1 Introduction

1.1 From Eq. (1.2), the value of γ is infinite if $v = c$, so there is no Lorentz transformation to an inertial frame corresponding to a rest frame for light.

1.2 Since $E = m\gamma$, for a 7 TeV proton,

$$\gamma = \frac{E}{m} = \frac{7 \times 10^{12}\,\text{eV}}{938.3 \times 10^{6}\,\text{eV}} = 7460.$$

Then from the definition of γ,

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = 0.999999991.$$

This is a speed that is only about 3 meters per second less than that of light.

1.3 This question is ambiguous, since it does not specify whether the curvature is that of the surface itself (which is called *intrinsic curvature*) or whether it is the apparent curvature of the surface seen embedded in a higher-dimensional euclidean space (which is called the *extrinsic curvature*). In general relativity the curvature of interest is usually intrinsic curvature. Then the sheet of paper can be laid out flat and is not curved, the cylinder is *also flat,* with no intrinsic curvature, because one can imagine cutting it longitudinally and rolling it out into a flat surface, but the sphere has finite intrinsic curvature because it cannot be cut and rolled out flat without distortion. The reason that the cylinder seems to be curved is because the 2D surface is being viewed embedded in 3D space, which gives a non-zero *extrinsic curvature,* but if attention is confined only to the 2D surface it has no *intrinsic curvature*. This is a rather qualitative discussion but in later chapters methods will be developed to quantify the amount of intrinsic curvature for a surface.

2 Coordinate Systems and Transformations

2.1 Utilizing Eq. (2.31) to integrate around the circumference of the circle,

$$C = \oint ds = \oint (dx^2 + dy^2)^{1/2} = 2\int_{-R}^{+R} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

subject to the constraint $R^2 = x^2 + y^2$, where the factor of two and the limits are because x ranges from $-R$ to $+R$ over half a circle. The constraints yield $dy/dx = -(R^2 - x^2)^{-1/2}x$, which permits the integral to be written as

$$C = 2\int_{-R}^{R} dx \sqrt{\frac{R^2}{R^2 - x^2}}.$$

Introducing a new integration variable a through $a \equiv x/R$ then gives

$$C = 2R\int_{-1}^{+1} \frac{da}{\sqrt{1 - a^2}} = 2\pi R,$$

since the integral is $\sin^{-1} a$. In plane polar coordinates the line element is given by Eq. (2.32) and proceeding as above the circumference is

$$\begin{aligned} C &= \oint ds = \oint (dr^2 + r^2 d\varphi^2)^{1/2} \\ &= \int_0^{2\pi} d\varphi \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2} = R\int_0^{2\pi} d\varphi = 2\pi R, \end{aligned}$$

where $r = R$ has been used, implying that $dr/d\varphi = 0$.

2.2 Under a Galilean transformation $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t$ and $t' = t$ it is clear that the acceleration $\boldsymbol{a}$ and the separation vector $\boldsymbol{r} = \Delta\boldsymbol{x}$ between two masses are unchanged. Thus the second law $\boldsymbol{F} = m\boldsymbol{a}$ and the gravitational law $\boldsymbol{F} = Gm_1m_2\hat{\mathbf{r}}/r^2$ are invariant under Galilean transformations.

2.3 Our solution follows Example 1.2.1 of Foster and Nightingale [88]. The tangent and dual basis vectors, and the products for $g_{ij} = g_{ji} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$, were worked out in Example 2.3. The elements for $g^{ij} = g^{ji} = \boldsymbol{e}^i \cdot \boldsymbol{e}^j$ can be obtained in a similar fashion. For example,

$$g^{12} = g^{21} = (\tfrac{1}{2}\boldsymbol{i} + \tfrac{1}{2}\boldsymbol{j}) \cdot (\tfrac{1}{2}\boldsymbol{i} - \tfrac{1}{2}\boldsymbol{j}) = \tfrac{1}{4} - \tfrac{1}{4} = 0,$$

where the orthonormality of the cartesian basis vectors has been used. Summarizing the results,

$$g_{ij} = \begin{pmatrix} 4v^2 + 2 & 4uv & 2v \\ 4uv & 4u^2 + 2 & 2u \\ 2v & 2u & 1 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} \frac{1}{2} & 0 & -v \\ 0 & \frac{1}{2} & -u \\ -v & -u & 2u^2 + 2v^2 + 1 \end{pmatrix}$$

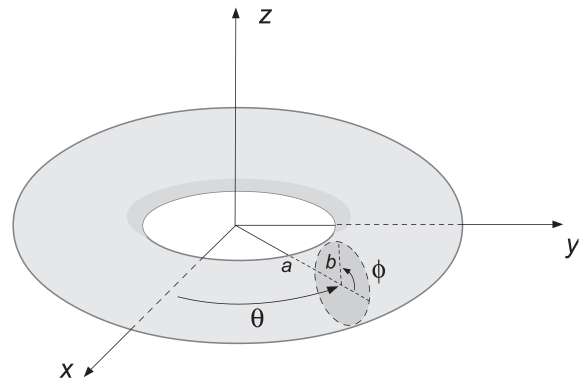

Fig. 2.1 Figure for Problem 2.5.

By direct multiplication the product of these two matrices is the unit matrix, verifying Eq. (2.26) explicitly for this case. Utilizing Eq. (2.29), the line element is

$$
\begin{aligned}
ds^2 &= g_{ij} du^i du^j \\
&= g_{uu} du^2 + 2g_{uv} du dv + 2g_{uw} du dw + g_{vv} dv^2 + 2g_{vw} dv dw + g_{ww} dw^2 \\
&= (4v^2+2) du^2 + 8uv du dv + 4v du dw + (4u^2+2) dv^2 + 4u dv dw + dw^2
\end{aligned}
$$

where $g_{ij} = g_{ji}$ has been used and no summation is implied by repeated indices.

2.4 Using the spherical coordinates

$$u^1 = r \qquad u^2 = \theta \qquad u^3 = \varphi$$

defined through Eq. (2.2) and the results of Example 2.2,

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = 1 \qquad \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = r^2 \qquad \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = r^2 \sin^2 \theta,$$

while all non-diagonal components vanish. Thus the metric tensor is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}.$$

The corresponding line element is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2,$$

where Eq. (2.29) has been used.

2.5 This solution is based on Problem 1.2 in Ref. [88]. From the parameterization $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ with

$$x = (a + b\cos\varphi)\cos\theta \qquad y = (a + b\cos\varphi)\sin\theta \qquad z = b\sin\varphi,$$

where the radius of the doughnut a and radius of the circle b are defined in Fig. 2.1 [this document], the tangent basis vectors are

$$\boldsymbol{e}_\theta = \frac{\partial \boldsymbol{r}}{\partial \theta} = -\sin\theta(a+b\cos\varphi)\,\boldsymbol{i} + (a+b\cos\varphi)\cos\theta\,\boldsymbol{j}$$
$$\boldsymbol{e}_\varphi = \frac{\partial \boldsymbol{r}}{\partial \varphi} = -(b\sin\varphi\cos\theta)\,\boldsymbol{i} - (b\sin\varphi\sin\theta)\,\boldsymbol{j} + (b\cos\varphi)\,\boldsymbol{k}.$$

The corresponding elements of the metric tensor $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ are

$$g_{\varphi\varphi} = b^2 \qquad g_{\varphi\theta} = g_{\theta\varphi} = 0 \qquad g_{\theta\theta} = (a+b\cos\varphi)^2.$$

2.6 The tangent basis vectors and metric tensor g_{ij} were given in Example 2.4. Since g^{ij} is the matrix inverse of g_{ij}, which is diagonal,

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad \longrightarrow \qquad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Then the dual basis may be obtained by raising indices with the metric tensor: $\boldsymbol{e}^i = g^{ij}\boldsymbol{e}_j$, giving

$$\boldsymbol{e}^1 = g^{11}\boldsymbol{e}_1 + g^{12}\boldsymbol{e}_2 = \boldsymbol{e}_1 \qquad \boldsymbol{e}^2 = g^{21}\boldsymbol{e}_1 + g^{22}\boldsymbol{e}_2 = \frac{1}{r^2}\boldsymbol{e}_2$$

for the elements of the dual basis.

2.7 For a constant displacement d in the x direction

$$x' = x - d \qquad y' = y \qquad z' = z.$$

Since d is constant

$$dx' = dx \qquad dy' = dy \qquad dz' = dz$$

and therefore $ds'^2 = ds^2$. From Eq. (2.41), a rotation in the $x-y$ plane may be written

$$x' = x\cos\theta + y\sin\theta \qquad y' = -x\sin\theta + y\cos\theta \qquad z' = z,$$

which gives the transformed line element

$$\begin{aligned} ds'^2 &= (dx')^2 + (dy')^2 + (dz')^2 \\ &= (\cos\theta dx + \sin\theta dy)^2 + (-\sin\theta dx + \cos\theta dy)^2 + dz^2 \\ &= (\cos^2\theta + \sin^2\theta)dx^2 + (\cos^2\theta + \sin^2\theta)dy^2 + dz^2 \\ &= dx^2 + dy^2 + dz^2 \\ &= ds^2. \end{aligned}$$

Therefore the euclidean spatial line element is invariant under displacements by a constant amount and under rotations.

2.8 Taking the scalar products using Eqs. (2.8), (2.9), and (2.20) gives

$$\boldsymbol{e}^i \cdot \boldsymbol{V} = \boldsymbol{e}^i \cdot (V^j \boldsymbol{e}_j) = V^j \boldsymbol{e}^i \cdot \boldsymbol{e}_j = V^j \delta^i_j = V^i,$$
$$\boldsymbol{e}_i \cdot \boldsymbol{V} = \boldsymbol{e}_i \cdot (V_j \boldsymbol{e}^j) = V_j \boldsymbol{e}_i \cdot \boldsymbol{e}^j = V_j \delta^j_i = V_i,$$

which is Eq. (2.22).

2.9 Utilizing that the angle θ between the basis vectors is determined by $\cos\theta = \boldsymbol{e}_1 \cdot \boldsymbol{e}_2/|\boldsymbol{e}_1||\boldsymbol{e}_2|$, the area of the parallelogram is

$$\begin{aligned} dA &= |\boldsymbol{e}_1||\boldsymbol{e}_2| \sin\theta \, dx^1 dx^2 \\ &= |\boldsymbol{e}_1||\boldsymbol{e}_2|(1-\cos^2\theta)^{1/2} dx^1 dx^2 \\ &= \left(|\boldsymbol{e}_1|^2|\boldsymbol{e}_2|^2 - (\boldsymbol{e}_1\cdot\boldsymbol{e}_2)^2\right)^{1/2} dx^1 dx^2. \end{aligned}$$

The components of the metric tensor g_{ij} are

$$\boldsymbol{e}_1\cdot\boldsymbol{e}_2 = g_{12} = g_{21} \qquad |\boldsymbol{e}_1||\boldsymbol{e}_1| = \boldsymbol{e}_1\cdot\boldsymbol{e}_1 = g_{11} \qquad |\boldsymbol{e}_2||\boldsymbol{e}_2| = \boldsymbol{e}_2\cdot\boldsymbol{e}_2 = g_{22},$$

so the area of the parallelogram may be expressed as

$$dA = \left(g_{11}g_{22} - g_{12}^2\right)^{1/2} dx^1 dx^2 = \sqrt{\det g}\, dx^1 dx^2,$$

where $\det g$ is the determinant of the metric tensor. This is the 2D version of the invariant 4D volume element given in Eq. (3.48).

3 Tensors and Covariance

3.1 For the three cases

$$T'^{\mu\nu} = V'^{\mu}V'^{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}V^{\alpha}V^{\beta} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}T^{\alpha\beta}$$

$$T'_{\mu\nu} = V'_{\mu}V'_{\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}V_{\alpha}V_{\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}T_{\alpha\beta}$$

$$T'^{\nu}_{\mu} = V'_{\mu}V'^{\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}V_{\alpha}V^{\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}T_{\alpha}{}^{\beta}.$$

3.2 From Eqs. (3.50) and (3.51) with indices suitably relabeled

$$\begin{aligned}
A'_{\mu,\nu} - \Gamma'^{\lambda}_{\mu\nu}A'_{\lambda} &= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} + A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&\quad - \left(\Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\kappa}} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\right)\frac{\partial x^{\gamma}}{\partial x'^{\lambda}}A_{\gamma} \\
&= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} + A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&\quad - \Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\kappa}}\frac{\partial x^{\gamma}}{\partial x'^{\lambda}}A_{\gamma} - \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\frac{\partial x^{\gamma}}{\partial x'^{\lambda}}A_{\gamma} \\
&= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} + A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&\quad - \Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}A_{\kappa} - A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} - \Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}A_{\kappa} \\
&= \left(A_{\alpha,\beta} - \Gamma^{\kappa}_{\alpha\beta}A_{\kappa}\right)\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}},
\end{aligned}$$

which is Eq. (3.52).

3.3 (a) Since δ^{ν}_{μ} is a rank-2 tensor with the same components in all coordinate systems (see Section 3.8), under a coordinate transformation $g_{\mu\alpha}g^{\alpha\nu} = \delta^{\nu}_{\mu}$ becomes $g'_{\mu\alpha}g'^{\alpha\nu} = \delta^{\nu}_{\mu}$. Since $g_{\mu\nu}$ is a tensor, if we assume $g^{\mu\nu}$ is also a tensor then

$$g'_{\mu\alpha} = \frac{\partial x^{\kappa}}{\partial x'^{\mu}}\frac{\partial x^{\eta}}{\partial x'^{\alpha}}g_{\kappa\eta}. \qquad g'^{\alpha\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}g^{\rho\sigma}.$$

Then evaluating $g'_{\mu\alpha}g'^{\alpha\nu}$,

$$g'_{\mu\alpha}\frac{\partial x'^{\alpha}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}g^{\rho\sigma} = \frac{\partial x^{\kappa}}{\partial x'^{\mu}}\frac{\partial x^{\eta}}{\partial x'^{\alpha}}g_{\kappa\eta}\frac{\partial x'^{\alpha}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}g^{\rho\sigma} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \delta^{\nu}_{\mu},$$