CHAPTER 1

1.1 A large number of people with personalities, backgrounds, etc. similar to Ralph's should be exposed to an identical situation in which theft is possible. The fraction of those that steal in relation to the whole would give some meaning -- in the relative frequency sense only -- to the phrase "Ralph is possibly guilty of theft.".

1.2 Note that $D \rightarrow \underline{3}$ but $\underline{3} \not\rightarrow D$ i.e., D implies $\underline{3}$ but not the other way around. Thus if we turn over card 2 and find a $\underline{3}$. So what? It was never stated that, a $\underline{3} \rightarrow F$. Likewise with card $\underline{3}$. On the other hand if we turn over card 4 and don't find a $\underline{3}$, then the rule is violated. Hence we must turn over card $\underline{4}$ and card $\underline{1}$, of course.

1.3 The number of favorable outcomes is four: 2, 4, 6, 8. The total number of outcomes is 9. Hence

$$P[\text{even}] = 4/9$$

The key assumption: all outcomes equally likely

1.4 There are $2^3 = 8$ outcomes: HHH HHT HTH THH HTT THT TTH TTT. The circled outcomes are the desired events. Hence

$$P[2 \text{ heads}, 1 \text{ tail}] = 3/8.$$

1.5 The possible outcomes are:

11	21	31
12	22	32
13	23	33

The circled outcomes are when the same ball is drawn twice. Hence

$$P[\text{drawing ball twice}] = 3/9 = \frac{1}{3}$$

1.6 Let $b_1, b_2, \cdots, b_6$ represent the six balls. Then

$\Omega_1 = \{ b_1b_2, b_1b_3, b_1b_4, b_1b_5, b_1b_6, b_2b_1, b_2b_3, b_2b_4, b_2b_5, b_2b_6, b_3b_1, b_3b_2, b_3b_4, b_3b_5, b_3b_6, b_4b_1, b_4b_2, b_4b_3, b_4b_5, b_4b_6, b_5b_1, b_5b_2, b_5b_3, b_5b_4, b_5b_6, b_6b_1, b_6b_2, b_6b_3, b_6b_4, b_6b_5 \}$.

If the first ball is replaced before the second draw then

$\Omega_2 = \Omega_1 \cup \{ b_1b_1, b_2b_2, b_3b_3, b_4b_4, b_5b_5, b_6b_6 \}$

1.7 Let h_M = height of man, h_W = height of woman. Then

a) $\Omega = \{ h_M, h_W : h_M > 0, h_W > 0 \}$

b) $E = \{ h_M, h_W : h_M > 0, h_W > 0, h_M < h_W \}$

1.8 To shorten the notation, the event of drawing a ball with the number i will be denoted as i. Then

$\Omega = \{1, 2, \cdots, 10\}$, $E = \{1,2,3,4,5\}$, $F = \{4,5,6,7,8\}$

Thus:

$E^c = \{6,7,8,9,10\}$, $F^c = \{1,2,3,9,10\}$

$EF = \{4,5\}$, $E \cup F = \{1,2,3,4,5,6,7,8\}$

$EF^c = \{1,2,3\}$, $E^cF = \{6,7,8\}$, $E^c \cup F^c = \{1,2,3,6,7,8,9,10\}$

$(EF^c) \cup (E^cF) = \{1,2,3,6,7,8\}$, $(EF) \cup (E^cF^c) = \{4,5,9,10\}$

$(E \cup F)^c = \{9,10\}$, $(EF)^c = \{1,2,3,6,7,8,9,10\}$,

where:

$E^c = \{$drawing a ball greater than 5$\}$

$F^c = \{$drawing a ball not in the range 4-8 inclusive$\}$

$EF = \{$drawing a ball greater than 3 but no greater than 5$\}$

The other events are similarly described.

1.9 For this experiment, all possible outcomes are

11	21	31	41	51
12	22	32	42	52
13	23	33	43	53
14	24	34	44	54
15	25	35	45	55

There are $5 \times 5 = 26$ possibilities. The favarable outcomes are 4: (1,4), (2,3), (3,2), (4,1) so $P = \frac{4}{25}$ is indeed correct.

"Dim" ignored the fact that (i,j) is different from (j,i); i.e., the outcome (2,3) is distinct from (3,2).

"Dense" ignored the fact that the 9 distinguishable outcomes are not equally likely.

1.10 a) $\Omega \cup \phi = \Omega \qquad \Omega\phi = \phi$

Hence $P[\Omega \cup \phi] = P(\Omega) + P(\phi) = P[\Omega] = 1$

Therefore $P(\phi) = 1 - P[\Omega] = 0$

b) $E = EF^c \cup EF \qquad [EF^c] \cap [EF] = \phi$

Hence $P(E) = P(EF^c) + P(EF)$ or

$P(EF^c) = P(E) - P(EF)$

c) $E \cup E^c = \Omega \qquad EE^c = \phi \qquad P[E] + P(E^c) = P[\Omega] = 1$

1.11 $E \oplus F = EF^c \cup FE^c$; but $(EF^c) \cap (FE^c) = \phi$

Hence $P[E \oplus F] = P(EF^c) + P(FE^c)$

1.12 $P(EF^c) = P(E) - P[EF]$ and

$P(FE^c) = P(F) - P(EF)$

Hence $P[E \oplus F] = P[E] + P[F] - 2P[EF]$

1.13 (a) For simplicity associate as follows: cat = 1, dog = 2, goat = 3, pig = 4. Then $\mathcal{Y}$ ($\mathcal{F}$) consists of 16 events:

$\{1\}$; $\{2\}$; $\{3\}$; $\{4\}$; $\{1,2\}$; $\{1,3\}$; $\{1,4\}$; $\{2,3\}$; $\{2,4\}$; $\{3,4\}$; $\{1,2,4\}$; $\{1,2,3\}$; $\{1,3,4\}$; $\{2,3,4\}$; $\{1,2,3,4\}$; ϕ.

Note that $\{1,2\} = \{1\} \cup \{2\}$ and

$$P[\{1,2\}] = P[\{1\}] + P[\{2\}] - \underbrace{P[\{1\}\cap\{2\}]}_{=0}$$

But $P[\{2\}] = 0.5$, therefore $P[\{1\}] = 0.9 - 0.5 = 0.4$

$$P[\{3,4\}] = 0.1 = P[\{4\}] + P[\{3\}] = 0.05 + P[\{3\}]$$

Therefore $P[\{3\}] = 0.05$; $P[\{2\}] = 0.5$; $P[\{4\}] = 0.05$; $P[\{1\}] = 0.4$. From these probabilities we can compute all other probabilities using $P[A\cup B] = P[A] + P[B]$ if $A\cap B = \phi$. For example $P[\{1,3,4\}]$

$$= P[\{1\}] + P[\{3\}] + P[\{4\}]$$

$$= 0.4 + 0.05 + 0.05 = 0.5.$$

The underlying experiment assumes the animals are in separate covered cages and a cage is chosen at random.

(b) We cannot use the full 16 event field of part (a) because we have less probability information now.

For example we can determine

$$P[\{cat\}] = P[\{cat, dog\} - \{dog\}]$$
$$= P[\{cat, dog\}] - P[\{dog\}]$$
$$= 0.9 - 0.5$$
$$= 0.4,$$

but we cannot determine

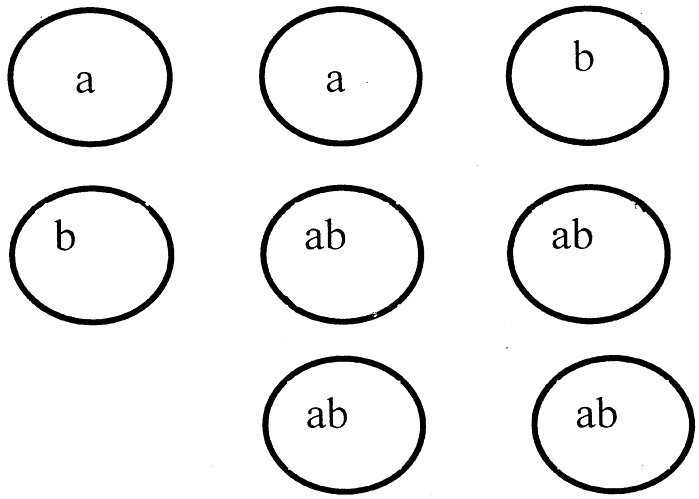
$P[\{pig\}]$, we now are only given $P[\{goat, pig\}]$ and there is no other relevant information. There must be specified a probability for every event!

So we define a field of just 8 events: treating {goat, pig} as a singleton or elementary event. Using the mapping event → # from part (a), we have,

$$\mathcal{F} = \{ \{1\}; \{2\}; \{1,2\}; \{3,4\}; \{1,3,4\}; \{2,3,4\}; \{1,2,3,4\}; \phi \}.$$

and each event has a probability.

1.14 The composition of the urn is:



$P[A]=6/8$; $P[B]=6/8$ $P[AB]=n_{ab}/n_T=4/8$ is not equal to $P[A]P[B]=9/16$
Therefore A and B are not independent.

1.15 Let n_i, $i=1,2$ represent the outcome of the ith toss. Since the tosses are independent:

$$P[n_1, n_2] = P[n_1]P[n_2] = \frac{1}{6}\cdot\frac{1}{6}$$

$$P[n_1+n_2=7 \mid n_1=3] = P[n_2=4 \mid n_1=3]$$

$$= \frac{P[n_1=3]\,P[n_2=4]}{P[n_1=3]} = \frac{1}{6}$$

1.16 Clearly

$$P[A] = \frac{4}{52}$$

$$P[B] = \frac{26}{52} = \frac{1}{2}$$

$P[AB] = P\{$picking one of two red aces in 52 cards$\} = \frac{2}{52}$

Is $P[AB] \stackrel{?}{=} P[A]\,P[B]$

$$\frac{2}{52} \stackrel{?}{=} \frac{4}{52}\times\frac{1}{2} \qquad \text{yes!}$$

Hence A and B are independent

1.17 All possible outcomes are given below:

$$\left.\begin{array}{ccccc} 11 & 21 & 31 & 41 \cdots & 91 \\ 12 & 22 & 32 & --- & 92 \\ 13 & 23 & \vdots & & \vdots \\ 14 & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & & \\ 19 & 29 & --- & & 99 \end{array}\right\} \text{81 possible outcomes.}$$

To get $\Sigma = 7$, we need one of the following outcomes: 61, 52, 43, 34, 25, 16. Thus there 6 ways of getting $\Sigma = 7$. Hence

$$P(\Sigma=7 \mid \Sigma \text{ odd}) = \frac{P(\Sigma=7, \Sigma \text{ odd})}{P(\Sigma \text{ odd})} = \frac{P(\Sigma 7)}{P(\Sigma \text{ odd})}$$

The only unknown is $P(\Sigma \text{ odd})$. Out of 81 outcomes, 40 produce odd Σ, 41 even. Thus

$$P(\Sigma=7 \mid \Sigma \text{ odd}) = \frac{6/81}{40/81} = 0.15 .$$

Now we want to compute

$$P(\text{at least one of } N_1, N_2 > 7 \mid \Sigma > 10) \triangleq P'.$$

We can write

$$P' = \frac{P(\text{at least one of } N_1, N_2 > 7, \Sigma > 10)}{P(\Sigma > 10)} .$$

From the table, $P(\Sigma > 10) = 36/81$. It is easy to see from the table that the only ways

1.17 contd.

$N_1 + N_2 = \Sigma > 10$ and both $N_1, N_2 \le 7$ are the following:
$\Sigma = 11 = 6+5 = 5+6 = 7+4 = 4+7$, $\Sigma = 12 = 5+7 = 7+5 = 6+6$,
$\Sigma = 13 = 6+7 = 7+6$, $\Sigma = 14 = 7+7$. that is, the 36 possible outcomes are reduced by 10 and

$$P(\text{at least one of } N_1, N_2 > 7, \Sigma > 10) = \frac{26}{81}$$

Hence

$$P' = \frac{26/81}{36/81} \doteq 0.72.$$

Finally to compute

$$P(\Sigma \text{ odd} \mid N_1 > 8) = \frac{P(\Sigma \text{ odd}, N_1 > 8)}{P(N_1 > 8)}$$

We know that $P(N_1 > 8) = P(N_1 = 9) = 1/9$. Also from the table we see that the event $\{\Sigma \text{ odd}, N_1 = 9\}$ can happen as follows: $9+2$, $9+4$, $9+6$, $9+8$.
Thus $P(\Sigma \text{ odd}, N_1 > 8) = 4/81$. Hence

$$P(\Sigma \text{ odd} \mid N_1 > 8) = \frac{4/81}{9/81} = \frac{4}{9} \simeq 0.44.$$