# Contents

# 2. Linear Systems and Stochastic Processes

## 2.12 Exercises

The following Exercises identify some of the basic ideas presented in this Chapter.

### 2.12.1 Problems

2.1. Does the equation of a straight line $y = \alpha x + \beta$, where $\alpha$ and $\beta$ are constants, represent a linear system? Show the proof.

2.2. Show how the z-Transform can become the DFT.

2.3. Which of the FIR filters defined by the following impulse responses are linear phase filters?

a. $h[n] = \{0.2, 0.3, 0.3, 0.2\}$
b. $h[n] = \{0.1, 0.2, 0.2, 0.1, 0.2, 0.2\}$
c. $h[n] = \{0.2, 0.2, 0.1, 0.1, 0.2, 0.2\}$
d. $h[n] = \{0.05, 0.15, 0.3, -0.15, -0.15\}$
e. $h[n] = \{0.05, 0.3, 0.0, -0.3, -0.05\}$

2.4. Given the following FIR filter impulse responses what are their $H(z)$ and $H(z^{-1})$? What are the zeros of the filters? Express the transfer functions in terms of zeros and poles? Prove that these filters have linear phases.

a. $h[n] = \{0.5, 0.5\}$
b. $h[n] = \{0.5, 0.0, -0.5\}$
c. $h[n] = \{0.5, 0.0, 0.5\}$
d. $h[n] = \{0.25, -0.5, 0.25\}$

2.5. Which of the following vector pairs are orthogonal or orthonormal?

a. $[1, -3, 5]^T$ and $[-1, -2, -1]^T$

b. $[0.6, 0.8]^T$ and $[4, -3]^T$
c. $[0.8, 0.6]^T$ and $[0.6, -0.8]^T$
d. $[1, 2, 3]^T$ and $[4, 5, 6]^T$

2.6. Which of the following matrices are Toeplitz?

$$\text{a. } \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{bmatrix}, \text{ b. } \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \text{ c. } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ d. } \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \text{ e. } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\text{f. } \begin{bmatrix} 1 & (1+j) & (1-j) \\ (1-j) & 1 & (1+j) \\ (1+j) & (1-j) & 1 \end{bmatrix}$$

What is special about matrices c, d, e, and f?

2.7. Which of the following matrices are orthogonal? Compute the matrix inverses of those that are orthogonal.

$$\text{a. } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ b. } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ c. } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ d. } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \text{ e. } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

2.8. Why are ergodic processes important?

2.9. Find the eigenvalues of the following 2 x 2 Toeplitz matrix,

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Find the eigenvectors for $a = 4$ and $b = 1$.

2.10. Compute the rounding quantization error variance for an Analogue to Digital Converter (ADC) with a quantization interval equal to $\Delta$. Assume that the signal distribution is uniform and that the noise is stationary white noise.

2.11. What is the mean and autocorrelation of the random phase sinusoid defined by, $x[n] = A\sin(n\omega_0 + \phi)$, given that $A$ and $\omega_0$ are fixed constants and $\phi$ is a random variable that is uniformly distributed over the interval $-\pi$ to $\pi$. The probability density function for $\phi$ is,

$$p_\phi(\alpha) = \begin{cases} \frac{1}{2\pi}, -\pi \le \alpha < \pi \\ 0 \quad , \text{elsewhere} \end{cases}$$

Repeat the computations for the harmonic process, $x[n] = Ae^{j(n\omega_0+\phi)}$.

2.12. Given the autocorrelation function for the random phase sinusoid in the previous Problem 2.11 compute the 2 x 2 autocorrelation matrix.

2.13. The autocorrelation sequence of a zero mean white noise process is $r_v(k) = \sigma_v^2 \delta(k)$ and the power spectrum is $P_v(e^{j\theta}) = \sigma_v^2$, where $\sigma_v^2$ is the variance of the process. For the random phase sinusoid the autocorrelation sequence is,

$$r_x(m) = \frac{1}{2} A^2 \cos(m\omega_0)$$

and the power spectrum is,

$$P_x(e^{j\theta}) = \frac{1}{2}\pi A^2 [u_0(\omega-\omega_0) + u_0(\omega+\omega_0)]$$

where:
$u_0(\omega-\omega_0)$ represents an impulse at frequency $\omega_0$.

What is the power spectrum of the first-order autoregressive process that has an autocorrelation sequence of,

$$r_x(m) = \alpha^{|m|}$$

where:
$|\alpha| < 1$

2.14. Let $x[n]$ be a random process that is generated by filtering white noise $w[n]$ with a first-order LSI filter having a system transfer function of,

$$H(z) = \frac{1}{1-0.25z^{-1}}$$

If the variance of the white noise is $\sigma_w^2 = 1$ what is the power spectrum of $x[n]$, $P_x(z)$? Find the autocorrelation of $x[n]$ from $P_x(z)$.

2.15. If $x[n]$ is a zero mean wide-sense stationary white noise process and $y[n]$ is formed by filtering $x[n]$ with a stable LSI filter $h[n]$ then is it true that,

$$\sigma_y^2 = \sigma_x^2 \sum_{n=-\infty}^{\infty} |h[n]|^2$$

where:

$\sigma_y^2$ and $\sigma_x^2$ are the variances of $x[n]$ and $y[n]$ respectively.

### 2.12.2 Answers

2.1. No, the equation does not represent a linear system for $\beta \neq 0$. It is called affine. If $\beta = 0$ it is linear. This type of system is often called linear but strictly it is not.

2.2.

$$H_p(k) = H_p(e^{j\theta})\Big|_{\theta=e^{\frac{2\pi k}{N}}} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\theta n}$$

$$\theta = \omega T = 2\pi f T$$

$$f = \frac{F_s k}{N}, T = \frac{1}{F_s}, \ fT = \frac{F_s k}{NF_s} = \frac{k}{N}$$

$$\therefore \theta = \frac{2\pi k}{N} \text{ and } H_p(k) = \sum_{n=-\infty}^{\infty} h[n]e^{-\frac{j2\pi kn}{N}}$$

2.3. A $N$-coefficient linear phase FIR filter must have an impulse response

where,

$$h^*[n] = h[N-1-n], \quad \text{Hermitian.}$$

or

$$h^*[n] = -h[N-1-n], \text{ anti-Hermitian.}$$

a. Yes, b. No, c. Yes, d. Yes, e. Yes.

2.4. a. $H(z) = (0.5z + 0.5)/z$, where $q = 1$ and $p = 0$
$H(z^{-1}) = 0.5 + 0.5z^{-1}$

There is one zero, $z_1 = -1$.
If $H(z)$ has zeros at $z = z_1$ then there must also be a zero at,

$$z = \frac{1}{z^*_1}, \quad \text{ie. } z = \frac{1}{z^*_1} = -1$$

$H(z^{-1}) = b_0(1-z_1z^{-1}) = 0.5(1+z^{-1})$

The filter has a linear phase if $H^*(z^*) = \pm z^{N-1} H\left(\frac{1}{z}\right)$

$H^*(z^*) = 0.5 + 0.5z$
$z^1 H(z^{-1}) = 0.5z + 0.5$

Problems b. to d. are left as exercises for you.

2.5. a. $[1, -3, 5]^T$ and $[-1, -2, -1]^T$ are orthogonal.
b. $[0.6, 0.8]^T$ and $[-4, 3]^T$ are orthogonal.
c. $[0.8, 0.6]^T$ and $[0.6, -0.8]^T$ are orthonormal.
d. $[1, 2, 3]^T$ and $[4, 5, 6]^T$ are neither orthogonal nor orthonormal.

2.6. a. Yes, d. No, c. Yes, symmetric, also perisymmetric Hankel, d. No, symmetric and perisymmetric Hankel, e. Yes, symmetrical, f. Yes, Hermitian.

2.7.

a. Yes, $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, b. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, c. No, d. No, e. No.

2.8. An ergodic process is a stationary process whose statistical properties can be estimated from the time averages of a single sample function. This means that since the statistics are invariant with respect to translation in time they can be determined by averaging outputs of the function at different times, which is possible to do in the real world.

2.9. Solve the characteristic 2nd order polynomial equation $p(\lambda)$=det($\mathbf{A}$-$\lambda\mathbf{I}$)=0.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{bmatrix} a-\lambda & b \\ b & a-\lambda \end{bmatrix}\right) = (a-\lambda)^2 - b^2$$

$$= \lambda^2 - 2a\lambda + (a^2 - b^2) = 0$$

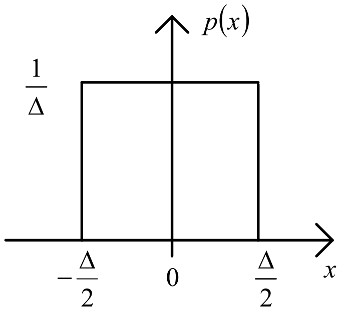
The eigenvalues are the roots $\lambda_1$ and $\lambda_2$ of the quadratic equation in $\lambda$ above. For $a = 4$ and $b = 1$, $\lambda^2 - 8\lambda + 15 = 0$ and $\lambda_1 = 5$ and $\lambda_2 = 3$.

The eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can be found by solving the two following equations sets,

$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$
$\mathbf{A}\mathbf{v}_1 = 5\mathbf{v}_1$
and
$\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$
$\mathbf{A}\mathbf{v}_2 = 3\mathbf{v}_2$

For the first equation the normalised $\mathbf{v}_1 = \frac{1}{\sqrt{2}}[1 \quad 1]^T$ and for the second the normalised $\mathbf{v}_2 = \frac{1}{\sqrt{2}}[1 \quad -1]^T$.

2.10. The PDF for the rounding quantization error is a tophat type as shown in the diagram below,




The variance equation for a noise signal $x(t)$ is,

$$\sigma_x^2 = E\{(x-\bar{x})^2\} = \int_{-\infty}^{+\infty} (x-\bar{x})^2 p(x)dx$$

For this case the mean is zero and the equation becomes,

$$\sigma_x^2 = E\{x^2\} = \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} x^2 p(x)dx = \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} x^2 \frac{1}{\Delta} dx = \frac{1}{\Delta}\left[\frac{x^3}{3}\right]_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}}$$

$$= \frac{\Delta^2}{12}$$

2.11. It is a zero mean process, i.e.,