Chapter One

1. See the accompanying MATHEMATICA notebook.

2. $AC\{D,B\} = ACDB + ACBD$, $A\{C,B\}D = ACBD + ABCD$, $C\{D,A\}B = CDAB + CADB$, and $\{C,A\}DB = CADB + ACDB$. Therefore $-AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB = -ACDB + ABCD - CDAB + ACDB = ABCD - CDAB = [AB, CD]$

3. Recall that $\left\langle(\Delta A)^2\right\rangle = \langle A^2\rangle - \langle A\rangle^2$ and $|S_x;+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ (1.4.17a). Since both parts of the problem deal with S_y, recall also that $S_y = \frac{\hbar}{2}[-i|+\rangle\langle-| + i|-\rangle\langle+|]$. Therefore $S_y|S_x;+\rangle = \frac{\hbar}{2}\left[\frac{i}{\sqrt{2}}|-\rangle - \frac{i}{\sqrt{2}}|+\rangle\right]$ and $S_y^2|S_x;+\rangle = \left(\frac{\hbar}{2}\right)^2\left[\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right]$. Thus $\langle S_y^2\rangle = \left(\frac{\hbar}{2}\right)^2$ and $\langle S_y\rangle = 0$, so $\left\langle(\Delta S_y)^2\right\rangle = \left(\frac{\hbar}{2}\right)^2$. This would seem to be a problem because $\left\langle(\Delta S_x)^2\right\rangle = 0$ and the left side of the inequality statement is zero. However $[S_x, S_y] = i\hbar S_z$ and $S_z|S_x;+\rangle = \frac{\hbar}{2}\left[\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right]$ so $\langle S_z\rangle = 0$ and both sides of the uncertainty inequality are zero.

For second part, $S_z^2|S_x;+\rangle = \left(\frac{\hbar}{2}\right)^2\left[\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right]$ so $\langle S_z^2\rangle = \left(\frac{\hbar}{2}\right)^2$ and $\left\langle(\Delta S_z)^2\right\rangle = \left(\frac{\hbar}{2}\right)^2$ and the left side of the uncertainty inequality is $\left(\frac{\hbar}{2}\right)^4$. The right side needs $[S_z, S_y] = -i\hbar S_x$ so is $\frac{1}{4}\hbar^2\left(\frac{\hbar}{2}\right)^2$ and the uncertainty relation is satisfied by the equality.

4. (a) $\mathsf{Tr}(X) = a_0\mathsf{Tr}(1) + \sum_\ell \mathsf{Tr}(\sigma_\ell)a_\ell = 2a_0$ since $\mathsf{Tr}(\sigma_\ell) = 0$. Also $\mathsf{Tr}(\sigma_k X) = a_0\mathsf{Tr}(\sigma_k) + \sum_\ell \mathsf{Tr}(\sigma_k\sigma_\ell)a_\ell = \frac{1}{2}\sum_\ell \mathsf{Tr}(\sigma_k\sigma_\ell + \sigma_\ell\sigma_k)a_\ell = \sum_\ell \delta_{k\ell}\mathsf{Tr}(1)a_\ell = 2a_k$. So, $a_0 = \frac{1}{2}\mathsf{Tr}(X)$ and $a_k = \frac{1}{2}\mathsf{Tr}(\sigma_k X)$. (b) Just do the algebra to find $a_0 = (X_{11} + X_{22})/2$, $a_1 = (X_{12} + X_{21})/2$, $a_2 = i(-X_{21} + X_{12})/2$, and $a_3 = (X_{11} - X_{22})/2$.

5. Since $\det(\boldsymbol{\sigma}\cdot\mathbf{a}) = -a_z^2 - (a_x^2 + a_y^2) = -|\mathbf{a}|^2$, the cognoscenti realize that this problem really has to do with rotation operators. From this result, and (3.2.44), we write

$$\det\left[\exp\left(\pm\frac{i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right)\right] = \cos\left(\frac{\phi}{2}\right) \pm i\sin\left(\frac{\phi}{2}\right)$$

and multiplying out determinants makes it clear that $\det(\boldsymbol{\sigma}\cdot\mathbf{a}') = \det(\boldsymbol{\sigma}\cdot\mathbf{a})$. Similarly, use (3.2.44) to explicitly write out the matrix $\boldsymbol{\sigma}\cdot\mathbf{a}'$ and equate the elements to those of $\boldsymbol{\sigma}\cdot\mathbf{a}$. With $\hat{\mathbf{n}}$ in the z-direction, it is clear that we have just performed a rotation (of the spin vector) through the angle ϕ.

6. (a) $\mathsf{Tr}(XY) \equiv \sum_a\langle a|XY|a\rangle = \sum_a\sum_b\langle a|X|b\rangle\langle b|Y|a\rangle$ by inserting the identity operator. Then commute and reverse, so $\mathsf{Tr}(XY) = \sum_b\sum_a\langle b|Y|a\rangle\langle a|X|b\rangle = \sum_b\langle b|YX|b\rangle = \mathsf{Tr}(YX)$.
(b) $XY|\alpha\rangle = X[Y|\alpha\rangle]$ is dual to $\langle\alpha|(XY)^\dagger$, but $Y|\alpha\rangle \equiv |\beta\rangle$ is dual to $\langle\alpha|Y^\dagger \equiv \langle\beta|$ and $X|\beta\rangle$ is dual to $\langle\beta|X^\dagger$ so that $X[Y|\alpha\rangle]$ is dual to $\langle\alpha|Y^\dagger X^\dagger$. Therefore $(XY)^\dagger = Y^\dagger X^\dagger$.
(c) $\exp[if(A)] = \sum_a \exp[if(A)]|a\rangle\langle a| = \sum_a \exp[if(a)]|a\rangle\langle a|$
(d) $\sum_a \psi_a^*(\mathbf{x}')\psi_a(\mathbf{x}'') = \sum_a\langle\mathbf{x}'|a\rangle^*\langle\mathbf{x}''|a\rangle = \sum_a\langle\mathbf{x}''|a\rangle\langle a|\mathbf{x}'\rangle = \langle\mathbf{x}''|\mathbf{x}'\rangle = \delta(\mathbf{x}'' - \mathbf{x}')$

7. For basis kets $|a_i\rangle$, matrix elements of $X \equiv |\alpha\rangle\langle\beta|$ are $X_{ij} = \langle a_i|\alpha\rangle\langle\beta|a_j\rangle = \langle a_i|\alpha\rangle\langle a_j|\beta\rangle^*$. For spin-1/2 in the $|\pm z\rangle$ basis, $\langle +|S_z = \hbar/2\rangle = 1$, $\langle -|S_z = \hbar/2\rangle = 0$, and, using (1.4.17a), $\langle \pm|S_x = \hbar/2\rangle = 1/\sqrt{2}$. Therefore

$$|S_z = \hbar/2\rangle\langle S_x = \hbar/2| \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

8. $A[|i\rangle + |j\rangle] = a_i|i\rangle + a_j|j\rangle \neq [|i\rangle + |j\rangle]$ so in general it is not an eigenvector, unless $a_i = a_j$. That is, $|i\rangle + |j\rangle$ is not an eigenvector of A unless the eigenvalues are degenerate.

9. Since the product is over a complete set, the operator $\prod_{a'}(A - a')$ will always encounter a state $|a_i\rangle$ such that $a' = a_i$ in which case the result is zero. Hence for any state $|\alpha\rangle$

$$\prod_{a'}(A - a')|\alpha\rangle = \prod_{a'}(A - a')\sum_i |a_i\rangle\langle a_i|\alpha\rangle = \sum_i \prod_{a'}(a_i - a')|a_i\rangle\langle a_i|\alpha\rangle = \sum_i 0 = 0$$

If the product instead is over all $a' \neq a_j$ then the only surviving term in the sum is

$$\prod_{a'}(a_j - a')|a_i\rangle\langle a_i|\alpha\rangle$$

and dividing by the factors $(a_j - a')$ just gives the projection of $|\alpha\rangle$ on the direction $|a'\rangle$. For the operator $A \equiv S_z$ and $\{|a'\rangle\} \equiv \{|+\rangle, |-\rangle\}$, we have

$$\begin{aligned}
\prod_{a'}(A - a') &= \left(S_z - \frac{\hbar}{2}\right)\left(S_z + \frac{\hbar}{2}\right) \\
\text{and} \qquad \prod_{a' \neq a''}\frac{A - a'}{a'' - a'} &= \frac{S_z + \hbar/2}{\hbar} \qquad \text{for } a'' = +\frac{\hbar}{2} \\
\text{or} \qquad &= \frac{S_z - \hbar/2}{-\hbar} \qquad \text{for } a'' = -\frac{\hbar}{2}
\end{aligned}$$

It is trivial to see that the first operator is the null operator. For the second and third, you can work these out explicitly using (1.3.35) and (1.3.36), for example

$$\frac{S_z + \hbar/2}{\hbar} = \frac{1}{\hbar}\left[S_z + \frac{\hbar}{2}1\right] = \frac{1}{2}\left[(|+\rangle\langle+|) - (|-\rangle\langle-|) + (|+\rangle\langle+|) + (|-\rangle\langle-|)\right] = |+\rangle\langle+|$$

which is just the projection operator for the state $|+\rangle$.

10. I don't see any way to do this problem other than by brute force, and neither did the previous solutions manual. So, make use of $\langle+|+\rangle = 1 = \langle-|-\rangle$ and$\langle+|-\rangle = 0 = \langle-|+\rangle$ and carry through six independent calculations of $[S_i, S_j]$ (along with $[S_i, S_j] = -[S_j, S_i]$) and the six for $\{S_i, S_j\}$ (along with $\{S_i, S_j\} = +\{S_j, S_i\}$).

11. From the figure $\hat{\mathbf{n}} = \hat{\mathbf{i}}\cos\alpha\sin\beta + \hat{\mathbf{j}}\sin\alpha\sin\beta + \hat{\mathbf{k}}\cos\beta$ so we need to find the matrix representation of the operator $\mathbf{S}\cdot\hat{\mathbf{n}} = S_x\cos\alpha\sin\beta + S_y\sin\alpha\sin\beta + S_z\cos\beta$. This means we need the matrix representations of S_x, S_y, and S_z. Get these from the prescription (1.3.19) and the operators represented as outer products in (1.4.18) and (1.3.36), along with the association (1.3.39a) to define which element is which. Thus

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We therefore need to find the (normalized) eigenvector for the matrix

$$\begin{pmatrix} \cos\beta & \cos\alpha\sin\beta - i\sin\alpha\sin\beta \\ \cos\alpha\sin\beta + i\sin\alpha\sin\beta & -\cos\beta \end{pmatrix} = \begin{pmatrix} \cos\beta & e^{-i\alpha}\sin\beta \\ e^{i\alpha}\sin\beta & -\cos\beta \end{pmatrix}$$

with eigenvalue $+1$. If the upper and lower elements of the eigenvector are a and b, respectively, then we have the equations $|a|^2 + |b|^2 = 1$ and

$$\begin{aligned} a\cos\beta + be^{-i\alpha}\sin\beta &= a \\ ae^{i\alpha}\sin\beta - b\cos\beta &= b \end{aligned}$$

Choose the phase so that a is real and positive. Work with the first equation. (The two equations should be equivalent, since we picked a valid eigenvalue. You should check.) Then

$$\begin{aligned} a^2(1-\cos\beta)^2 &= |b|^2\sin^2\beta = (1-a^2)\sin^2\beta \\ 4a^2\sin^4(\beta/2) &= (1-a^2)4\sin^2(\beta/2)\cos^2(\beta/2) \\ a^2[\sin^2(\beta/2)+\cos^2(\beta/2)] &= \cos^2(\beta/2) \\ a &= \cos(\beta/2) \\ \text{and so} \qquad b &= ae^{i\alpha}\frac{1-\cos\beta}{\sin\beta} = \cos(\beta/2)e^{i\alpha}\frac{2\sin^2(\beta/2)}{2\sin(\beta/2)\cos(\beta/2)} \\ &= e^{i\alpha}\sin(\beta/2) \end{aligned}$$

which agrees with the answer given in the problem.

12. Use simple matrix techniques for this problem. The matrix representation for H is

$$H \doteq \begin{bmatrix} a & a \\ a & -a \end{bmatrix}$$

Eigenvalues E satisfy $(a-E)(-a-E) - a^2 = -2a^2 + E^2 = 0$ or $E = \pm a\sqrt{2}$. Let x_1 and x_2 be the two elements of the eigenvector. For $E = +a\sqrt{2} \equiv E^{(1)}$, $(1-\sqrt{2})x_1^{(1)} + x_2^{(1)} = 0$, and for $E = -a\sqrt{2} \equiv E^{(2)}$, $(1+\sqrt{2})x_1^{(2)} + x_2^{(2)} = 0$. So the eigenstates are represented by

$$|E^{(1)}\rangle \doteq N^{(1)}\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix} \qquad \text{and} \qquad |E^{(2)}\rangle \doteq N^{(2)}\begin{bmatrix} -1 \\ \sqrt{2}+1 \end{bmatrix}$$

where ${N^{(1)}}^2 = 1/(4-2\sqrt{2})$ and ${N^{(2)}}^2 = 1/(4+2\sqrt{2})$.

13. It is of course possible to solve this using simple matrix techniques. For example, the characteristic equation and eigenvalues are

$$\begin{aligned} 0 &= (H_{11}-\lambda)(H_{22}-\lambda) - H_{12}^2 \\ \lambda &= \frac{H_{11}+H_{22}}{2} \pm \left[\left(\frac{H_{11}-H_{22}}{2}\right)^2 + H_{12}^2\right]^{1/2} \equiv \lambda_\pm \end{aligned}$$

You can go ahead and solve for the eigenvectors, but it is tedious and messy. However, there is a strong hint given that you can make use of spin algebra to solve this problem, another two-state system. The Hamiltonian can be rewritten as

$$H \doteq A\mathsf{1} + B\sigma_z + C\sigma_x$$

where $A \equiv (H_{11}+H_{22})/2$, $B \equiv (H_{11}-H_{22})/2$, and $C \equiv H_{12}$. The eigenvalues of the first term are both A, and the eigenvalues for the sum of the second and third terms are those of $\pm(2/\hbar)$ times a spin vector multiplied by $\sqrt{B^2+C^2}$. In other words, the eigenvalues of the full Hamiltonian are just $A \pm \sqrt{B^2+C^2}$ in full agreement with what we got with usual matrix techniques, above. From the hint (or Problem 9) the eigenvectors must be

$$|\lambda_+\rangle = \cos\frac{\beta}{2}|1\rangle + \sin\frac{\beta}{2}|2\rangle \qquad \text{and} \qquad |\lambda_-\rangle = -\sin\frac{\beta}{2}|1\rangle + \cos\frac{\beta}{2}|2\rangle$$

where $\alpha = 0$, $\tan\beta = C/B = 2H_{12}/(H_{11}-H_{22})$, and we do $\beta \to \pi - \beta$ to "flip the spin."

14. Using the result of Problem 9, the probability of measuring $+\hbar/2$ is

$$\left|\left[\frac{1}{\sqrt{2}}\langle+| + \frac{1}{\sqrt{2}}\langle-|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right]\right|^2 = \frac{1}{2}\left[\sqrt{\frac{1+\cos\gamma}{2}} + \sqrt{\frac{1-\cos\gamma}{2}}\right]^2 = \frac{1+\sin\gamma}{2}$$

The results for $\gamma = 0$ (i.e. $|+\rangle$), $\gamma = \pi/2$ (i.e. $|S_x+\rangle$), and $\gamma = \pi$ (i.e. $|-\rangle$) are 1/2, 1, and 1/2, as expected. Now $\langle(S_x - \langle S_x\rangle)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$, but $S_x^2 = \hbar^2/4$ from Problem 8 and

$$\begin{aligned} \langle S_x\rangle &= \left[\cos\frac{\gamma}{2}\langle+| + \sin\frac{\gamma}{2}\langle-|\right]\frac{\hbar}{2}\left[|+\rangle\langle-| + |-\rangle\langle+|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right] \\ &= \frac{\hbar}{2}\left[\cos\frac{\gamma}{2}\langle-| + \sin\frac{\gamma}{2}\langle+|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right] = \hbar\cos\frac{\gamma}{2}\sin\frac{\gamma}{2} = \frac{\hbar}{2}\sin\gamma \end{aligned}$$

so $\langle(S_x - \langle S_x\rangle)^2\rangle = \hbar^2(1-\sin^2\gamma)/4 = \hbar^2\cos^2\gamma/4 = \hbar^2/4, 0, \hbar^2 4$ for $\gamma = 0, \pi/2, \pi$.

15. All atoms are in the state $|+\rangle$ after emerging from the first apparatus. The second apparatus projects out the state $|S_n+\rangle$. That is, it acts as the projection operator

$$|S_n+\rangle\langle S_n+| = \left[\cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}|-\rangle\right]\left[\cos\frac{\beta}{2}\langle+| + \sin\frac{\beta}{2}\langle-|\right]$$

and the third apparatus projects out $|-\rangle$. Therefore, the probability of measuring $-\hbar/2$ after the third apparatus is

$$P(\beta) = |\langle+|S_n+\rangle\langle S_n+|-\rangle|^2 = \cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2} = \frac{1}{4}\sin^2\beta$$

The maximum transmission is for $\beta = 90°$, when 25% of the atoms make it through.

16. The characteristic equation is $-\lambda^3 - 2(-\lambda)(1/\sqrt{2})^2 = \lambda(1-\lambda^2) = 0$ so the eigenvalues are $\lambda = 0, \pm1$ and there is no degeneracy. The eigenvectors corresponding to these are

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

The matrix algebra is not hard, but I did this with MATLAB using

```
M=[[0 1 0];[1 0 1];[0 1 0]]/sqrt(2)
[V,D]=eig(M)
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These are the eigenvectors corresponding to the a spin-one system, for a measurement in the x-direction in terms of a basis defined in the z-direction. I'm not sure if there is enough information in Chapter One, though, in order to deduce this.

17. The answer is *yes*. The identity operator is $1 = \sum_{a',b'} |a',b'\rangle\langle a',b'|$ so

$$AB = AB1 = AB\sum_{a',b'}|a',b'\rangle\langle a',b'| = A\sum_{a',b'}b'|a',b'\rangle\langle a',b'| = \sum_{a',b'}b'a'|a',b'\rangle\langle a',b'| = BA$$

Completeness is powerful. It is important to note that the sum must be over both a' and b' in order to span the complete set of sets.

18. Since $AB = -BA$ and $AB|a,b\rangle = ab|a,b\rangle = BA|a,b\rangle$, we must have $ab = -ba$ where both a and b are real numbers. This can only be satisfied if $a = 0$ or $b = 0$ or both.

19. Assume there is no degeneracy and look for an inconsistency with our assumptions. If $|n\rangle$ is a nondegenerate energy eigenstate with eigenvalue E_n, then it is the *only* state with this energy. Since $[H.A_1] = 0$, we must have $HA_1|n\rangle = A_1H|n\rangle = E_nA_1|n\rangle$. That is, $A_1|n\rangle$ is an eigenstate of energy with eigenvalue E_n. Since H and A_1 commute, though, they may have simultaneous eigenstates. Therefore, $A_1|n\rangle = a_1|n\rangle$ since there is only one energy eigenstate.

Similarly, $A_2|n\rangle$ is also an eigenstate of energy with eigenvalue E_n, and $A_2|n\rangle = a_2|n\rangle$. But $A_1A_2|n\rangle = a_2A_1|n\rangle = a_2a_1|n\rangle$ and $A_2A_1|n\rangle = a_1a_2|n\rangle$, where a_1 and a_2 are real numbers. This cannot be true, in general, if $A_1A_2 \neq A_2A_1$ so our assumption of "no degeneracy" must be wrong. There is an out, though, if $a_1 = 0$ or $a_2 = 0$, since one operator acts on zero.

The example given is from a "central forces" Hamiltonian. (See Chapter Three.) The Hamiltonian commutes with the orbital angular momentum operators L_x and L_y, but $[L_x, L_y] \neq 0$. Therefore, in general, there is a degeneracy in these problems. The degeneracy is avoided, though for S-states, where the quantum numbers of L_x and L_y are both necessarily zero.

20. The positivity postulate says that $\langle\gamma|\gamma\rangle \geq 0$, and we apply this to $|\gamma\rangle \equiv |\alpha\rangle + \lambda|\beta\rangle$. The text shows how to apply this to prove the Schwarz Innequality $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$, from which one derives the generalized uncertainty relation (1.4.53), namely

$$\langle(\Delta A)^2(\Delta B)^2\rangle \geq \frac{1}{4}\left|\langle[A,B]\rangle\right|^2$$

Note that $[\Delta A, \Delta B] = [A - \langle A\rangle, B - \langle B\rangle] = [A, B]$. Taking $\Delta A|\alpha\rangle = \lambda\Delta B|\alpha\rangle$ with $\lambda^* = -\lambda$, as suggested, so $\langle\alpha|\Delta A = -\lambda\langle\alpha|\Delta B$, for a particular state $|\alpha\rangle$. Then

$$\langle\alpha|[A,B]|\alpha\rangle = \langle\alpha|\Delta A\Delta B - \Delta B\Delta A|\alpha\rangle = -2\lambda\langle\alpha|(\Delta B)^2|\alpha\rangle$$

and the equality is clearly satisfied in (1.4.53). We are now asked to verify this relationship for a state $|\alpha\rangle$ that is a gaussian wave packet when expressed as a wave function $\langle x'|\alpha\rangle$. Use

$$\begin{aligned}
\langle x'|\Delta x|\alpha\rangle &= \langle x'|x|\alpha\rangle - \langle x\rangle\langle x'|\alpha\rangle = (x' - \langle x\rangle)\langle x'|\alpha\rangle \\
\text{and}\qquad \langle x'|\Delta p|\alpha\rangle &= \langle x'|p|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle = \frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle \\
\text{with}\qquad \langle x'|\alpha\rangle &= (2\pi d^2)^{-1/4}\exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x' - \langle x\rangle)^2}{4d^2}\right] \\
\text{to get}\qquad \frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle &= \left[\langle p\rangle - \frac{\hbar}{i}\frac{1}{2d^2}(x' - \langle x\rangle)\right]\langle x'|\alpha\rangle \\
\text{and so}\qquad \langle x'|\Delta p|\alpha\rangle &= i\frac{\hbar}{2d^2}(x' - \langle x\rangle)\langle x'|\alpha\rangle = \lambda\langle x'|\Delta x|\alpha\rangle
\end{aligned}$$

where λ is a purely imaginary number. The conjecture is satisfied.

It is very simple to show that this condition is satisfied for the ground state of the harmonic oscillator. Refer to (2.3.24) and (2.3.25). Clearly $\langle x\rangle = 0 = \langle p\rangle$ for any eigenstate $|n\rangle$, and $x|0\rangle$ is proportional to $p|0\rangle$, with a proportionality constant that is purely imaginary.

21. Have $S_x^2 = \hbar^2/4 = S_y^2 = S_z^2$, also $[S_x, S_y] = i\hbar S_z$, all from Problem 8. Now $\langle S_x\rangle = \langle S_y\rangle = 0$ for the $|+\rangle$ state. Then $\langle(\Delta S_x)^2\rangle = \hbar^2/4 = \langle(\Delta S_y)^2\rangle$, and $\langle(\Delta S_x)^2\langle(\Delta S_y)^2 = \hbar^4/16$. Also $|\langle[S_x, S_y]\rangle|^2/4 = \hbar^2|\langle S_z\rangle|^2/4 = \hbar^4/16$ and the generalized uncertainty principle is satisfied by the equality. On the other hand, for the $|S_x+\rangle$ state, $\langle(\Delta S_x)^2\rangle = 0$ and $\langle S_z\rangle = 0$, and again the generalized uncertainty principle is satisfied with an equality.