a = \frac{\omega}{g}A\left(\frac{1}{r}\right)^{1/2} \tag{5}$$

Then, $\zeta_a$ decays as $\left(\frac{1}{r}\right)^{1/2}$

## Solution exercise 2.3

a) Assume infinite depth. A wave front propagates with the group velocity which is equal to the half of the phase velocity on deep waters. Then

$$C_g = \frac{1}{2} \cdot \frac{\omega}{k} \tag{1}$$

Free surface condition:

$$\omega^2 = kg \quad \Rightarrow \quad \frac{\omega}{k} = \frac{g}{\omega} \tag{2}$$

Then

$$C_g = \frac{1}{2} \cdot \frac{g}{\omega} = \frac{1}{2} \cdot \frac{gT}{2\pi} = 1.561 \ m/s \tag{3}$$

Propagation time $T_A$

$$T_A = \frac{100}{1.561} = 64 \ s \tag{4}$$

b) Transportation of a cork. Due to equation 2.21 in the text book, the Stoke's drift velocity is

$$\zeta_a^2 \ \omega k e^{2kz_0} \tag{5}$$

Assume $z_0 = 0$, then the drift time $T_B$ is given as

$$T_B = \frac{100}{\zeta_a^2 \ \omega k} = \frac{100}{\zeta_a^2 \cdot \omega \cdot \frac{\omega^2}{g}} = 506{,}2 \ s \tag{6}$$

c) Maximum fluid velocity in the tank. Fluid velocity amplitude as function of depth:

$$\omega \zeta_a e^{kz} \tag{7}$$

Assume $z = 0$, then the maximum velocity is given as:

$$\omega\zeta_a = \frac{2\pi}{T}\zeta_a = 0.7854\ m/s \tag{8}$$

d) The wave crests move to the wave front with the phase velocity. The time between two crests is then

$$T_c = \frac{\lambda k}{\omega} = \frac{2\pi}{\omega} = T = 2\ s \tag{9}$$

Phase of wave 1.5 m close to the wave maker. Investigate the argument of trigonometric function which is $\omega t - kx$. The kx term controls the phase variation in space.

$$kx = \frac{\omega^2}{g}\cdot x = \frac{1}{g}\left(\frac{2\pi}{T}\right)^2 x = \frac{\pi^2}{g}x = 1.509\ [-] \tag{10}$$

This is approximately $\frac{\pi}{2}$ which corresponds to a phase shift of $\lambda/4$

e) Time between two passing wave crests. Have to consider the relative velocity between the wave crest and observer.

$$V = +1\ m/s \Rightarrow V_{eff} = \frac{\omega}{k} + 1.0\ m/s = 4.123\ m/s \tag{11}$$

$$T_c = \frac{\lambda}{V_{eff}} = \frac{2\pi g}{\omega^2}\cdot\frac{1}{V_{eff}} = 1.51\ s \tag{12}$$

$$V = -1\ m/s \Rightarrow V_{eff} = 2{,}123\ m/s \tag{13}$$

$$T_c = 2{,}94\ s \tag{14}$$

The phase shift is unchanged. This subject is called the frequency of encounter effect and will be investigated later in the book.

f) It is left to the reader to obtain numerical answers, but note the following facts:

- Energy as well as the wave front moves with the group velocity.

- The wave crests moves with the phase velocity which is higher than the group velocity. Hence; the wave crests propagate to the wave front were they break.

- The group velocity for deep water is half the phase velocity. For shallow waters the group and phase velocities are equal.

# Solution exercise 2.4

a) Definition of $T_1$. Note that

$$m_n = \int_0^\infty \omega^n S(\omega) d\omega \tag{1}$$

$$T_1 = 2\pi \frac{m_0}{m_1} = 2\pi \frac{(a-0.5)b \cdot \frac{2\pi}{T_1} \cdot H_{1/3}^2 \cdot T_1}{\frac{1}{2}(a^2-0.5^2)\left(\frac{2\pi}{T_1}\right)^2 \cdot b \cdot H_{1/3}^2 \cdot T_1} = 2T_1 \frac{1}{(a+0.5)} \tag{2}$$

Hence;

$$a + 0.5 = 2.0 \Rightarrow a = 1.5 \tag{3}$$

Further;

$$H_{1/3}^2 = 16 m_o = 16 \cdot (a-0.5)\, b \cdot \frac{2\pi}{T_1} \cdot H_{1/3}^2 \cdot T_1 = 32\pi \cdot b\, H_{1/3}^2 \tag{4}$$

$$\Rightarrow b = \frac{1}{32\pi} \tag{5}$$

b) Relationship between $T_2$ and $T_1$ :

$$T_2 = 2\pi\sqrt{\frac{m_0}{m_2}} = 2\pi\sqrt{\frac{(a-0.5)b \cdot \frac{2\pi}{T_1} \cdot H_{1/3}^2 \cdot T_1}{\frac{1}{3}(a^3-0.5^3)\left(\frac{2\pi}{T_1}\right)^3 b \cdot H_{1/3}^2 \cdot T_1}} = T_1 \cdot \sqrt{\frac{3}{1.5^3-0.5^3}} = 0.961 \cdot T_1 \tag{6}$$

**Solution exercise 2.5**

a) Standard deviation of horizontal velocity as well as acceleration.

$$\sigma_u^2 = \int_0^\infty \left(\frac{u}{\zeta_a}\right)^2 S(\omega)d\omega \tag{1}$$

$$\sigma_{a_1}^2 = \int_0^\infty \left(\frac{a_1}{\zeta_a}\right)^2 S(\omega)d\omega \tag{2}$$

Note that $A_j \to \zeta_a$ as $N \to \infty$ , $\Delta\omega \to 0$

b) Assume

$$\left(\frac{a_1}{\zeta_a}\right)^2 = \omega^4 e^{2kz} \tag{3}$$

For z = 0 the standard deviation is given as

$$\sigma_{a_1}^2 = \int_0^\infty \omega^4 S(\omega)\ d\omega \tag{4}$$

The behavior of the integrand near $\omega = 0$ is $\sim\frac{1}{\omega}$. Hence; the integral does not exist for z = 0. Elsewhere the integral exists since the integrand behaves like $\frac{1}{\omega} \cdot e^{\frac{\omega^2 z}{g}}$ which is finite near $\omega$=0.

In order to get the standard deviation of fluid acceleration to exist in the whole fluid domain, the wave spectrum has to be proportional to $\omega^{-n}$ where $n \leq 4$

c) Consistent with equation 2.38 in the text book we assume a seastate build up by N frequency components and K wave directions. Then we have:

$$\sigma_u^2 = \sum_{j=1}^N \sum_{k=1}^K \left(\frac{u}{\zeta_a}\right)^2 S(\omega_j,\theta_k)\ \Delta\omega_j\ \Delta\ \theta_k \tag{5}$$