CHAPTER 1: Introduction

1.1. Show that the diffusion equation (heat conduction is one example) $u_{,xx} = \alpha u_{,t}$, where α is a positive constant, is parabolic.

Solution of 1.1.

$$u_{,xx} = \alpha u_{,t}$$

The above equation can be reduced to a first order form following the same procedure presented in Section 1.5. We let $f = u_{,x}$ and $g = u_{,t}$ from which we obtain the two first-order equations:

$$f_{,x} = \alpha g$$

$$f_{,t} = g_{,x}$$

Expressing the derivatives of the dependent variables as

$$f_{,s} = f_{,x}x_{,s} + f_{,t}t_{,s}$$

$$g_{,s} = g_{,x}x_{,s} + g_{,t}t_{,s}$$

And writing the above system is matrix form,

$$\boldsymbol{Az} = \begin{bmatrix} \alpha^{-1} & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ x_{,s} & t_{,s} & 0 & 0 \\ 0 & 0 & x_{,s} & t_{,s} \end{bmatrix} \begin{Bmatrix} f_{,x} \\ f_{,t} \\ g_{,x} \\ g_{,t} \end{Bmatrix} = \begin{Bmatrix} g \\ 0 \\ f_{,s} \\ g_{,s} \end{Bmatrix}$$

$$det(\boldsymbol{A}) = \alpha^{-1}\left(t_{,s}\right)^2 \Rightarrow t_{,s} = 0$$

Dividing by $x_{,s}$

$$t_{,x} = 0$$

From which we conclude that the diffusion equation is parabolic.

1.2. Determine the classification of the equation for the dynamics of beams, $u_{,xxxx} = \alpha u_{,tt}$.

<u>*Solution of 1.2.*</u>

Considering the solution for exercise 1, by inspection, the equation for the dynamics of beams is also parabolic. Note that, if $f = u_{,xx}$ and $g = u_{,t}$ we get

$$f_{,xx} = \alpha g_{,t}$$

CHAPTER 2: Lagrangian and Eulerian finite elements in one dimension

2.1. Transform the principle of virtual work to the principle of virtual power by letting $\delta u = \delta v$ and using the conservation of mass and the transformations for the stresses. (Note that this is possible since the admissibility conditions on the two sets of test and trial function spaces are identical).

Solution of 2.1.

The Principle of virtual work is:

$$\int_{X_1}^{X_2} (\delta u)_{,X} \, PA_0 \, dX - (\delta u A_0 n^0 P)|_{\Gamma_t} - \int_{X_1}^{X_2} \delta u \, \rho_0 b A_0 dX + \int_{X_1}^{X_2} \delta u \, \rho_0 A_0 \ddot{u} dX = 0$$

The transformation to the principle of virtual power is possible by letting $\delta u = \delta v$, using the conservation of mass, $\rho_0 A_0 dX = \rho A dx$, the transformation for the stresses, $PA_0 = \sigma A$, and using the chain rule $\frac{\partial}{\partial X} = \frac{\partial}{\partial x}\frac{\partial x}{\partial X}$

$$\int_{x_1}^{x_2} (\delta v)_{,x} \frac{\partial x}{\partial X} \, \sigma A dX - (\delta v \, A \, n \, \sigma)|_{\Gamma_t} - \int_{x_1}^{x_2} \delta v \rho b A \, dx + \int_{x_1}^{x_2} \delta u \rho A \dot{v} \, dx = 0$$

$$\int_{x_1}^{x_2} [(\delta v)_{,x} \, \sigma A - \delta v(\rho b A + \rho A \dot{v})] dx - (\delta v \, A \, n \, \sigma)|_{\Gamma_t} = 0$$

2.2. **Consider a tapered two-node element with a linear displacement field as in Example 2.1 where the cross-sectional area** $A_0 = A_{01}(1 - \xi) + A_{02}\xi$, **where** A_{01} **and** A_{02} **are the initial cross-sectional areas at nodes 1 and 2. Assume that the nominal stress** P **is also linear in the element, i.e.** $P = P_1(1 - \xi) + P_2\xi$.

(a) **Using the total Lagrangian formulation, develop expressions for the internal nodal forces. For a constant body force, develop the external nodal forces. Compare the internal and external nodal forces for the case when** $A_{01} = A_{02} = A_0$ **and** $P_1 = P_2$ **to the results in Example 2.1.**

(b) **Develop the consistent mass matrix. Then obtain a diagonal form of the mass matrix by the row-sum technique. Find the frequencies of a single element with consistent mass and the diagonal mass by solving the eigenvalue problem**

$$\mathbf{K}\mathbf{y} = \omega^2 \mathbf{M}\mathbf{y} \quad \text{where } \mathbf{K} = \frac{E^{PF}(A_{01} + A_{02})}{2\ell_0}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

<u>*Solution of 2.2a).*</u>

Similarly to the example 2.1, the displacement field is given by the linear Lagrange interpolant expressed in terms of the material coordinates:

$$u(X,t) = \frac{1}{l_0}\begin{bmatrix} X_2 - X & X - X_1 \end{bmatrix}\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}$$

where $\boldsymbol{N} = \frac{1}{l_0}\begin{bmatrix} X_2 - X & X - X_1 \end{bmatrix}$ and $l_0 = X_2 - X_1$.

The strain measure is evaluated in terms of the nodal displacements by using $\varepsilon = \sum_I \frac{\partial N_I}{\partial X} u_I^e = \boldsymbol{B_0}\, \boldsymbol{u^e}$

$$\varepsilon(X,t) = u_{,X} = \sum_I \frac{\partial N_I}{\partial X} u_I^e = \frac{1}{l_0}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}$$

where $\boldsymbol{B}_0 = \frac{1}{l_0}\begin{bmatrix} -1 & 1 \end{bmatrix}$.

However, the displacement field can also be expressed in parent coordinates

$$u(\xi,t) = \begin{bmatrix} 1 - \xi & \xi \end{bmatrix}\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}$$

where $\boldsymbol{N}(\xi) = [1-\xi \quad \xi]$, with $\xi = \frac{X-X_1}{l_0} \Rightarrow X_{,\xi} = l_0$.

Using parent coordinates, the displacement field becomes

$$\varepsilon(\xi,t) = X_{,\xi}^{-1}\,\boldsymbol{N}_{,\xi}(\xi)\boldsymbol{u}^e(t) = \frac{1}{l_0}\,[-1 \quad 1]\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}$$

where $\boldsymbol{B}_0 = X_{,\xi}^{-1}\,\boldsymbol{N}_{,\xi}(\xi) = \frac{1}{l_0}[-1 \quad 1]$

We can now obtain the internal nodal forces:

$$\boldsymbol{f}_e^{int} = \int_{\Omega_0^e} \boldsymbol{B}_o^T P\, d\,\Omega_0 = \int_0^1 \boldsymbol{B}_0^T(\xi) P\, A_0\, X_{,\xi} d\xi$$

$$= \int_0^1 \frac{1}{l_0}\begin{bmatrix} -1 \\ 1 \end{bmatrix}(P_1(1-\xi) + P_2\xi)(A_{01}(1-\xi) + A_{02}\xi) l_0 d\xi$$

$$\boldsymbol{f}_e^{int} = \frac{A_{01}(2P_1+P_2) + A_{02}(P_1+2P_2)}{6}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

The external nodal forces are:

$$\boldsymbol{f}_e^{ext} = \int_{\Omega_0^e} \rho_0 \boldsymbol{N}^T b\, A_0 dX = \int_0^1 \rho_0\, \boldsymbol{N}^T\,(\xi)\, bA_0\, X_{,\xi} d\xi$$

$$= \int_0^1 \rho_0 \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} b(A_{01}(1-\xi) + A_{02}\xi) l d\xi$$

$$\boldsymbol{f}_e^{ext} = \frac{\rho_0 b\, l_0}{6}\begin{Bmatrix} 2A_{01} + A_{02} \\ A_{01} + 2A_{02} \end{Bmatrix}$$

Finally, we can compare these results with the ones obtained in example 2.1. The internal and external nodal forces for the case when $A_{01} = A_{02} = A_0$ and $P_1 = P_2 = P$ become:

$$\boldsymbol{f}_e^{int} = A_0 P \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{f}_e^{ext} = \frac{\rho_0 A_0 l_0}{2}\begin{bmatrix} b \\ b \end{bmatrix}$$

Which are the same results obtained in example 2.1 (provided that $b_1 = b_2 = b$).

Solution of 2.2b).

Consistent mass matrix:

$$\boldsymbol{M}_{\boldsymbol{e}}^C = \int_{X_1}^{X_2} \rho_0 \boldsymbol{N}^T \boldsymbol{N} A_0 dX = \int_0^1 \rho_0\, \boldsymbol{N}^T(\xi)\boldsymbol{N}(\xi)\, A_0(\xi) X_{,\xi} d\xi$$

$$\boldsymbol{M}_{\boldsymbol{e}}^C = \frac{\rho_0 l}{12}\begin{bmatrix} 3A_{01} + A_{02} & A_{01} + A_{02} \\ A_{01} + A_{02} & A_{01} + 3A_{02} \end{bmatrix}$$

Diagonal Mass matrix ($M_{II} = \sum M_{IJ}$)

$$\boldsymbol{M}_{\boldsymbol{e}}^D = \frac{\rho_0 l}{6}\begin{bmatrix} 2A_{01} + A_{02} & 0 \\ 0 & A_{01} + 2A_{02} \end{bmatrix}$$

In order to find the natural frequencies, we need to solve the eigenvalue problem using the stiffness matrix provided in the problem statement:

$$\mathbf{K}\,\mathbf{y} = \omega^2 \boldsymbol{M}\boldsymbol{y} \Rightarrow (\boldsymbol{k} - \omega^2 \boldsymbol{M})\boldsymbol{y} = 0$$

Hence,

$$\det\,(\mathbf{k} - \omega^2 \boldsymbol{M}) = 0$$

Solving for the consistent mass matrix $\boldsymbol{M}_{\boldsymbol{e}}^C$, we find (besides the trivial $\omega_C = 0$):

$$\omega_C = \frac{2\sqrt{6}(A_{01} + A_{02})}{l_0(A_{01} + 2A_{02})}\sqrt{\frac{E^{PF}}{\rho_0}}$$

Solving for the diagonal mass matrix $\boldsymbol{M}_{\boldsymbol{e}}^{D}$, we find (besides the trivial $\omega_D = 0$):

$$\omega_D = \frac{3(A_{01} + A_{02})}{l_0\sqrt{(A_{01} + 2A_{02})(2A_{01} + A_{02})}}\sqrt{\frac{E^{PF}}{\rho_0}}$$

Note that $\omega_C > \omega_D$.

2.3. Consider a tapered two-node element with a linear displacement field in the updated Lagrangian formulation as in Example 2.4. Let the current cross-sectional area be given by $A = A_1(1-\xi) + A_2\xi$, where A_1 and A_2 are the current cross-sectional areas at nodes 1 and 2. Develop the internal nodal forces in terms of the Cauchy stress for the updated Lagrangian formulation assuming $\sigma = \sigma_1(1-\xi) + \sigma_2\xi$ where σ_1 and σ_2 are the Cauchy stresses at the two nodes. Develop the nodal external forces for a constant body force.

Solution of 2.3.

Internal nodal forces:

$$\boldsymbol{f}^{int} = \int_{x_1}^{x_2}\frac{\partial \boldsymbol{N}^T}{\partial x}\sigma\, A dx = \int_0^1 \frac{\partial \boldsymbol{N}^T}{\partial \xi}\left(x_{,\xi}\right)^{-1}\sigma\, A(\xi) x_{,\xi} d\xi$$

$$\boldsymbol{N} = \begin{bmatrix}1-\xi & \xi\end{bmatrix}$$

$$x = \boldsymbol{N}^T\begin{Bmatrix}x_1\\x_2\end{Bmatrix} = (1-\xi)x_1 + \xi x_2$$

$$x_{,\xi} = -x_1 + x_2 = l$$

$$\boldsymbol{f}^{int} = \int_0^1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{1}{l} \left(\sigma_1(1-\xi) + \sigma_2\xi\right)\left(A_1(1-\xi) + A_2\xi\right) l \, d\xi$$

$$\boldsymbol{f}^{int} = \left[\frac{2A_1}{3}(4\sigma_1 - \sigma_2) + \frac{2A_2}{3}(\sigma_2 - \sigma_1)\right] \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

External nodal forces:

$$\boldsymbol{f}^{ext} = \int_{x_1}^{x_2} \rho \boldsymbol{N}^T b \, A dx = \int_0^1 \rho \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} b\left(A_1(1-\xi) + A_2\xi\right) l \, d\xi$$

$$\boldsymbol{f}^{ext} = \frac{\rho b l}{3} \begin{Bmatrix} A_1 + \dfrac{A_2}{2} \\ \dfrac{A_1}{2} + A_2 \end{Bmatrix}$$

2.4. Consider a 2-element mesh consisting of elements of length ℓ with constant cross-sectional area A. Assemble a consistent mass matrix and a stiffness matrix and obtain the frequency for the two element mesh with all nodes free (the eigenvalue problem is 3 × 3). The frequency analysis assumes a linear response so the initial and current geometry are identical. Repeat the same problem with a lumped mass. Compare the frequencies for the lumped and consistent mass matrices to the exact frequency for a free-free rod, $\omega = n\frac{\pi c}{L}$, where $n = 0, 1, \ldots$ Observe that the consistent mass frequencies are above the exact, whereas the diagonal mass frequencies are below the exact.

<u>*Solution of 2.4.*</u>

Calculating the consistent mass matrix for each element:

$$\boldsymbol{M}^C_{(1)} = \boldsymbol{M}^C_{(2)} = \boldsymbol{M}^C_e = \int_{x_1}^{x_2} \rho \boldsymbol{N}^{\boldsymbol{T}} \boldsymbol{N}\, A\, dx = \int_0^1 \rho A \boldsymbol{N}^{\boldsymbol{T}} \boldsymbol{N}\, x_{,\xi}\, dA$$

For a two-node element with length l:

$$\boldsymbol{N} = \begin{bmatrix} 1-\xi & \xi \end{bmatrix}$$

$$x = \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = (1-\xi)x_1 + \xi x_2$$

$$x_{,\xi} = -x_1 + x_2 = l$$

Then,

$$\boldsymbol{M}^C_e = \int_{x_1}^{x_2} \rho\, A \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} l\, d\xi = \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Assemble the mass matrix for the two element mesh according to the connectivity matrices:

$$\boldsymbol{M}^C = \boldsymbol{L}_{(1)} \boldsymbol{M}^C_{(1)} + \boldsymbol{L}_{(2)} \boldsymbol{M}^C_{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\boldsymbol{M}^C = \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The stiffness matrix for each element (considering linear elastic isotropic material):

$$\boldsymbol{K}_{(1)} = \boldsymbol{K}_{(2)} = \boldsymbol{K}_e = \int_{x_1}^{x_2} \frac{\partial \boldsymbol{N}^T}{\partial x} \frac{\partial \boldsymbol{N}}{\partial x} EA\, dx = \frac{E\, A}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assembling the stiffness matrix for the two element mesh:

$$\boldsymbol{K} = \boldsymbol{L}_{(1)} \boldsymbol{K}_{(1)} + \boldsymbol{L}_{(2)} \boldsymbol{K}_{(2)} = \frac{EA}{l} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

We can now determine the natural frequencies solving the eigenvalue problem as in exercise 2:

$$\boldsymbol{K}\,\boldsymbol{y} = \omega^2\,\boldsymbol{M}\,\boldsymbol{y} \;\Rightarrow (\boldsymbol{k} - \omega^2\,\boldsymbol{M})\boldsymbol{y} = 0$$

Using the consistent mass matrix:

$$det(\boldsymbol{k} - \omega^2\boldsymbol{M}^C\,) = 0 \quad \Rightarrow \quad \omega_1^2 = 0 \;\vee\; \omega_2^2 = \frac{3E}{l^2\rho} \;\vee\; \omega_2^2 = \frac{12E}{l^2\rho}$$

$$\omega_1 = 0 \;\vee\; \omega_2 = \frac{\sqrt{3}}{l}\sqrt{\frac{E}{\rho}} \cong 0.551\,\frac{\pi c}{l} \;\vee\; \omega_3 = \frac{2\sqrt{3}}{l}\sqrt{\frac{E}{\rho}} \cong\; 1.103\,\frac{\pi c}{l}$$

Using the lumped mass matrix:

$$\boldsymbol{M}^D = \frac{\rho A l}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(\boldsymbol{k} - \omega^2\boldsymbol{M}^D) = 0 \quad \Rightarrow \quad \omega_1^2 = 0 \vee \omega_2^2 = \frac{2E}{l^2\rho} \vee \omega_3^2 = \frac{4E}{l^2\rho}$$

$$\omega_1 = 0 \;\vee\; \omega_2 = \frac{\sqrt{2}}{l}\sqrt{\frac{E}{\rho}} \cong\; 0.450\,\frac{\pi c}{l} \;\vee\; \omega_3 = \frac{2}{l}\sqrt{\frac{E}{\rho}} \cong\; 0.637\frac{\pi c}{l}$$

The exact solution for the first 3 natural frequencies is $(L = 2l)$:

$$\omega_1 = 0 \vee \omega_2 = 0.5\frac{\pi c}{l} \vee\; \omega_3 = \frac{\pi c}{l}$$

So we see that the frequencies obtained with the consistent mass matrix are above the exact solution and the frequencies for the lumped mass matrix are below.

2.5. Repeat Example 2.6 for spherical symmetry, where

$$\mathbf{D} = \begin{Bmatrix} D_{rr} \\ D_{\theta\theta} \\ D_{\phi\phi} \end{Bmatrix}, \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{\phi\phi} \end{Bmatrix}, D_{rr} = v_{r,r}, D_{\theta\theta} = D_{\phi\phi} = \frac{1}{r} vr$$

Solution of 2.5.

For spherical symmetry,

$$\boldsymbol{D} = \begin{Bmatrix} D_{rr} \\ D_{\theta\theta} \\ D_{\phi\phi} \end{Bmatrix} = \begin{Bmatrix} v_{r,r} \\ \frac{1}{r} v_r \\ \frac{1}{r} v_r \end{Bmatrix}$$

Momentum equation in spherical coordinates, for this problem:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + \rho b_r = \rho\, \dot{v}_r$$

$$\delta P^{int} = \int_0^{2\pi}\int_0^{\pi}\int_{r_1^e}^{r_2^e} (\delta D_r \sigma_r + \delta D_{\theta\theta} + \delta\, D_{\emptyset}\sigma_{\emptyset})\, r^2 \sin\phi \;\, dr d\phi d\theta$$

$$\delta P^{int} = \int_{r_1^e}^{r_2^e} \delta \boldsymbol{D}^T \boldsymbol{\sigma}\, 4\pi r^2 dr$$

Considering a linear velocity field:

$$v(\xi, t) = \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \begin{Bmatrix} v_1(t) \\ v_2(t) \end{Bmatrix}$$