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Quantum-Enabled Technologies

The exercises below involve some of the basic mathematics as well as the basic concepts and methods of classical mechanics and electromagnetism that are used throughout the book, rather than the main subject of this chapter. For more details on classical mechanics, i.e., the Newtonian, Lagrangian, and Hamiltonian formulations of classical mechanics, see Appendix A, which includes additional exercises. For the classical theory of waves see Appendix B and, for the basics of Maxwell's theory of electromagnetism, see Appendix C.

Exercise 1.1 (Euler's formula)

Use Euler's formula to prove the following addition theorems:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Solution:

Using Euler's formula, $e^{ix} = \cos x + i \sin x$, we can write

$$e^{i(x+y)} = \cos(x+y) + i \sin(x+y).$$

On the other hand, since $e^{i(x+y)} = e^{ix}e^{iy}$, we have

$$\begin{aligned} e^{i(x+y)} &= (\cos x + i \sin x)(\cos y + i \sin y) \\ &= (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y). \end{aligned}$$

By equating the real and imaginary parts of the above two equations, we obtain the addition theorems.

Exercise 1.2 (Partial derivatives)

When

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}},$$

show that $\nabla^2 f = 0$, where ∇^2 is the Laplacian operator.

Solution:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(-\frac{1}{2}\frac{1}{(x^2+y^2+z^2)^{3/2}}2x\right) = \frac{\partial}{\partial x}\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right) \\ &= \frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} \\ &= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{-x^2+2y^2-z^2}{(x^2+y^2+z^2)^{5/2}}, \\ \frac{\partial^2 f}{\partial z^2} &= \frac{-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}}.\end{aligned}$$

Thus,

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{-x^2+2y^2-z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}} = 0.\end{aligned}$$

Exercise 1.3 (Vector calculus)

In a vacuum, Maxwell's equations are written as $\nabla \cdot \boldsymbol{\mathcal{E}} = \rho/\varepsilon_0$, $\nabla \cdot \boldsymbol{B} = 0$, $\nabla \times \boldsymbol{\mathcal{E}} = -\partial \boldsymbol{B}/\partial t$, and $\nabla \times \boldsymbol{B} = (1/c^2)\partial \boldsymbol{\mathcal{E}}/\partial t$, where $\boldsymbol{\mathcal{E}} = -\nabla\phi - \partial \boldsymbol{A}/\partial t$ is the electric field vector, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is the magnetic field vector, ϕ and $\boldsymbol{A}$ are the scalar and vector potentials, respectively, ρ is the charge density (assumed to be a constant here), ε_0 is the vacuum permittivity, and c is the speed of light. Using the Lorentz condition $\nabla \cdot \boldsymbol{A} + (1/c^2)\partial\phi/\partial t = 0$, derive the following equations:

$$\begin{aligned}\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \boldsymbol{A} &= \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2}\end{aligned}$$

Solution:

From Faraday's law, we have

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial(\nabla \times \boldsymbol{A})}{\partial t} = -\nabla \times \frac{\partial \boldsymbol{A}}{\partial t}.$$

Thus,

$$0 = \nabla \times \left(\boldsymbol{\mathcal{E}} + \frac{\partial \boldsymbol{A}}{\partial t}\right).$$

Because the curl is 0, there must be some scalar function ϕ such that

$$-\nabla\phi = \boldsymbol{\mathcal{E}} + \frac{\partial \boldsymbol{A}}{\partial t}.$$

Rearranging for $\boldsymbol{\mathcal{E}}$ and plugging into the Ampére's law, we get

$$\begin{aligned}
\nabla \times \boldsymbol{B} =& \frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t},\\
\nabla \times (\nabla \times \boldsymbol{A}) =& -\frac{1}{c^2}\left(\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \frac{\partial(\nabla\phi)}{\partial t}\right),\\
\nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} =& -\frac{1}{c^2}\left(\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \frac{\partial(\nabla\phi)}{\partial t}\right),\\
\nabla\left(-\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) - \nabla^2\boldsymbol{A} =& -\frac{1}{c^2}\left(\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \frac{\partial(\nabla\phi)}{\partial t}\right),\\
\nabla^2\boldsymbol{A} =& \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}.
\end{aligned}$$

Similarly, using Coulomb's law, we obtain

$$\begin{aligned}
\nabla\cdot\boldsymbol{\mathcal{E}} =& \frac{\rho}{\epsilon_0},\\
\nabla\cdot\left(-\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t}\right) =& \frac{\rho}{\epsilon_0},\\
-\nabla^2\phi - \frac{\partial(\nabla\boldsymbol{A})}{\partial t} =& \frac{\rho}{\epsilon_0}.
\end{aligned}$$

Finally, using the Lorentz condition, $\nabla\boldsymbol{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$, we can write

$$\begin{aligned}
-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} =& \frac{\rho}{\epsilon_0},\\
\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} =& -\frac{\rho}{\epsilon_0}.
\end{aligned}$$

Exercise 1.4 (Differential equations)

A particle of mass m (> 0) connected to a spring with spring constant K (> 0) obeys Newton's equation of motion:

$$-Kx = m\frac{d^2x}{dt^2}.$$

Here, $x(t)$ is the position of the particle measured from its equilibrium position at time t.

(a) Set up and solve the characteristic equation of this second-order differential equation. Namely, assume $x(t) = e^{\lambda t}$ and obtain the roots λ_1 and λ_2.

Solution:

We rewrite the equation of motion as

$$\frac{d^2x(t)}{dt^2} = -\frac{K}{m}\,x(t).$$

Assuming $x(t) = e^{\lambda t}$, we obtain

$$\frac{d^2x(t)}{dt^2} = \frac{d^2}{dt^2}e^{\lambda t} = \lambda^2 e^{\lambda t} = -\frac{K}{m}\,e^{\lambda t}$$
$$\Rightarrow \lambda_{1,2} = \pm\sqrt{\frac{-K}{m}} = \pm i\sqrt{\frac{K}{m}}$$

(b) What is the general solution to the equation of motion?

Solution:

$$x(t) = A_1\,e^{\lambda_1 t} + A_2\,e^{\lambda_2 t} = A_1\,e^{i\sqrt{\frac{K}{m}}\,t} + A_2\,e^{-i\sqrt{\frac{K}{m}}\,t}$$
$$= c_1\cos\left(\sqrt{\frac{K}{m}}\,t\right) + c_2\sin\left(\sqrt{\frac{K}{m}}\,t\right)$$

(c) What is the specific solution to the equation of motion that satisfies the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$?

Solution:

$$x(0) = c_1\cos(0) + c_2\sin(0) = c_1 = x_0.$$

$$\dot{x}(0) = -x_0\sqrt{\frac{K}{m}}\,\sin(0) + c_2\sqrt{\frac{K}{m}}\,\cos(0) = c_2\sqrt{\frac{K}{m}} = v_0.$$
$$\Rightarrow c_2 = v_0\sqrt{\frac{m}{K}}.$$

Therefore, the specific solution is

$$x(t) = x_0\cos\left(\sqrt{\frac{K}{m}}\,t\right) + v_0\sqrt{\frac{m}{K}}\sin\left(\sqrt{\frac{K}{m}}\,t\right).$$

Exercise 1.5 (Wave motion analysis)

Given the wavefunctions

$$\psi_1 = 4\sin 2\pi(0.2x - 3t),$$
$$\psi_2 = \frac{\sin(7x + 3.5t)}{2.5}$$

determine in each case the values of (a) frequency, (b) wavelength, (c) period, (d) amplitude, (e) phase velocity, and (f) direction of motion. Time t is given in seconds, and x is given in meters.

Solution:
For ψ_1

(a)

$$f = \frac{1}{T} = \frac{1}{\frac{2\pi}{2\pi \times 3}} = 3$$

(b)

$$\lambda = \frac{2\pi}{2\pi \times 0.2} = 5$$

(c)

$$T = \frac{2\pi}{2\pi \times 3} = \frac{1}{3} = 0.33$$

(d)

$$A = 4$$

(e)

$$v_p = \frac{\lambda}{T} = \frac{5}{0.33} = 15$$

(f) The wave moves in the positive x-direction

For ψ_2

(a)

$$f = \frac{1}{T} = \frac{2\pi}{3.5} = \frac{7\pi}{4}$$

(b)

$$\lambda = \frac{2\pi}{7}$$

(c)

$$T = \frac{2\pi}{3.5} = \frac{4\pi}{7}$$

(d)

$$A = \frac{1}{2.5} = 0.4$$

(e)

$$v_p = \frac{\lambda}{T} = \frac{2\pi/7}{4\pi/7} = \frac{1}{2}$$

(f) The wave moves in the negative x-direction

Exercise 1.6 (Three-dimensional wave equation)

The electric field of an electromagnetic wave in vacuum traveling in the positive x-direction is given by

$$\boldsymbol{\mathcal{E}} = \mathcal{E}_0 \boldsymbol{e}_y \sin\left(\frac{\pi z}{z_0}\right) \cos\left(kx - \omega t\right)$$

where $\boldsymbol{e}_y$ is a unit vector in the y-direction and z_0 is a constant.

(a) Using the wave equation, obtain an expression for $k = k(\omega)$.

<u>*Solution*</u>:

$$\nabla^2 \boldsymbol{\mathcal{E}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \mathcal{E}_0 \boldsymbol{e}_y \sin\left(\frac{\pi z}{z_0}\right) \cos\left(kx - \omega t\right)$$
$$= -k^2 \boldsymbol{\mathcal{E}} - \frac{\pi^2}{z_0^2} \boldsymbol{\mathcal{E}},$$
$$\frac{\partial}{\partial t} \boldsymbol{\mathcal{E}} = \omega \mathcal{E}_0 \boldsymbol{e}_y \sin\left(\frac{\pi z}{z_0}\right) \sin(kx - \omega t),$$
$$\frac{\partial^2}{\partial t^2} \boldsymbol{\mathcal{E}} = -\omega^2 \mathcal{E}_0 \boldsymbol{e}_y \sin\left(\frac{\pi z}{z_0}\right) \cos(kx - \omega t) = -\omega^2 \boldsymbol{\mathcal{E}}.$$

In vacuum, the conductivity $\sigma = 0$ and the permittivity $\varepsilon = \varepsilon_0$, so the wave equation is given by

$$\nabla^2 \boldsymbol{\mathcal{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2}.$$

Therefore,

$$-k^2 \boldsymbol{\mathcal{E}} - \frac{\pi^2}{z_0^2} \boldsymbol{\mathcal{E}} = -\frac{\omega^2}{c^2} \boldsymbol{\mathcal{E}}.$$

By equating the coefficient of $\boldsymbol{\mathcal{E}}$ on both sides, we get the dispersion relationship

$$\omega = c\sqrt{k^2 + \frac{\pi^2}{z_0^2}}.$$

(b) Find the phase velocity of the wave.

<u>*Solution*</u>:

$$v_\mathrm{p} = \frac{\omega}{k} = \frac{c}{k}\sqrt{k^2 + \frac{\pi^2}{z_0^2}} = c\sqrt{1 + \frac{\pi^2}{z_0^2 k^2}}.$$

Exercise 1.7 (Wavepacket)

Let us take the superposition of an infinite number of plane waves

$$\psi(x,t) = \int_{-\infty}^{\infty} A(k) e^{-i(\omega t - kx)} dk$$

where $A(k)$ represents a spectral distribution function.