

Chapter 2

2.1 a. Spring constant, k : The change in the force per unit length change of the spring.

b. Coefficient of subgrade reaction, k' :

Spring constant divided by the foundation contact area, $k' = \frac{k}{A}$

c. Undamped natural circular frequency: $\omega_n = \sqrt{\frac{k}{m}}$ rad/s

where $m = \text{mass} = \frac{W}{g}$

d. Undamped natural frequency: $f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$ (in Hz)

Note: Circular frequency defines the rate of oscillation in term of radians per unit time; 2π radians being equal to one complete cycle of rotation.

e. Period, T : The time required for the motion to begin repeating itself.

f. Resonance: Resonance occurs when $\frac{\omega_n}{\omega} = 1$

g. Critical damping coefficient: $c_c = 2\sqrt{km}$

where $k = \text{spring constant}$; $m = \text{mass} = \frac{W}{g}$

h. Damping ratio: $D = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}$

where $c = \text{viscous damping coefficient}$; $c_c = \text{critical damping coefficient}$

i. Damped natural frequency:

$$\omega_d = \omega_n \sqrt{1 - D^2}$$

$$f_d = \sqrt{1 - D^2} f_n$$

- 2.2** Weight of machine + foundation, $W = 400$ kN
Spring constant, $k = 100,000$ kN/m

$$\text{Mass of the machine + foundation, } m = \frac{W}{g} = \frac{400}{9.81} = 40.77 \frac{\text{kN}}{\text{m/s}^2}$$

Natural frequency of undamped free vibration is [Eq. (2.19)]

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{100,000}{40.77}} = \mathbf{7.88 \text{ Hz}}$$

$$\text{From Eq. (2.18), } T = \frac{1}{f_n} = \frac{1}{7.88} = \mathbf{0.127 \text{ s}}$$

- 2.3** Weight of machine + foundation, $W = 400$ kN
Spring constant, $k = 100,000$ kN/m

Static deflection of foundation is [Eq. (2.2)]

$$z_s = \frac{W}{k} = \frac{400}{100,000} = 4 \times 10^{-3} \text{ m} = \mathbf{4 \text{ mm}}$$

- 2.4** External force to which the foundation is subjected, $Q = 35.6 \sin \omega t$ kN
 $f = 13.33$ Hz
Weight of the machine + foundation, $W = 178$ kN
Spring constant, $k = 70,000$ kN/m

For this foundation, let time $t = 0$, $z = z_0 = 0$, $\dot{z} = v_0 = 0$

a. Mass of the machine + foundation, $m = \frac{W}{g} = \frac{178}{9.81} = 18.145 \frac{\text{kN}}{\text{m/s}^2}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{70,000}{18.145}} = 62.11 \text{ rad/s}$$

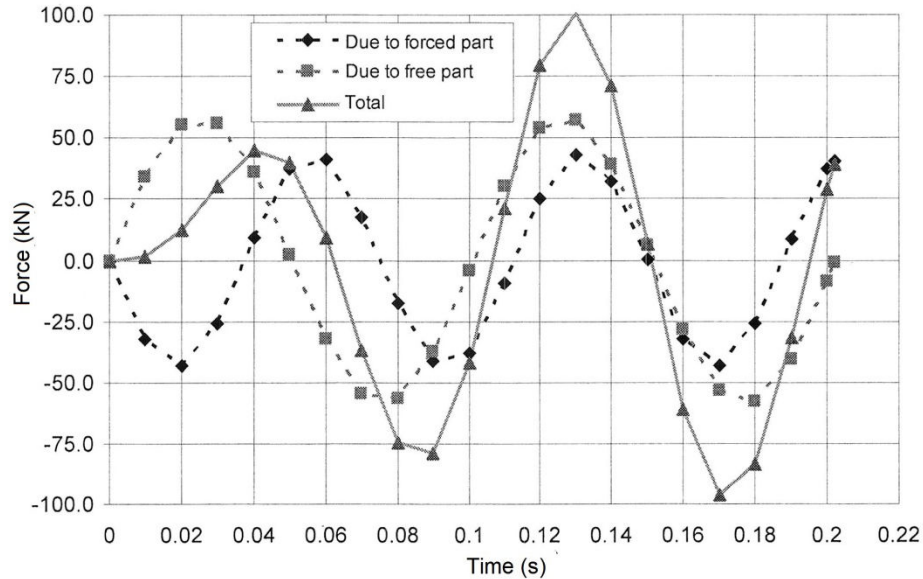
$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{62.11} = \mathbf{0.101 \text{ s}}$$

- b.** The frequency of loading, $f = 13.33$ Hz

$$\omega = 2\pi f = 2\pi(13.33) = 83.75 \text{ rad/s}$$

$$\begin{aligned} \text{Force due to forced part, } F_1 &= k \left(\frac{Q_0/k}{1 - \omega^2/\omega_n^2} \right) \sin \omega t \\ &= (70,000) \left(\frac{35.6/70,000}{1 - 83.75^2/62.11^2} \right) \sin(83.75t) \\ &= \mathbf{43.51 \sin(83.75t) \text{ kN}} \end{aligned}$$

See the plot below for F_1 vs. t



c. Force due to free part, $F_2 = k \left(\frac{Q_0/k}{1 - \omega^2/\omega_n^2} \right) \left(-\frac{\omega}{\omega_n} \sin \omega_n t \right)$

$$\begin{aligned} &= 70,000 \left(\frac{35.6/70,000}{1 - 83.75^2/62.11^2} \right) \left(-\frac{83.75}{62.11} \sin(62.11t) \right) \\ &= \mathbf{58.67 \sin(62.11t) \text{ kN}} \end{aligned}$$

See the plot above in Part **b** for F_2 vs. t .

d. Total dynamic force on the subgrade:

$$F = F_1 + F_2 = \mathbf{-43.51 \sin(83.75t) + 58.67 \sin(62.11t) \text{ kN}}$$

The plot of variation of the dynamic force on the subgrade of the foundation due to (a) forced part, (b) free part, and (c) total of the response for time $t = 0$ to $t = 2T$ is shown in the figure above (Part **b**).

2.5 The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k_{eq}}{m}} \quad \text{where } k_{eq} = \text{equivalent stiffness of the spring system}$$

For springs attached in series, the equivalent stiffness is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}, \text{ or } \frac{1}{k_{eq}} = \frac{k_1 k_2}{k_1 + k_2}$$

The natural frequency of the given undamped free vibration spring mass system is

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k_1 k_2}{k_1 + k_2}} \times \frac{1}{m}$$

2.6 The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k_{eq}}{m}} \quad \text{where } k_{eq} = \text{equivalent stiffness of the spring system}$$

For springs attached in parallel, the equivalent stiffness is given by

$$k_{eq} = k_1 + k_2$$

The natural frequency of the given undamped free vibration spring mass system is

$$f_n = \frac{1}{2\pi} \sqrt{\frac{(k_1 + k_2)}{m}}$$

2.7 The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k_{eq}}{m}} \quad \text{where } k_{eq} = \text{equivalent stiffness of the spring system}$$

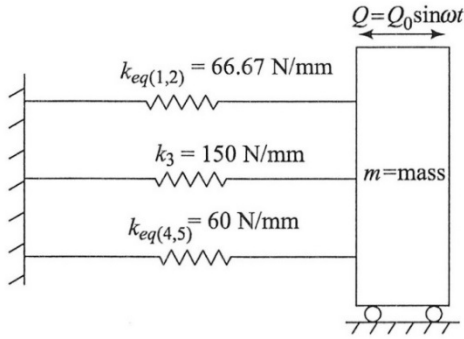
In the given spring-mass system, springs with stiffness k_1 and k_2 are in series. Hence, their equivalent stiffness is

$$k_{eq(1,2)} = \frac{k_1 k_2}{k_1 + k_2} = \frac{100 \times 200}{100 + 200} = \frac{20,000}{300} = 66.67 \text{ N/mm}$$

Similarly, springs with stiffness k_4 and k_5 are in series. Hence, their equivalent stiffness is

$$k_{eq(4,5)} = \frac{k_4 k_5}{k_4 + k_5} = \frac{100 \times 150}{100 + 150} = 60 \text{ N/mm}$$

Now, the given spring system can be reduced to three springs in series.



The resulting system will be three springs in parallel. Their equivalent stiffness is given by

$$k_{eq} = k_{eq(1,2)} + k_3 + k_{eq(4,5)} = 66.67 + 150 + 60 = 276.67 \text{ N/mm} = 276.67 \text{ kN/m}$$

The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k_{eq}}{m}} = \frac{1}{2\pi} \sqrt{\frac{276.67 \times 1000}{100}} = \mathbf{8.37 \text{ Hz}}$$

$$\text{Time period } T = (1/f_n) = (1/8.37) = \mathbf{0.119 \text{ s}}$$

2.8 Sinusoidal-varying force, $Q = 50 \sin \omega t \text{ N}$; $Q_0 = 50 \text{ N}$; $\omega = 47 \text{ rad/s}$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{276.67 \times 1000}{100}} = 52.6 \text{ rad/s}$$

Amplitude of vibration = static deflection z_s \times magnification M

$$z_s = \frac{Q_0}{k_{eq}} = \frac{50}{276.67} = 0.1807 \text{ mm}$$

From Eq. (2.34),

$$M = \frac{1}{1 - (\omega/\omega_n)^2} = \frac{1}{1 - (47/52.6)^2} = 4.96$$

$$\text{Amplitude of vibration} = 0.1807 \times 4.96 = \mathbf{0.896 \text{ mm}}$$

2.9 Weight of the body, $W = 135 \text{ N}$

$$\text{Mass of the body, } m = W/g = 135/9.81 = 13.76 \text{ kg}$$

$$\text{Spring constant, } k = 2600 \text{ N/m}$$

$$\text{Dashpot resistance, } c = 0.7/(60/1000) = 11.67 \text{ N-s/m}$$

- a. Damped natural frequency [Eq. (2.67)]

$$f_d = \sqrt{1 - D^2} f_n$$

$$D = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{11.67}{2\sqrt{2600 \times 13.76}} = 0.031$$

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{2600}{13.76}} = 2.19 \text{ Hz}$$

$$f_d = \sqrt{1 - 0.031^2} \times (2.19) = \mathbf{2.18 \text{ Hz}}$$

- b. Damping ratio [Eq. (2.47b)],

$$D = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{11.67}{2\sqrt{2600 \times 13.76}} = \mathbf{0.031}$$

- c. Ratio of successive amplitudes of the body is given by [Eq. (2.70)],

$$\frac{Z_n}{Z_{n+1}} = e^\delta$$

$$\text{where } \delta = \ln\left(\frac{Z_n}{Z_{n+1}}\right) = \frac{2\pi D}{\sqrt{1 - D^2}} = \frac{2\pi \times 0.031}{\sqrt{1 - 0.031^2}} = 0.195$$

$$\frac{Z_n}{Z_{n+1}} = e^\delta = e^{0.195} = \mathbf{1.215}$$

- d. At time $t = 0$ s, amplitude $Z_0 = 25$ mm.
 After n cycles of disturbance

$$\frac{1}{n} \ln \frac{Z_0}{Z_n} = \frac{2\pi D}{\sqrt{1 - D^2}}; \quad \ln \frac{Z_0}{Z_n} = \frac{2\pi n D}{\sqrt{1 - D^2}}$$

With $n = 5$,

$$\ln \frac{Z_0}{Z_5} = \frac{2\pi \times 5 \times D}{\sqrt{1 - D^2}} = \frac{2\pi \times 5 \times 0.031}{\sqrt{1 - 0.031^2}} = 0.974$$

$$\frac{Z_0}{Z_5} = e^{0.974} = 2.649; \quad Z_5 = \frac{25}{2.649} = 9.44 \text{ mm}$$

After 5 cycles of disturbance, the amplitude of vibration = **9.44 mm**

- 2.10** $Q_0 = 6.7$ kN
 $\omega = 3100$ rad/min = 51.67 rad/s
 Weight of machine + foundation, $W = 290$ kN
 Spring constant, $k = 875$ MN/m = 875,000 kN/m

Natural angular frequency, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{875,000 \times 10^3}{290 \times 10^3 / 9.81}} = 172.04$ rad/s

From Eq. (2.43), $F_{\text{dynam}} = \frac{Q_0}{1 - (\omega/\omega_n)} = \frac{6.7}{1 - (51.67/172.04)} = 9.58$ kN

Maximum force on the subgrade = 290 + 9.58 = **299.58 kN**

Minimum force on the subgrade = 290 - 9.58 = **280.42 kN**

- 2.11** $Q_0 = 200$ kN
 $\omega = 6000$ rad/min = 100 rad/s
 Weight of machine + foundation, $W = 400$ kN
 Spring constant, $k = 120,000$ kN/m

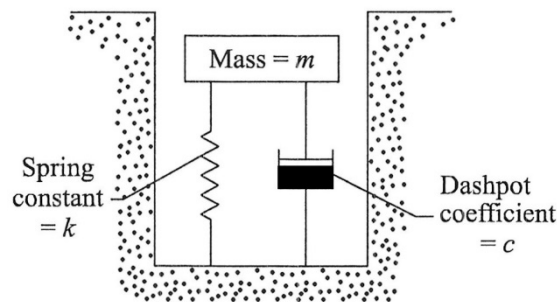
Natural angular frequency, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{120,000 \times 10^3}{400 \times 10^3 / 9.81}} = 54.25$ rad/s

Dynamic force, $F_{\text{dynam}} = \frac{Q_0}{1 - \omega/\omega_n} = \frac{200}{1 - (100/54.25)} = 237.16$ kN

Maximum force on the subgrade = 400 + 237.16 = **637.16 kN**

Minimum force on the subgrade = 400 - 237.16 = **162.84 kN**

- 2.12** Weight of the body, $W = 800$ kN
 Spring constant, $k = 200,000$ kN/m
 Dashpot coefficient, $c = 2340$ kN-s/m



- a. $c_c = 2\sqrt{km} = 2\sqrt{200,000 \times 800/9.81} = 8077.1 \text{ kN-s/m}$
- b. Damping ratio, $D = \frac{c}{c_c} = \frac{2340}{8077.1} = 0.29$
- c. $\delta = \frac{2\pi D}{\sqrt{1-D^2}} = \frac{2\pi \times 0.29}{\sqrt{1-0.29^2}} = 1.9$
- d. $f_d = \sqrt{1-D^2} f_n; f_n = \frac{1}{2\pi} \sqrt{\frac{200,000 \times 9.81}{800}} = 7.88 \text{ Hz}$
- $f_d = \sqrt{1-0.29^2} \times 7.88 = 7.54 \text{ Hz}$

- 2.13** Weight of the body, $W = 800 \text{ kN}$
 Spring constant, $k = 200,000 \text{ kN/m}$
 Dashpot coefficient, $c = 2340 \text{ kN-s/m}$
 $Q_0 = 25 \text{ kN}$
 Operating frequency, $\omega = 100 \text{ rad/s}$

- a. Natural circular frequency, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200,000 \times 9.81}{800}} = 49.52 \text{ rad/s}$

From Problem 2.12, damping ratio, $D = 0.29$

From Eq. (2.28), the amplitude of vertical vibration of the foundation is

$$\begin{aligned} Z &= \frac{(Q_0/k)}{\sqrt{[1 - (\omega^2/\omega_n^2)]^2 + 4D^2(\omega^2/\omega_n^2)}} \\ &= \frac{(25/200,000)}{\sqrt{[1 - (100^2/49.52^2)]^2 + 4 \times 0.29^2(100^2/49.52^2)}} \\ &= 3.795 \times 10^{-5} \text{ m} = 3.795 \times 10^{-2} \text{ mm} \end{aligned}$$

- b. From Eq. (2.90), the maximum dynamic force transmitted to the subgrade is

$$Z\sqrt{k^2 + (c\omega)^2} = (3.795 \times 10^{-5})\sqrt{200,000^2 + (2340 \times 100)^2} = 11.68 \text{ kN}$$