Mathematical Preliminaries

Note: An asterisk (*) before an exercise indicates that there is a solution in the Student Study Guide.

Exercise Set 1.1, page 14

*1. For each part, $f \in C[a, b]$ on the given interval. Since $f(a)$ and $f(b)$ are of opposite sign, the Intermediate Value Theorem implies that a number c exists with $f(c) = 0$.

2. (a) $[0, 1]$

   (b) $[0, 1]$, $[4, 5]$, $[-1, 0]$

   *(c) $[-2, -2/3]$, $[0, 1]$, $[2, 4]$

   (d) $[-3, -2]$, $[-1, -0.5]$, and $[-0.5, 0]$

3. For each part, $f \in C[a, b]$, f' exists on (a, b) and $f(a) = f(b) = 0$. Rolle's Theorem implies that a number c exists in (a, b) with $f'(c) = 0$. For part (d), we can use $[a, b] = [-1, 0]$ or $[a, b] = [0, 2]$.

4. The maximum value for $|f(x)|$ is given below.

   *(a) $(2 \ln 2)/3 \approx 0.4620981$

   (b) 0.8

   (c) 5.164000

   (d) 1.582572

*5. For $x < 0$, $f(x) < 2x + k < 0$, provided that $x < -\frac{1}{2}k$. Similarly, for $x > 0$, $f(x) > 2x + k > 0$, provided that $x > -\frac{1}{2}k$. By Theorem 1.11, there exists a number c with $f(c) = 0$. If $f(c) = 0$ and $f(c') = 0$ for some $c' \neq c$, then by Theorem 1.7, there exists a number p between c and c' with $f'(p) = 0$. However, $f'(x) = 3x^2 + 2 > 0$ for all x.

6. Suppose p and q are in $[a, b]$ with $p \neq q$ and $f(p) = f(q) = 0$. By the Mean Value Theorem, there exists $\xi \in (a, b)$ with

$$f(p) - f(q) = f'(\xi)(p - q).$$

   But, $f(p) - f(q) = 0$ and $p \neq q$. So $f'(\xi) = 0$, contradicting the hypothesis.

7. (a) $P_2(x) = 0$

   (b) $R_2(0.5) = 0.125$; actual error $= 0.125$

(c) $P_2(x) = 1 + 3(x-1) + 3(x-1)^2$

(d) $R_2(0.5) = -0.125$; actual error $= -0.125$

8. $P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

x	0.5	0.75	1.25	1.5
$P_3(x)$	1.2265625	1.3310547	1.5517578	1.6796875
$\sqrt{x+1}$	1.2247449	1.3228757	1.5	1.5811388
$\lvert\sqrt{x+1} - P_3(x)\rvert$	0.0018176	0.0081790	0.0517578	0.0985487

*9. Since

$$P_2(x) = 1 + x \quad \text{and} \quad R_2(x) = \frac{-2e^{\xi}(\sin\xi + \cos\xi)}{6}x^3$$

for some ξ between x and 0, we have the following:

(a) $P_2(0.5) = 1.5$ and $|f(0.5) - P_2(0.5)| \le 0.0932$;

(b) $|f(x) - P_2(x)| \le 1.252$;

(c) $\int_0^1 f(x)\ dx \approx 1.5$;

(d) $|\int_0^1 f(x)\ dx - \int_0^1 P_2(x)\ dx| \le \int_0^1 |R_2(x)|\,dx \le 0.313$, and the actual error is 0.122.

10. $P_2(x) = 1.461930 + 0.617884\left(x - \frac{\pi}{6}\right) - 0.844046\left(x - \frac{\pi}{6}\right)^2$ and $R_2(x) = -\frac{1}{3}e^{\xi}(\sin\xi + \cos\xi)\left(x - \frac{\pi}{6}\right)^3$ for some ξ between x and $\frac{\pi}{6}$.

(a) $P_2(0.5) = 1.446879$ and $f(0.5) = 1.446889$. An error bound is 1.01×10^{-5}, and the actual error is 1.0×10^{-5}.

(b) $|f(x) - P_2(x)| \le 0.135372$ on $[0, 1]$

(c) $\int_0^1 P_2(x)\ dx = 1.376542$ and $\int_0^1 f(x)\ dx = 1.378025$

(d) An error bound is 7.403×10^{-3}, and the actual error is 1.483×10^{-3}.

11. $P_3(x) = (x-1)^2 - \frac{1}{2}(x-1)^3$

(a) $P_3(0.5) = 0.312500$, $f(0.5) = 0.346574$. An error bound is $0.291\overline{6}$, and the actual error is 0.034074.

(b) $|f(x) - P_3(x)| \le 0.291\overline{6}$ on $[0.5, 1.5]$

(c) $\int_{0.5}^{1.5} P_3(x)\ dx = 0.08\overline{3}$, $\int_{0.5}^{1.5} (x-1)\ln x\ dx = 0.088020$

(d) An error bound is $0.058\overline{3}$, and the actual error is 4.687×10^{-3}.

12. (a) $P_3(x) = -4 + 6x - x^2 - 4x^3$; $P_3(0.4) = -2.016$

(b) $|R_3(0.4)| \le 0.05849$; $|f(0.4) - P_3(0.4)| = 0.013365367$

(c) $P_4(x) = -4 + 6x - x^2 - 4x^3$; $P_4(0.4) = -2.016$

(d) $|R_4(0.4)| \le 0.01366$; $|f(0.4) - P_4(0.4)| = 0.013365367$

13. $P_4(x) = x + x^3$
   (a) $|f(x) - P_4(x)| \le 0.012405$
   (b) $\int_0^{0.4} P_4(x)\ dx = 0.0864,\ \int_0^{0.4} xe^{x^2}\ dx = 0.086755$
   (c) 8.27×10^{-4}
   (d) $P_4'(0.2) = 1.12,\ f'(0.2) = 1.124076$. The actual error is 4.076×10^{-3}.

*14. First we need to convert the degree measure for the sine function to radians. We have $180° = \pi$ radians, so $1° = \frac{\pi}{180}$ radians. Since,

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad \text{and} \quad f'''(x) = -\cos x,$$

we have $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$.

The approximation $\sin x \approx x$ is given by

$$f(x) \approx P_2(x) = x, \quad \text{and} \quad R_2(x) = -\frac{\cos \xi}{3!}x^3.$$

If we use the bound $|\cos \xi| \le 1$, then

$$\left|\sin \frac{\pi}{180} - \frac{\pi}{180}\right| = \left|R_2\left(\frac{\pi}{180}\right)\right| = \left|\frac{-\cos \xi}{3!}\left(\frac{\pi}{180}\right)^3\right| \le 8.86 \times 10^{-7}.$$

15. Since $42° = 7\pi/30$ radians, use $x_0 = \pi/4$. Then

$$\left|R_n\left(\frac{7\pi}{30}\right)\right| \le \frac{\left(\frac{\pi}{4} - \frac{7\pi}{30}\right)^{n+1}}{(n+1)!} < \frac{(0.053)^{n+1}}{(n+1)!}.$$

For $|R_n(\frac{7\pi}{30})| < 10^{-6}$, it suffices to take $n = 3$. To 7 digits,

$$\cos 42° = 0.7431448 \quad \text{and} \quad P_3(42°) = P_3(\frac{7\pi}{30}) = 0.7431446,$$

so the actual error is 2×10^{-7}.

*16. (a) $P_3(x) = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{23}{648}x^3$
   (b) We have

$$f^{(4)}(x) = \frac{-119}{1296}e^{x/2}\sin\frac{x}{3} + \frac{5}{54}e^{x/2}\cos\frac{x}{3},$$

so

$$\left|f^{(4)}(x)\right| \le \left|f^{(4)}(0.60473891)\right| \le 0.09787176, \quad \text{for} \ \ 0 \le x \le 1,$$

and

$$|f(x) - P_3(x)| \le \frac{|f^{(4)}(\xi)|}{4!}|x|^4 \le \frac{0.09787176}{24}(1)^4 = 0.004077990.$$

17. (a) $P_3(x) = \ln(3) + \frac{2}{3}(x-1) + \frac{1}{9}(x-1)^2 - \frac{10}{81}(x-1)^3$
    (b) $\max_{0\le x\le 1} |f(x) - P_3(x)| = |f(0) - P_3(0)| = 0.02663366$
    (c) $\tilde{P}_3(x) = \ln(2) + \frac{1}{2}x^2$
    (d) $\max_{0\le x\le 1} |f(x) - \tilde{P}_3(x)| = |f(1) - \tilde{P}_3(1)| = 0.09453489$
    (e) $P_3(0)$ approximates $f(0)$ better than $\tilde{P}_3(1)$ approximates $f(1)$.

18. $P_n(x) = \sum_{k=0}^{n} x^k, \; n \ge 19$

19. $P_n(x) = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}x^k, \; n \ge 7$

20. For n odd, $P_n(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + \frac{1}{n}(-1)^{(n-1)/2}x^n$. For n even, $P_n(x) = P_{n-1}(x)$.

21. A bound for the maximum error is 0.0026.

22. (a) $P_n^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, 1, \ldots, n$. The shapes of P_n and f are the same at x_0.
    (b) $P_2(x) = 3 + 4(x-1) + 3(x-1)^2$.

23. (a) The assumption is that $f(x_i) = 0$ for each $i = 0, 1, \ldots, n$. Applying Rolle's Theorem on each on the intervals $[x_i, x_{i+1}]$ implies that for each $i = 0, 1, \ldots, n-1$ there exists a number z_i with $f'(z_i) = 0$. In addition, we have

$$a \le x_0 < z_0 < x_1 < z_1 < \cdots < z_{n-1} < x_n \le b.$$

    (b) Apply the logic in part (a) to the function $g(x) = f'(x)$ with the number of zeros of g in $[a, b]$ reduced by 1. This implies that numbers w_i, for $i = 0, 1, \ldots, n-2$ exist with

$$g'(w_i) = f''(w_i) = 0, \quad \text{and} \quad a < z_0 < w_0 < z_1 < w_1 < \cdots w_{n-2} < z_{n-1} < b.$$

    (c) Continuing by induction following the logic in parts (a) and (b) provides $n+1-j$ distinct zeros of $f^{(j)}$ in $[a, b]$.
    (d) The conclusion of the theorem follows from part (c) when $j = n$, for in this case there will be (at least) $(n+1) - n = 1$ zero in $[a, b]$.

*24. First observe that for $f(x) = x - \sin x$ we have $f'(x) = 1 - \cos x \ge 0$, because $-1 \le \cos x \le 1$ for all values of x. Also, the statement clearly holds when $|x| \ge \pi$, because $|\sin x| \le 1$.

    (a) The observation implies that $f(x)$ is non-decreasing for all values of x, and in particular that $f(x) > f(0) = 0$ when $x > 0$. Hence for $x \ge 0$, we have $x \ge \sin x$, and when $0 \le x \le \pi$, $|\sin x| = \sin x \le x = |x|$.
    (b) When $-\pi < x < 0$, we have $\pi \ge -x > 0$. Since $\sin x$ is an odd function, the fact (from part (a)) that $\sin(-x) \le (-x)$ implies that $|\sin x| = -\sin x \le -x = |x|$.
    As a consequence, for all real numbers x we have $|\sin x| \le |x|$.

25. Since $R_2(1) = \frac{1}{6}e^{\xi}$, for some ξ in $(0, 1)$, we have $|E - R_2(1)| = \frac{1}{6}|1 - e^{\xi}| \le \frac{1}{6}(e-1)$.

26. (a) Use the series

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} \quad \text{to integrate} \quad \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

and obtain the result.

(b) We have

$$\frac{2}{\sqrt{\pi}}e^{-x^2}\sum_{k=0}^{\infty}\frac{2^k x^{2k+1}}{1\cdot 3\cdots(2k+1)} = \frac{2}{\sqrt{\pi}}\left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^7 + \frac{1}{24}x^8 + \cdots\right]$$
$$\cdot\left[x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + \frac{8}{105}x^7 + \frac{16}{945}x^9 + \cdots\right]$$
$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 + \cdots\right] = \text{erf }(x)$$

(c) 0.8427008

(d) 0.8427069

(e) The series in part (a) is alternating, so for any positive integer n and positive x we have the bound

$$\left|\text{erf}(x) - \frac{2}{\sqrt{\pi}}\sum_{k=0}^{n}\frac{(-1)^k x^{2k+1}}{(2k+1)k!}\right| < \frac{x^{2n+3}}{(2n+3)(n+1)!}\ .$$

We have no such bound for the positive term series in part (b).

27. (a) Let x_0 be any number in $[a,b]$. Given $\epsilon > 0$, let $\delta = \epsilon/L$. If $|x - x_0| < \delta$ and $a \le x \le b$, then $|f(x) - f(x_0)| \le L|x - x_0| < \epsilon$.

(b) Using the Mean Value Theorem, we have

$$|f(x_2) - f(x_1)| = |f'(\xi)||x_2 - x_1|,$$

for some ξ between x_1 and x_2, so

$$|f(x_2) - f(x_1)| \le L|x_2 - x_1|.$$

(c) One example is $f(x) = x^{1/3}$ on $[0,1]$.

*28. (a) The number $\frac{1}{2}(f(x_1) + f(x_2))$ is the average of $f(x_1)$ and $f(x_2)$, so it lies between these two values of f. By the Intermediate Value Theorem 1.11 there exist a number ξ between x_1 and x_2 with

$$f(\xi) = \frac{1}{2}(f(x_1) + f(x_2)) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

(b) Let $m = \min\{f(x_1), f(x_2)\}$ and $M = \max\{f(x_1), f(x_2)\}$. Then $m \le f(x_1) \le M$ and $m \le f(x_2) \le M$, so

$$c_1 m \le c_1 f(x_1) \le c_1 M \quad \text{and} \quad c_2 m \le c_2 f(x_2) \le c_2 M.$$

Thus

$$(c_1 + c_2)m \le c_1 f(x_1) + c_2 f(x_2) \le (c_1 + c_2)M$$

and

$$m \le \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} \le M.$$

By the Intermediate Value Theorem 1.11 applied to the interval with endpoints x_1 and x_2, there exists a number ξ between x_1 and x_2 for which

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

(c) Let $f(x) = x^2 + 1$, $x_1 = 0$, $x_2 = 1$, $c_1 = 2$, and $c_2 = -1$. Then for all values of x,

$$f(x) > 0 \quad \text{but} \quad \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} = \frac{2(1) - 1(2)}{2 - 1} = 0.$$

29. (a) Since f is continuous at p and $f(p) \neq 0$, there exists a $\delta > 0$ with

$$|f(x) - f(p)| < \frac{|f(p)|}{2},$$

for $|x - p| < \delta$ and $a < x < b$. We restrict δ so that $[p - \delta, p + \delta]$ is a subset of $[a, b]$. Thus, for $x \in [p - \delta, p + \delta]$, we have $x \in [a, b]$. So

$$-\frac{|f(p)|}{2} < f(x) - f(p) < \frac{|f(p)|}{2} \quad \text{and} \quad f(p) - \frac{|f(p)|}{2} < f(x) < f(p) + \frac{|f(p)|}{2}.$$

If $f(p) > 0$, then

$$f(p) - \frac{|f(p)|}{2} = \frac{f(p)}{2} > 0, \quad \text{so} \quad f(x) > f(p) - \frac{|f(p)|}{2} > 0.$$

If $f(p) < 0$, then $|f(p)| = -f(p)$, and

$$f(x) < f(p) + \frac{|f(p)|}{2} = f(p) - \frac{f(p)}{2} = \frac{f(p)}{2} < 0.$$

In either case, $f(x) \neq 0$, for $x \in [p - \delta, p + \delta]$.

(b) Since f is continuous at p and $f(p) = 0$, there exists a $\delta > 0$ with

$$|f(x) - f(p)| < k, \quad \text{for} \quad |x - p| < \delta \quad \text{and} \quad a < x < b.$$

We restrict δ so that $[p - \delta, p + \delta]$ is a subset of $[a, b]$. Thus, for $x \in [p - \delta, p + \delta]$, we have

$$|f(x)| = |f(x) - f(p)| < k.$$

Exercise Set 1.2, page 28

1. We have

	Absolute error	Relative error
(a)	0.001264	4.025×10^{-4}
(b)	7.346×10^{-6}	2.338×10^{-6}
(c)	2.818×10^{-4}	1.037×10^{-4}
(d)	2.136×10^{-4}	1.510×10^{-4}
(e)	2.647×10^{1}	1.202×10^{-3}
(f)	1.454×10^{1}	1.050×10^{-2}
(g)	420	1.042×10^{-2}
(h)	3.343×10^{3}	9.213×10^{-3}

2. The largest intervals are:
   (a) $(3.1412784, 3.1419068)$
   (b) $(2.7180100, 2.7185536)$
   *(c) $(1.4140721, 1.4143549)$
   (d) $(1.9127398, 1.9131224)$

3. The largest intervals are
   (a) (149.85,150.15)
   (b) (899.1, 900.9)
   (c) (1498.5, 1501.5)
   (d) (89.91,90.09)

4. The calculations and their errors are:
   (a) (i) 17/15 (ii) 1.13 (iii) 1.13 (iv) both 3×10^{-3}
   (b) (i) 4/15 (ii) 0.266 (iii) 0.266 (iv) both 2.5×10^{-3}
   (c) (i) 139/660 (ii) 0.211 (iii) 0.210 (iv) 2×10^{-3}, 3×10^{-3}
   (d) (i) 301/660 (ii) 0.455 (iii) 0.456 (iv) 2×10^{-3}, 1×10^{-4}

5. We have

	Approximation	Absolute error	Relative error
(a)	134	0.079	5.90×10^{-4}
(b)	133	0.499	3.77×10^{-3}
(c)	2.00	0.327	0.195
(d)	1.67	0.003	1.79×10^{-3}
*(e)	1.80	0.154	0.0786
(f)	-15.1	0.0546	3.60×10^{-3}
(g)	0.286	2.86×10^{-4}	10^{-3}
(h)	0.00	0.0215	1.00

6. We have

	Approximation	Absolute error	Relative error
(a)	133.9	0.021	1.568×10^{-4}
(b)	132.5	0.001	7.55×10^{-6}
(c)	1.700	0.027	0.01614
(d)	1.673	0	0
(e)	1.986	0.03246	0.01662
(f)	-15.16	0.005377	3.548×10^{-4}
(g)	0.2857	1.429×10^{-5}	5×10^{-5}
(h)	-0.01700	0.0045	0.2092

7. We have

	Approximation	Absolute error	Relative error
(a)	133	0.921	6.88×10^{-3}
(b)	132	0.501	3.78×10^{-3}
(c)	1.00	0.673	0.402
(d)	1.67	0.003	1.79×10^{-3}
*(e)	3.55	1.60	0.817
(f)	-15.2	0.0454	0.00299
(g)	0.284	0.00171	0.00600
(h)	0	0.02150	1

8. We have

	Approximation	Absolute error	Relative error
(a)	133.9	0.021	1.568×10^{-4}
(b)	132.5	0.001	7.55×10^{-6}
(c)	1.600	0.073	0.04363
(d)	1.673	0	0
(e)	1.983	0.02945	0.01508
(f)	-15.15	0.004622	3.050×10^{-4}
(g)	0.2855	2.143×10^{-4}	7.5×10^{-4}
(h)	-0.01700	0.0045	0.2092

9. We have

	Approximation	Absolute error	Relative error
*(a)	3.14557613	3.983×10^{-3}	1.268×10^{-3}
(b)	3.14162103	2.838×10^{-5}	9.032×10^{-6}

10. We have

	Approximation	Absolute error	Relative error
(a)	2.7166667	0.0016152	5.9418×10^{-4}
(b)	2.718281801	2.73×10^{-8}	1.00×10^{-8}

11. (a) We have

$$\lim_{x\to 0}\frac{x\cos x-\sin x}{x-\sin x}=\lim_{x\to 0}\frac{-x\sin x}{1-\cos x}=\lim_{x\to 0}\frac{-\sin x-x\cos x}{\sin x}=\lim_{x\to 0}\frac{-2\cos x+x\sin x}{\cos x}=-2$$

(b) $f(0.1)\approx -1.941$

(c) $\dfrac{x(1-\frac{1}{2}x^2)-(x-\frac{1}{6}x^3)}{x-(x-\frac{1}{6}x^3)}=-2$

(d) The relative error in part (b) is 0.029. The relative error in part (c) is 0.00050.

12. (a) $\lim_{x\to 0}\dfrac{e^x-e^{-x}}{x}=\lim_{x\to 0}\dfrac{e^x+e^{-x}}{1}=2$

(b) $f(0.1)\approx 2.05$

(c) $\dfrac{1}{x}\left(\left(1+x+\dfrac{1}{2}x^2+\dfrac{1}{6}x^3\right)-\left(1-x+\dfrac{1}{2}x^2-\dfrac{1}{6}x^3\right)\right)=\dfrac{1}{x}\left(2x+\dfrac{1}{3}x^3\right)=2+\dfrac{1}{3}x^2$;
using three-digit rounding arithmetic and $x=0.1$, we obtain 2.00.

(d) The relative error in part (b) is = 0.0233. The relative error in part (c) is = 0.00166.

13. We have

	x_1	Absolute error	Relative error	x_2	Absolute error	Relative error
(a)	92.26	0.01542	1.672×10^{-4}	0.005419	6.273×10^{-7}	1.157×10^{-4}
(b)	0.005421	1.264×10^{-6}	2.333×10^{-4}	−92.26	4.580×10^{-3}	4.965×10^{-5}
(c)	10.98	6.875×10^{-3}	6.257×10^{-4}	0.001149	7.566×10^{-8}	6.584×10^{-5}
(d)	−0.001149	7.566×10^{-8}	6.584×10^{-5}	−10.98	6.875×10^{-3}	6.257×10^{-4}

14. We have

	Approximation for x_1	Absolute error	Relative error
(a)	92.24	0.004580	4.965×10^{-5}
(b)	0.005417	2.736×10^{-6}	5.048×10^{-4}
(c)	10.98	6.875×10^{-3}	6.257×10^{-4}
(d)	−0.001149	7.566×10^{-8}	6.584×10^{-5}

	Approximation for x_2	Absolute error	Relative error
(a)	0.005418	2.373×10^{-6}	4.377×10^{-4}
(b)	−92.25	5.420×10^{-3}	5.875×10^{-5}
(c)	0.001149	7.566×10^{-8}	6.584×10^{-5}
(d)	−10.98	6.875×10^{-3}	6.257×10^{-4}

15. The machine numbers are equivalent to

    (a) 3224

    (b) −3224

    *(c) 1.32421875

    (d) 1.3242187500000002220446049250313080847263336181640625

16. (a) Next Largest: 3224.00000000000045474735088646411895751953125;
    Next Smallest: 3223.99999999999954525264911353588104248046875

    (b) Next Largest: −3224.00000000000045474735088646411895751953125;
    Next Smallest: −3223.99999999999954525264911353588104248046875

    *(c) Next Largest: 1.3242187500000002220446049250313080847263336181640625;
    Next Smallest: 1.3242187499999997779553950749686919152736663818359375

    (d) Next Largest: 1.324218750000000444089209850062616169452667236328125;
    Next Smallest: 1.32421875

17. (b) The first formula gives −0.00658, and the second formula gives −0.0100. The true three-digit value is −0.0116.

18. (a) −1.82

    (b) 7.09×10^{-3}

    (c) The formula in (b) is more accurate since subtraction is not involved.

19. The approximate solutions to the systems are

    (a) $x = 2.451$, $y = -1.635$

    (b) $x = 507.7$, $y = 82.00$

20. (a) $x = 2.460 \quad y = -1.634$

    (b) $x = 477.0 \quad y = 76.93$

*21. (a) In nested form, we have $f(x) = (((1.01e^x - 4.62)e^x - 3.11)e^x + 12.2)e^x - 1.99$.

    (b) −6.79

    (c) −7.07

    (d) The absolute errors are

$$|-7.61 - (-6.71)| = 0.82 \quad \text{and} \quad |-7.61 - (-7.07)| = 0.54.$$

    Nesting is significantly better since the relative errors are

$$\left|\frac{0.82}{-7.61}\right| = 0.108 \quad \text{and} \quad \left|\frac{0.54}{-7.61}\right| = 0.071,$$

22. We have $39.375 \leq \text{Volume} \leq 86.625 \quad \text{and} \quad 71.5 \leq \text{Surface Area} \leq 119.5$.

23. (a) $n = 77$

(b) $n = 35$

*24. When $d_{k+1} < 5$,

$$\left|\frac{y - fl(y)}{y}\right| = \frac{0.d_{k+1}\ldots \times 10^{n-k}}{0.d_1 \ldots \times 10^n} \le \frac{0.5 \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

When $d_{k+1} > 5$,

$$\left|\frac{y - fl(y)}{y}\right| = \frac{(1 - 0.d_{k+1}\ldots) \times 10^{n-k}}{0.d_1 \ldots \times 10^n} < \frac{(1 - 0.5) \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

25. (a) $m = 17$

(b) We have

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1)\cdots(m-k-1)(m-k)!}{k!(m-k)!} = \left(\frac{m}{k}\right)\left(\frac{m-1}{k-1}\right)\cdots\left(\frac{m-k-1}{1}\right)$$

(c) $m = 181707$

(d) 2,597,000; actual error 1960; relative error 7.541×10^{-4}

26. (a) The actual error is $|f'(\xi)\epsilon|$, and the relative error is $|f'(\xi)\epsilon| \cdot |f(x_0)|^{-1}$, where the number ξ is between x_0 and $x_0 + \epsilon$.

(b) (i) 1.4×10^{-5}; 5.1×10^{-6} (ii) 2.7×10^{-6}; 3.2×10^{-6}

(c) (i) 1.2; 5.1×10^{-5} (ii) 4.2×10^{-5}; 7.8×10^{-5}

27. (a) 124.03

(b) 124.03

(c) −124.03

(d) −124.03

(e) 0.0065

(f) 0.0065

(g) −0.0065

(h) −0.0065

*28. Since $0.995 \le P \le 1.005$, $0.0995 \le V \le 0.1005$, $0.082055 \le R \le 0.082065$, and $0.004195 \le N \le 0.004205$, we have $287.61° \le T \le 293.42°$. Note that $15°\text{C} = 288.16\text{K}$.

When P is doubled and V is halved, $1.99 \le P \le 2.01$ and $0.0497 \le V \le 0.0503$ so that $286.61° \le T \le 293.72°$. Note that $19°\text{C} = 292.16\text{K}$. The laboratory figures are within an acceptable range.

Exercise Set 1.3, page 39

1. (a) $\frac{1}{1}+\frac{1}{4}\ldots+\frac{1}{100}=1.53;\quad \frac{1}{100}+\frac{1}{81}+\ldots+\frac{1}{1}=1.54.$

   The actual value is 1.549. Significant round-off error occurs much earlier in the first method.

   (b) The following algorithm will sum the series $\sum_{i=1}^{N} x_i$ in the reverse order.

   INPUT $N; x_1, x_2, \ldots, x_N$
   OUTPUT SUM

   STEP 1 Set $SUM = 0$
   STEP 2 For $j = 1, \ldots, N$ set $i = N - j + 1$
   $SUM = SUM + x_i$
   STEP 3 OUTPUT(SUM);
   STOP.

2. We have

	Approximation	Absolute Error	Relative Error
(a)	2.715	3.282×10^{-3}	1.207×10^{-3}
(b)	2.716	2.282×10^{-3}	8.394×10^{-4}
(c)	2.716	2.282×10^{-3}	8.394×10^{-4}
(d)	2.718	2.818×10^{-4}	1.037×10^{-4}

*3. (a) 2000 terms

   (b) 20,000,000,000 terms

4. 4 terms

*5. 3 terms

6. (a) $O\left(\frac{1}{n}\right)$

   (b) $O\left(\frac{1}{n^2}\right)$

   (c) $O\left(\frac{1}{n^2}\right)$

   (d) $O\left(\frac{1}{n}\right)$

7. The rates of convergence are:

   (a) $O(h^2)$

   (b) $O(h)$

   (c) $O(h^2)$

   (d) $O(h)$

*8. (a) $n(n+1)/2$ multiplications; $(n+2)(n-1)/2$ additions.

(b) $\sum_{i=1}^{n} a_i \left(\sum_{j=1}^{i} b_j \right)$ requires n multiplications; $(n+2)(n-1)/2$ additions.

9. The following algorithm computes $P(x_0)$ using nested arithmetic.

INPUT $n, a_0, a_1, \ldots, a_n, x_0$
OUTPUT $y = P(x_0)$

STEP 1 Set $y = a_n$.
STEP 2 For $i = n-1, n-2, \ldots, 0$ set $y = x_0 y + a_i$.
STEP 3 OUTPUT (y);
STOP.

*10. The following algorithm uses the most effective formula for computing the roots of a quadratic equation.

INPUT A, B, C.
OUTPUT x_1, x_2.

STEP 1 If $A = 0$ then
if $B = 0$ then OUTPUT ('NO SOLUTIONS');
STOP.
else set $x_1 = -C/B$;
OUTPUT ('ONE SOLUTION',x_1);
STOP.

STEP 2 Set $D = B^2 - 4AC$.

STEP 3 If $D = 0$ then set $x_1 = -B/(2A)$;
OUTPUT ('MULTIPLE ROOTS', x_1);
STOP.

STEP 4 If $D < 0$ then set
$b = \sqrt{-D}/(2A)$;
$a = -B/(2A)$;
OUTPUT ('COMPLEX CONJUGATE ROOTS');
$x_1 = a + bi$;
$x_2 = a - bi$;
OUTPUT (x_1, x_2);
STOP.

STEP 5 If $B \geq 0$ then set
$d = B + \sqrt{D}$;
$x_1 = -2C/d$;
$x_2 = -d/(2A)$
else set
$d = -B + \sqrt{D}$;
$x_1 = d/(2A)$;
$x_2 = 2C/d$.

STEP 6 OUTPUT (x_1, x_2);
STOP.

11. The following algorithm produces the product $P = (x - x_0), \ldots, (x - x_n)$.

INPUT $n, x_0, x_1, \cdots, x_n, x$
OUTPUT P.

STEP 1 Set $P = x - x_0$;
$i = 1$.

STEP 2 While $P \neq 0$ and $i \leq n$ set
$P = P \cdot (x - x_i)$;
$i = i + 1$

STEP 3 OUTPUT (P);
STOP.

12. The following algorithm determines the number of terms needed to satisfy a given tolerance.

INPUT number x, tolerance TOL, maximum number of iterations M.
OUTPUT number N of terms or a message of failure.

STEP 1 Set $SUM = (1 - 2x)/\left(1 - x + x^2\right)$;
$S = (1 + 2x)/\left(1 + x + x^2\right)$;
$N = 2$.

STEP 2 While $N \leq M$ do Steps 3–5.

STEP 3 Set $j = 2^{N-1}$;
$y = x^j$
$t_1 = \dfrac{jy}{x}(1 - 2y)$;
$t_2 = y(y - 1) + 1$;
$SUM = SUM + \dfrac{t_1}{t_2}$.

STEP 4 If $|SUM - S| < TOL$ then
OUTPUT (N);
STOP.

STEP 5 Set $N = N + 1$.

STEP 6 OUTPUT('Method failed');
STOP.

When $TOL = 10^{-6}$, we need to have $N \geq 4$.

13. (a) If $|\alpha_n - \alpha|/(1/n^p) \leq K$, then

$$|\alpha_n - \alpha| \leq K(1/n^p) \leq K(1/n^q) \quad \text{since} \quad 0 < q < p.$$

Thus

$$|\alpha_n - \alpha|/(1/n^p) \leq K \quad \text{and} \quad \{\alpha_n\}_{n=1}^{\infty} \to \alpha$$

with rate of convergence $O(1/n^p)$.

(b)

n	$1/n$	$1/n^2$	$1/n^3$	$1/n^5$
5	0.2	0.04	0.008	0.0016
10	0.1	0.01	0.001	0.0001
50	0.02	0.0004	8×10^{-6}	1.6×10^{-7}
100	0.01	10^{-4}	10^{-6}	10^{-8}

The most rapid convergence rate is $O(1/n^4)$.

14. (a) If $F(h) = L + O(h^p)$, there is a constant $k > 0$ such that

$$|F(h) - L| \leq kh^p,$$

for sufficiently small $h > 0$. If $0 < q < p$ and $0 < h < 1$, then $h^q > h^p$. Thus, $kh^p < kh^q$, so

$$|F(h) - L| \leq kh^q \quad \text{and} \quad F(h) = L + O(h^q).$$

(b) For various powers of h we have the entries in the following table.

h	h^2	h^3	h^4
0.5	0.25	0.125	0.0625
0.1	0.01	0.001	0.0001
0.01	0.0001	0.00001	10^{-8}
0.001	10^{-6}	10^{-9}	10^{-12}

The most rapid convergence rate is $O\left(h^4\right)$.

*15. Suppose that for sufficiently small $|x|$ we have positive constants k_1 and k_2 independent of x, for which

$$|F_1(x) - L_1| \leq K_1|x|^\alpha \quad \text{and} \quad |F_2(x) - L_2| \leq K_2|x|^\beta.$$

Let $c = \max(|c_1|, |c_2|, 1)$, $K = \max(K_1, K_2)$, and $\delta = \max(\alpha, \beta)$.

(a) We have

$$\begin{aligned}|F(x) - c_1L_1 - c_2L_2| &= |c_1(F_1(x) - L_1) + c_2(F_2(x) - L_2)| \\ &\leq |c_1|K_1|x|^\alpha + |c_2|K_2|x|^\beta \leq cK[|x|^\alpha + |x|^\beta] \\ &\leq cK|x|^\gamma[1 + |x|^{\delta-\gamma}] \leq \tilde{K}|x|^\gamma,\end{aligned}$$

for sufficiently small $|x|$ and some constant $\tilde{K}$. Thus, $F(x) = c_1L_1 + c_2L_2 + O(x^\gamma)$.

(b) We have

$$\begin{aligned}|G(x) - L_1 - L_2| &= |F_1(c_1x) + F_2(c_2x) - L_1 - L_2| \\ &\le K_1|c_1x|^\alpha + K_2|c_2x|^\beta \le Kc^\delta[|x|^\alpha + |x|^\beta] \\ &\le Kc^\delta|x|^\gamma[1 + |x|^{\delta-\gamma}] \le \tilde{K}|x|^\gamma,\end{aligned}$$

for sufficiently small $|x|$ and some constant $\tilde{K}$. Thus, $G(x) = L_1 + L_2 + O(x^\gamma)$.

*16. Since

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x \quad \text{and} \quad x_{n+1} = 1 + \frac{1}{x_n},$$

we have

$$x = 1 + \frac{1}{x}, \quad \text{so} \quad x^2 - x - 1 = 0.$$

The quadratic formula implies that

$$x = \frac{1}{2}\left(1 + \sqrt{5}\right).$$

This number is called the *golden ratio*. It appears frequently in mathematics and the sciences.

*17. (a) To save space we will show the Maple output for each step in one line. Maple would produce this output on separate lines.

$n := 98; f := 1; s := 1$

$$n := 98 \quad f := 1 \quad s := 1$$

for i from 1 *to n do*
$l := f + s; f := s; s := l; od :$

$l :=2 \quad f := 1 \quad s := 2$
$l :=3 \quad f := 2 \quad s := 3$
$\vdots$
$l :=218922995834555169026 \quad f := 135301852344706746049 \quad s := 218922995834555169026$
$l :=354224848179261915075$

(b) $F100 := \dfrac{1}{sqrt(5)}\left(\left(\dfrac{(1 + sqrt(5))}{2}\right)^{100} - \left(\dfrac{1 - sqrt(5)}{2}\right)^{100}\right)$

$$F100 := \frac{1}{\sqrt{5}}\left(\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^{100} - \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^{100}\right)$$

evalf$(F100)$

$$0.3542248538 \times 10^{21}$$