1.22 Taylor series expansion of the function $f(x) = e^x$ is:

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \tag{1.1}$$

Use Eq. (1.21) to calculate the value of e^{-2} for the following cases. Use decimal numbers with six significant numbers (apply rounding at each step). In each case calculate also the true relative error. Use MATLAB with `format long` to calculate the true value of e^{-2}.
(*a*) Use the first four terms. (*b*) Use the first six terms. (*c*) Use the first eight terms.

Solution

(*a*) Using the first four terms and 6 significant figures,

$$e^{-2} = 1 - 2 + 2 - \frac{8}{6} = 1 - 1.33333 = -0.33333$$

Before calculating the true relative error, note that the sign of the answer is completely wrong since $e^x > 0$ for all real values of x. Use of format long in MATLAB yields:

```
>> format compact
>> format long
>> exp(-2)
ans =
   0.13533528323661
```

Therefore, the true relative error is:

$$TrueRelativeError = \left|\frac{0.13533528323661 - (-0.33333)}{0.13533528323661}\right| = 3.46$$

or 346 %!

(*b*) With the first 6 terms and retaining 6 significant figures,

$$e^{-2} = 1 - 2 + 2 - \frac{8}{6} + \frac{16}{24} - \frac{32}{120} = 1 - 1.33333 + 0.666667 - 0.266667$$

$$= 0.0666700$$

which yields a true relative error of $\left|\frac{0.13533528323661 - 0.0666700}{0.13533528323661}\right| = 0.50737$ or less than 50.74%.

(*c*) With the first 8 terms and retaining 6 significant figures,

$$e^{-2} = 1 - 2 + 2 - \frac{8}{6} + \frac{16}{24} - \frac{32}{120} + \frac{64}{720} - \frac{128}{5040}$$

$$= 1 - 1.33333 + 0.666667 - 0.266667 + 0.0888889 - 0.0252968$$

$$= 0.130262$$

which yields a true relative error of $\left|\frac{0.13533528323661 - 0.130262}{0.13533528323661}\right| = 0.03749$ or less than 3.75%.

It can be clearly seen from the answers of parts (*a*), (*b*), and (*c*), that as more terms are retained, the better the accuracy.