

1.17 Write the number 0.2 in binary form with sufficient number of digits so that the true relative error is less than 0.01.

Solution

The number 0.2 in binary form is quite long. To express it in binary form with a true relative error of less than 0.005, we can use Eq. (1.17), but replacing the numerical solution by the approximate solution:

$$\text{TrueRelativeError} = \left| \frac{\text{GivenNumber} - \text{ApproximateNumber}}{\text{GivenNumber}} \right| < 0.005$$

or, $\left| \frac{0.2 - x}{0.2} \right| < 0.005$, which means that $-0.005 < \frac{0.2 - x}{0.2} < 0.005$ or $0.199 < x < 0.201$. So the decimal equivalent of the binary number we seek with the desired relative error must be between 0.199 and 0.201. The largest power of 2 that can be divided into 0.2 without exceeding it is 2^{-3} or 0.125. Next, subtract $0.2 - 0.125 = 0.075$. The highest power of 2 that can be divided into 0.075 is 2^{-4} or 0.0625. This means that the binary approximation $2^{-3} + 2^{-4} = 0.1875$ does not meet our relative error requirement because it is not between 0.199 and 0.201. Now, subtract $0.075 - 0.0625 = 0.0125$. The largest power of 2 that can be divided into 0.0125 is 2^{-7} or 0.0078125. Now, $2^{-3} + 2^{-4} + 2^{-7} = 0.1953125$ which is still not between 0.199 and 0.201. So, subtract $0.0125 - 0.0078125 = 0.0046875$. The largest power of 2 that will divide into 0.0046875 is 2^{-8} or 0.00390625. Now, $2^{-3} + 2^{-4} + 2^{-7} + 2^{-8} = 0.19921875$. Since this result is between 0.199 and 0.201, the number 0.2 expressed in binary form with a relative error of less than 0.005 is:

$$0.2 \approx 2^{-3} + 2^{-4} + 2^{-7} + 2^{-8} \quad \text{or} \quad 0.00110011$$

As a check, let us calculate the true relative error:

$$\left| \frac{0.2 - 0.19921875}{0.2} \right| = 0.00390625 < 0.005$$

Excerpts from this work may be reproduced by instructors for distribution on a not-for-profit basis for testing or instructional purposes only to students enrolled in courses for which the textbook has been adopted. Any other reproduction or translation of this work beyond that permitted by Sections 107 or 108 of the 1976 United States Copyright Act without the permission of the copyright owner is unlawful.