# Solutions to Chapter 1

1. The intent of this rather vague problem is to get you to compare the two notions, probability as intuition and relative frequency theory. There are many possible answers to how to make the statement "Ralph is probably guilty of theft" have a numerical value in the relative frequency theory. First step is to define a repeatable experiment along with its outcomes. The favorable outcome in this case would be 'guilty.' Repeating this experiment a large number of times would then give the desired probability in a relative frequency sense. We thus see that it may entail a lot of work to attach an objective numerical value to such a subjective statement, if in fact it can be done at all.

   One possible approach would be to look through courthouse statistics for cases similar to Ralph's, similar both in terms of the case itself and the defendant. If we found a sufficiently large number of these cases, ten at least, we could then form the probability $p = n_F/n$, where $n_F$ is the number of favorable (guilty) verdicts, and $n$ is the total number of found cases. Here we effectively assume that the judge and jury are omniscient.

   Another possibility is to find a large number of people with personalities and backgrounds similar to Ralph's, and to expose them to a very similar situation in which theft is possible. The fraction of these people that then steal in relation to the total number of people, would then give an objective meaning to the phrase "Ralph is probably guilty of theft."

2. Note that $D \rightarrow 3$, but $3 \nrightarrow D$, i.e., $D$ implies 3 but not the other way around. Thus if we turn over card 2 and find a 3. So what? It was <u>never stated that</u> a $3 \rightarrow F$. Likewise, with card 3. On the other hand, if we turn over card 4 and find a $D$, then the rule is violated. Hence, we must turn over card 4 and card 1, of course.

3. Favourable Outcomes = {1, 3, 5, 7, 9 } .

Total number of outcomes is 9 . Hence $P[\text{ odd }] = \frac{5}{9}$

The key assumption : all outcomes equally likely .

4. There are $2^3 = 8$ outcomes : *HHH, HHT, HTH, THH, HTT, THT, TTH, TTT*

The circled events are the desired events. Hence

$P[\text{ 2 tails, 1 head }] = \frac{3}{8}$

5. The possible outcomes are :

| | | | | |
|---|---|---|---|---|
| 111 | 211 | 221 | 311 | 331 |
| 122 | 222 | 223 | 323 | 332 |
| 133 | 233 | 333 | ... | |

The total numbers of outcomes = 27

when the same ball is drawn is drawn thrice, 111, 222, 333

Hence $P[\text{ drawing a ball thrice }] = \frac{3}{27} = \frac{1}{9}$

6. Let $b_1 , b_2 , b_3 , \ldots\ldots , b_5$ represent 5 balls. Then

$\Omega_1 = \{ b_1 b_2, b_1 b_3, b_1 b_4, b_1 b_5, b_2 b_1, b_2 b_3, b_2 b_4, b_2 b_5, b_3 b_1, b_3 b_2, b_3 b_4, b_3 b_5, b_4 b_1, b_4 b_2, b_4 b_3, b_4 b_5, b_5 b_1, b_5 b_2, b_5 b_3, b_5 b_4 \}$

If the first ball is replaced before the second draw then

$\Omega_2 = \Omega_1 \cup \{ b_1 b_1, b_2 b_2, b_3 b_3, b_4 b_4, b_5 b_5 \}$ .

7. Let $h_M$ be the height of the man and $h_W$ be the height of the woman. Each outcome of the experiment can be expressed as a two-tuple $(h_M, h_W)$. Thus

   (a) The sample space $\Omega$ is the set of all possible pairs of heights for the man and woman. This is given as
   $$\Omega = \{(h_M, h_W) : h_M > 0, h_W > 0\}.$$

   (b) The event $E$, which is a subset of $\Omega$ is given by
   $$E = \{(h_M, h_W) : h_M > 0, h_W > 0, h_M < h_W\}.$$

8. The word problem describes the physical experiment of drawing numbered balls from an urn. We need to find a corresponding mathematical model. First we form an appropriate event space with meaningful outcomes. Here the physical experiment is 'draw ball from urn,' so the outcome in words is 'particular labeled ball drawn,' which we can identify with its label. So we select as *outcome* in our mathematical model, the number on the drawn ball's face, i.e. the particular label. The outcomes are thus the integers 1,2,3,4,5,6,7,8, 9, and 10. The *sample space* is then $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and is the set of all ten outcomes. We are told that $E$ is 'the event of drawing a ball numbered no greater than 5.' Thus we define in our event field $E = \{1, 2, 3, 4, 5\}$. The other event specified in the word problem is $F$ 'the event of drawing a ball greater than 3 but less than 9.' In our mathematical event field this corresponds to $F = \{4, 5, 6, 7, 8\}$. Having constructed our sample space with indicated events, we can use elementary set theory to determine the following answers:

$$E^c = \{6, 7, 8, 9, 10\}, \qquad F^c = \{1, 2, 3, 9, 10\},$$
$$EF = \{4, 5\}, \qquad E \cup F = \{1, 2, 3, 4, 5, 6, 7, 8\},$$
$$EF^c = \{1, 2, 3\}, \qquad E^cF = \{6, 7, 8\},$$
$$E^c \cup F^c = \{1, 2, 3, 6, 7, 8, 9, 10\},$$
$$(EF^c) \cup (E^cF) = \{1, 2, 3, 6, 7, 8\}, \qquad (EF) \cup (E^cF^c) = \{4, 5, 9, 10\}$$
$$(E \cup F)^c = \{9, 10\}, \qquad (EF)^c = \{1, 2, 3, 6, 7, 8, 9, 10\}.$$

The last part of the problem asks us to 'express these events in words.' Since we have a mathematical model, we should really more precisely ask what each of these events *corresponds to in words*. We know of course that $E$ corresponds to 'drawing a ball numbered no greater than 5.' We can thus loosely write $E = \{\text{'drawing a ball numbered no greater than 5'}\}$, although in our mathematical model $E$ is just the set of integers $\{1, 2, 3, 4, 5\}$. So when we write $E =$\{ 'drawing a ball numbered no greater than 5'\}, what we really mean is that the event $E$ in our mathematical model corresponds to the physical event 'drawing a ball numbered no greater than 5' mentioned in the word problem. With this caveat in mind, we can then write:

$$\begin{aligned} E^c &= \{\text{'drawing a ball greater than 5'}\}, \\ F^c &= \{\text{'drawing a ball not in the range 4-8 inclusive'}\}, \\ EF &= \{\text{'drawing a ball greater than 3 and no greater than 5'}\}, \\ &\quad \text{etc.} \end{aligned}$$

See end of document for solution 8.

9. The sample space containing four equally likely outcomes is given by $\Omega = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$. Two events $A = \{\zeta_1, \zeta_2\}$ and $B = \{\zeta_2, \zeta_3\}$ are given. The required events can be easily obtained by observation.

$AB^c$ = set of outcomes in $A$ and not in $B = \{\zeta_1\}$.
$BA^c$ = set of outcomes in $B$ and not in $A = \{\zeta_3\}$.
$AB$ = set of outcomes in $A$ and $B = \{\zeta_2\}$.
$A \cup B$ = set of outcomes in $A$ or in $B = \{\zeta_1, \zeta_2, \zeta_3\}$.

10. $A = AB \cup AB^c$. This can be proved using the distributive law on

$$A = A\Omega = A(B \cup B^c) = AB \cup AB^c.$$

$A \cup B = (AB^c) \cup (BA^c) \cup (AB)$. Here we first write $A = A(B \cup B^c)$ and $B = B(A \cup A^c)$. Then we can write

$$\begin{aligned} A \cup B &= (A(B \cup B^c)) \cup (B(A \cup A^c)) \\ &= (AB \cup AB^c) \cup (BA \cup BA^c) \\ &= AB \cup AB^c \cup BA \cup BA^c \\ &= AB \cup AB^c \cup BA^c, \end{aligned}$$

using the above laws and formulas. Notice that the above two decompositions are into disjoint sets. From the third axiom of probability, we know that the probability of union of disjoint sets is the sum of the probabilities of the disjoint sets. Therefore, we can add the probabilities over the unions.

11. In a given random experiment there are four *equally likely* outcomes $\zeta_1, \zeta_2, \zeta_3$, and $\zeta_4$. Let the event $A \triangleq \{\zeta_1, \zeta_2\}$.
$P[A] = P[\{\zeta_1, \zeta_2\}] = P[\{\zeta_1\}] + P[\{\zeta_2\}] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. $A^c = \{\zeta_3, \zeta_4\}$,
$P[A^c] = P[\{\zeta_3, \zeta_4\}] = P[\{\zeta_3\}] + P[\{\zeta_4\}] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
Note that we are told that the four outcomes are equally likely. This means that the four singleton (atomic) events have equal probability. $P[A] = \frac{1}{2} = 1 - P[A^c] = 1 - \frac{1}{2}$.

12. (a) The three axioms of probability are given below

   (a) [label=()]

   (b) For any event $A$, the probability of the even occuring is always non-negative.

$$P[A] \geq 0.$$

   This ensures that probability is never negative.

   (c) The probability of occurence of the sample space event $\Omega$ is one.

$$P[\Omega] = 1.$$

   This ensures that probability of no event exceeds one. The first two axioms ensures that the probability is a quantity between 0 and 1, inclusive.

   (d) For any two events $A, B$ that are disjoint, the probability of the union of the events is the sum of the probabilities of the two events.

$$P[A \cup B] = P[A] + P[B], \text{ when } AB = \phi.$$

   This axiom tells us that the probability of any event can be obtained by the sum disjoint events that constitute the event.

(b) The event $A \cup B$ can be obtained as the disjoint union of the three sets $AB, AB^c, A^cB$. Hence by applying the third axiom of probability, we obtain

$$\begin{aligned} P[A \cup B] &= P[AB \cup (AB^c \cup A^cB)] \\ &= P[AB] + P[AB^c \cup A^cB] \\ &= P[AB] + P[AB^c] + P[A^cB]. \end{aligned}$$

Now the event $A$ can be written as the disjoint union of $AB$ and $AB^c$ (Axiom 3). Therefore

$$P[A] = P[AB] + P[AB^c] \implies P[AB^c] = P[A] - P[AB]$$

Similarly

$$P[B] = P[AB] + P[A^cB] \implies P[A^cB] = P[B] - P[AB].$$

Therefore $P[A \cup B] = P[AB] + (P[A] - P[AB]) + (P[B] - P[AB]) = P[A] + P[B] - P[AB]$.

13. We first form our mathematical model by setting outcomes $\boldsymbol{\varsigma} = (\varsigma_1, \varsigma_2)$, where $\varsigma_1$ corresponds to the label on the first ball drawn, and $\varsigma_2$ corresponds to the label on the second ball drawn. We can also write the outcomes as strings $\boldsymbol{\varsigma} = \varsigma_1\varsigma_2$. The sample space $\Omega$ can then be identified with the 2-D array

$$\begin{matrix} 11 & 12 & 13 & 14 & 15 \\ 21 & 22 & 23 & 24 & 25 \\ 31 & 32 & 33 & 34 & 35 \\ 41 & 42 & 43 & 44 & 45 \\ 51 & 52 & 53 & 54 & 55 \end{matrix}.$$

There are thus 25 outcomes in the sample space. Now the word problem statement uses the phrase 'at random' to describe the drawing. This is a technical term that can be read 'equally likely.' Thus all the elementary events $\{\varsigma_1\varsigma_2\}$ in our mathematical model must have equal probability, i.e. $P[\{\varsigma_1\varsigma_2\}] = 1/25$. Armed thusly we can attack the given problem as follows. Define the event $E$ ={'sum of labels equals five'}, or precisely $E = \{41, 32, 23, 14\}$. Then we decompose this event into four singleton events as

$$E = \{41\} \cup \{32\} \cup \{23\} \cup \{14\}.$$

Since different singleton events are disjoint, probability adds, and we have

$$\begin{aligned} P[E] &= \frac{1}{25} + \frac{1}{25} + \frac{1}{25} + \frac{1}{25} \\ &= \frac{4}{25}. \end{aligned}$$

"Dim" ignored that outcome $ij$ is different (distinguishable) from outcome $ji$. "Dense" talked about the sums and correctly noted that there were nine of them. However, he incorrectly assumed that each sum was equally likely. Looking at our sample space above, we can see that the sum 2 has only one favorable outcome 11, while the sum 6 has five favorable outcomes, just looking at the anti-diagonals of this matrix.

14. We first show that

$A\cup(B\cap C)\leq(A\cup B)\cap(A\cup C)$ .

Let $x \in A\cup(B\cap C)$.

Then $x \in A$ or $x \in B\cap C$ .

If $x \in A$, then $x \in A\cup B$ and $x \in A\cup C$ , as

$A \leq A\cup B$ and $A \leq A\cup C$ .

$\therefore\ x \in (A\cup B)\cap(A\cup C)$ .

If $x \in B\cap C$ , then $x \in B$ and $x \in C$ .

Since $x \in B$, $x \in A\cup B$ and also since $x \in C$ ,

$x \in A\cup C$ .

$\therefore\ x \in (A\cup B)\cap(A\cup C)$ .

In both cases $x \in (A\cup B)\cap(A\cup C)$ . Thus $x \in A\cap(B\cup C) \Rightarrow x \in (A\cup B)\cap(A\cup C)$

$\therefore\ A\cup(B\cap C) \leq (A\cup B)\cap(A\cup C)$ (1)

Conversely, let $x \in (A\cup B)\cap(A\cup C)$.

Then $x \in A\cup B$ and $x \in A\cup C$ .

If $x \in A$, then clearly $x \in A\cup(B\cap C)$.

If $x \notin A$, from $x \in A\cup B \notin$ and $x \in A\cup C$, it

follows that $x \in B$ and $x \in C$ .

ie $x \in B\cap C$ .

Then if $x \in (A\cup B)\cap(A\cup C)$., then

$x \in A$ or $x \in B\cap C$

ie $x \in A\cup(B\cap C)$

$\therefore (A\cup B)\cap(A\cup C) \leq A\cup(B\cap C)$ (2)

from (1) and (2), we have

$$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)\ .$$

$(A \cap B) \cup (A \cap C)$.



15. We use the set identity $\Omega = A \cup A^c$. Since this union is disjoint, by the additivity of probability (i.e. axiom 3), we get $1 = P[\Omega] = P[A] + P[A^c]$, which with rearranging becomes the desired result.

16. (a) $A \cap C = \{1,2\} \cap \{4,5,6\} = \phi$. Therefore,

$$\begin{aligned} P[A \cap C] &= P[\phi] \\ &= 1 - P[\Omega] \\ &\quad (\because \Omega \cap \Phi = \phi, 1 = P[\Omega \cup \phi] = P[\Omega] + P[\phi]) \\ &= 1 - 1 \text{ ( because } P[\Omega] = 1) \\ &= 0. \end{aligned}$$

(b) $P[A \cup B \cup C] = P[\{1,2\} \cup \{2,3\} \cup \{4,5,6\} = P[\{1,2,3,4,5,6\}] = P[\Omega] = 1$.

(c) We see that $B \cap C = \phi$ and so $P[BC] = 0$. For $B$ and $C$ to be independent, $P[BC] = P[B]P[C]$. Therefore, if either $P[B] = 0$ or $P[C] = 0$ or both are zeros, $B$ and $C$ will be independent.

17. This problem uses only set theory and just two axioms of probability to get these general results.

(a) We need to show $P[\phi] = 0$. We write the disjoint decomposition $\Omega = \Omega \cup \phi$ and then use the additivity of probability (axiom 3) to get

$$\begin{aligned} P[\Omega] &= P[\Omega \cup \phi] \\ &= P[\Omega] + P[\phi]. \end{aligned}$$

So we must have $P[\phi] = 0$.

(b) Using set theory, we can write the disjoint decomposition

$$E = EF^c \cup EF.$$

Then by axiom 3, the additivity of probability, we have

$$\begin{aligned} P[E] &= P[EF^c \cup EF] \\ &= P[EF^c] + P[EF], \end{aligned}$$

or what is the same $P[EF^c] = P[E] - P[EF]$.

(c) Here we simply note $E \cup E^c = \Omega$ is a disjoint decomposition, so that again by axiom 3,

$$\begin{aligned} P[\Omega] &= P[E] + P[E^c] \\ &= 1, \quad \text{by axiom 2,} \end{aligned}$$

which is the same as $P[E] = 1 - P[E^c]$.

18. The outcome is the result of a probabilistic experiment. An event is a collection (set) of outcomes. The field of events is the complete collection of events that are relevant for the given probability problem.

19. We start with the mutually exclusive decomposition

$$A \cup B = AB^c \cup AB \cup A^cB,$$

yielding $P[A \cup B] = P[AB^c] + P[AB] + P[A^cB]$. Then consider the two simple disjoint decompositions

$$A = AB^c \cup AB \quad \text{and} \quad B = A^cB \cup AB,$$

which yield $P[A] = P[AB^c] + P[AB]$ and $P[B] = P[A^cB] + P[AB]$. Putting them all together, we have

$$\begin{aligned} P[A \cup B] &= P[AB^c] + P[AB] + P[A^cB] \\ &= (P[A] - P[AB]) + P[AB] + (P[B] - P[AB]) \\ &= P[A] + P[B] - P[AB]. \end{aligned}$$

20. From Eq. 1.4-3, we see that $E \oplus F = (E - F) \cup (F - E) = EF^c \cup E^cF$. We see that $EF^c$ and $E^cF$ are disjoint, i.e., $(EF^c) \cap (E^cF) = \phi$. Therefore, the probability of the union of $EF^c$ and $E^cF$ are the sum of the probabilities of the two events. In other words,

$$P(E \oplus F) = P(EF^c \cup E^cF) = P(EF^c) + P(E^cF).$$

21. We have already (Problem 17) seen that we can write $P[EF^c] = P[E] - P[EF]$ and $P[E^cF] = P[F] - P[EF]$. Therefore, $P(E \oplus F) = P(EF^c) + P(E^cF) = P[E] + P[F] - 2P[EF]$.

22. (a) For simplicity associate as follows: cat=1, dog=2, goat=3, and pig=4. The outcomes $\xi$ then become the integers 1,2,3, and 4. The sample space $\Omega = \{1, 2, 3, 4\}$. For probability information we are given:

$$P[\{1,2\}] = 0.9, P[\{3,4\}] = 0.1, P[\{4\}] = 0.05, \text{ and } P[\{2\}] = 0.5.$$

Now for every event in our field of events, we must be able to specify the probability. This is equivalent to being able to supply the probability for all the singleton events. To see if we can do this, we note that singleton events $\{1\}$ and $\{3\}$ are missing probabilities, so we first write

$$\begin{aligned} \{1\} &= \{1,2\} - \{2\}, \text{ so that} \\ P[\{1\}] &= P[\{1,2\}] - P[\{2\}] = 0.9 - 0.5 = 0.4. \end{aligned}$$

Doing the same for the other missing singleton probability $P[\{3\}]$, we write

$$\begin{aligned}\{3\} &= \{3,4\} - \{4\}, \text{ so that} \\ P[\{3\}] &= P[\{3,4\}] - P[\{4\}] = 0.1 - 0.05 = 0.05.\end{aligned}$$

Thus we have enough probability information for all the singleton events, and hence all $16 = 2^4$ subsets of $\Omega = \{1,2,3,4\}$. The appropriate field $\mathcal{F}$ of events then consists of the following events along with their probabilities:

$$\begin{array}{cc}
\{1\}, & P[\{1\}] = 0.4, \\
\{2\}, & P[\{2\}] = 0.5, \\
\{3\}, & P[\{3\}] = 0.05, \\
\{4\}, & P[\{4\}] = 0.05, \\
\{1,2\}, & P[\{1,2\}] = 0.9, \\
\{1,3\}, & P[\{1,3\}] = 0.45, \\
\{1,4\}, & P[\{1,4\}] = 0.45, \\
\{2,3\}, & P[\{2,3\}] = 0.55, \\
\{2,4\}, & P[\{2,4\}] = 0.55, \\
\{3,4\}, & P[\{3,4\}] = 0.1, \\
\{1,2,3\}, & P[\{1,2,3\}] = 0.95, \\
\{1,2,4\}, & P\{1,2,4\}] = 0.95, \\
\{1,3,4\}, & P[\{1,3,4\}] = 0.5, \\
\{2,3,4\}, & P[\{2,3,4\}] = 0.6, \\
\{1,2,3,4\}(=\Omega), & P[\{1,2,3,4\}] = 1 = P[\Omega], \\
\phi, & P[\phi] = 0.
\end{array}$$

(b) Now the above is not an appropriate field of events if some of the events do not have known probabilities. So if $P[\text{'pig'} = \{4\}] = 0.05$ is removed, then we cannot determine the probabilities of some of the above events. In particular we cannot find $P[\{3\}]$. The alternative then is to treat $\{3,4\}$, whose probability is still given, as a singleton and form a smaller field with just the 8 events formed by unions of $\{1\}$, $\{2\}$, and $\{3,4\}$. The resulting field, along with its probabilities is as follows:

$$\begin{array}{cc}
\{1\}, & P[\{1\}] = 0.4, \\
\{2\}, & P[\{2\}] = 0.5, \\
\{1,2\}, & P[\{1,2\}] = 0.9, \\
\{3,4\}, & P[\{3,4\}] = 0.1, \\
\{1,3,4\}, & P[\{1,3,4\}] = 0.5, \\
\{2,3,4\}, & P[\{2,3,4\}] = 0.6, \\
\{1,2,3,4\}(=\Omega), & P[\{1,2,3,4\}] = 1 = P[\Omega], \\
\phi, & P[\phi] = 0.
\end{array}$$

23. First we show that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
Suppose $x \in A \cap (B \cup C)$
Then $x \in A$
Therefore $x \in (A \cup B)$, and $x \in (A \cup C)$
Hence, $x \in (A \cup B) \cap (A \cup C)$.

Now we show that $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.
Suppose $x \in (A \cup B) \cap (A \cup C)$

Then $x \in (A \cup B)$ and $x \in (A \cup C)$
$x \in AorB$ and $x \in AorC$
If $x \in A$, then $x \in A \cup (B \cap C)$ (because $A \subset (A \cup (B \cap C))$)
If $x \in A$, then $x \in B$ and $x \in C$.
Or in other words, $x \in (B \cap C)$
$x \in A \cup (B \cap C)$.

Thus we have shown that both the sets are contained in each other. Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

24. The probability of $A$ is $P[A] = P[\{\zeta_1, \zeta_2\}] = P[\{\zeta_1\}] + P[\{\zeta_2\}] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. The event (set) $A^c$ in terms of the outcomes is $A^c = \{\zeta_3, \zeta_4\}$. The probabilty of $A^c$ is $P[A^c] = P[\{\zeta_3, \zeta_4\}] = P[\{\zeta_3\}] + P[\{\zeta_4\}] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Note that we are told that the four outcomes are equally likely. This means that the four singleton (atomic) eve
$P[A] = \frac{1}{2} = 1 - P[A^c] = 1 - \frac{1}{2}$.

See end of document for solution 24.

25. The composition of the urn is: $(a)$, $(a)$, $(b)$, $(b)$, $(ab)$, $(ab)$, $(ab)$, $(ab)$. $P[A] = 6/8$, $P[B] = 6/8$, $P[AB] = n_{ab}/n_T = 4/8$ is not equal to $P[A]P[B] = 9/16$. Therefore $A$ and $B$ are not independent.

26. Let $n_i, i = 1, 2$ represent the outcome of the $i$th toss. Since the tosses are independent:

$$P[n_1, n_2] = P[n_1]P[n_2] = \frac{1}{6} \cdot \frac{1}{6}$$

$$\begin{aligned} P[n_1 + n_2 = 7 | n_1 = 3] &= P[n_2 = 4 | n_1 = 3] \\ &= \frac{P[n_1 = 3, n_2 = 4]}{P[n_1 = 3]} \\ &= \frac{P[n_1 = 3]P[n_2 = 4]}{P[n_1 = 3]} \\ &\quad \text{(because tosses are independent)} \\ &= \frac{1}{6} \end{aligned}$$

27. The sample space $S$ consists of 36 elements, namely

$S = \{ (i, j): i, j = 1, 2, 3, 4, 5, 6\}$

ie $$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6), \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6), \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6), \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6), \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6), \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6), \end{Bmatrix}$$

Then

$$P(A) = P(B) = P(C) = \frac{18}{36} = \frac{1}{2}$$

Also $P(A \cap B) = P(A \cap C) = P(B \cap C) = \dfrac{9}{36} = \dfrac{1}{4}$

Thus

$$P(A \cap B) = \frac{1}{4} = P(A)\,P(B)$$

$$P(A \cap C) = \frac{1}{4} = P(A)\,P(B)$$

$$P(B \cap C) = \frac{1}{4} = P(B)\,P(C)\,.$$

Hence the events *A, B* and *C* are pair wise independent.

However, since the sum of two odd numbers is even ; the set $A \cap B \cap C = \phi$

$$\therefore \quad P(A \cap B \cap C) = 0.$$

Now $P(A)\,P(B)\,P(C) = \dfrac{1}{2} \times 1/2 \times 1/2 = \dfrac{1}{8}$

$$\neq 0 \; (= P(A \cap B \cap C))$$

Which shows that *A, B* and *C* are not independent.