1 Newtonian Particle Mechanics

1.1 Problems and Solutions

★ **Problem 1.1** A meterstick is at rest in a primed frame of reference, with one end at the origin and the other at $x' = 1.0$ m. (a) Using the Galilean transformation find the location of each end of the stick in the unprimed frame at a particular time t, and then find the length of the meter stick in the unprimed frame. (b) Repeat for the case that the stick is laid out along the positive y' axis, with one end at the origin and the other at $y' = 1.0$ m. What is the length of the stick in the unprimed frame?

Solution

(a) $x = x' + vt' = x' + vt$, so the left end has $x_l = 0 + vt = vt$, and the right end has $x_r = 1.0\text{ m} + vt$. Therefore the length $= x_r - x_l = 1.0$ m.

(b) $x' = 0$ for both ends in this case, and $y' = 0$ and $y' = 1.0$ m always. Therefore $x = x' + vt = vt$ for both ends, and $y = y' = 0$ and 1.0 m for the two ends, so the length in the unprimed frame is $\Delta y = \Delta y' = 1.0$ m. ■

★ **Problem 1.2** A river of width D flows uniformly at speed V relative to the shore. A swimmer swims always at speed $2V$ relative to the water. (a) If the swimmer dives in from one shore and swims in a direction perpendicular to the shoreline in the reference frame of the flowing river, how long does it take her to reach the opposite shore, and how far downstream has she been swept relative to the shore? (b) If instead she wants to swim to a point on the opposite shore directly across from her starting point, at what angle should she swim relative to the direction of the river flow, and how long would it take her to swim across?

Solution

(a) Her velocity perpendicular to the shoreline is $2V$, so the time to reach the opposite shore is $t = \frac{D}{2V}$. During this time, she is also swept downstream a distance $d = Vt = \frac{D}{2}$.

(b) She must have an upstream component of velocity V to make up for the river flow. From the Pythagorean theorem, her velocity component across the river is $\sqrt{(2V)^2 - V^2} = \sqrt{3}V$ and the angle

$$\theta = \tan^{-1}\frac{V}{\sqrt{3}V} = \sin^{-1}\frac{V}{2V} = \sin^{-1}\frac{1}{2} = 30°$$

Therefore her angle relative to the flow direction is $30° + 90° = 120°$. Her time to swim across is $t = \frac{D}{\sqrt{3}V}$. ■

★ **Problem 1.3** The crews of two eight-man sculls decide to race one another on a river of width D that flows at uniform velocity V_0. The crew of scull A rows downstream a distance D and then back upstream, while the crew of scull B rows to a point on the opposite shore directly across from the starting point, and then back to the starting point. They begin simultaneously, and each crew rows at the same speed V relative to the water, with $V > V_0$. Who wins the race, and by how much time?

Solution

A: Relative to the shore, A has velocity $V_0 + V$ downstream and $V - V_0$ upstream. The time spent downstream is $\frac{D}{V_0+V}$ and upstream $\frac{D}{V-V_0}$, so the total time for A is

$$\frac{D}{V_0 + V} + \frac{D}{V - V_0} = \frac{D(V - V_0 + V + V_0)}{V^2 - V_0^2} = \frac{2DV}{V^2 - V_0^2} = t_A$$

B: The velocity of B relative to the shore is $\sqrt{V^2 - V_0^2}$, so the total time across the stream and back for B is $t_B = \frac{2D}{\sqrt{V^2-V_0^2}}$. Therefore

$$t_A - t_B = 2D\left[\frac{V}{V^2 - V_0^2} - \frac{1}{\sqrt{V^2 - V_0^2}}\right] = \frac{2D}{V^2 - V_0^2}\left[V - \sqrt{V^2 - V_0^2}\right] > 0.$$

So B wins the race by $\Delta t = \frac{2D(V-\sqrt{V^2-V_0^2})}{V^2-V_0^2}$. ■

★ **Problem 1.4** Passengers standing in a coasting spaceship observe a distant star at the zenith, *i.e.*, directly overhead. If the spaceship then accelerates to speed $c/100$ where c is the speed of light, at what angle to the zenith (to three significant figures) do the passengers now see the star?

Solution

Note that $\sin\theta = \frac{c/100}{c} = \frac{1}{100} \simeq \theta$ ($\sin\theta \simeq \theta$ for $\theta \ll 1$). Alternatively, perhaps the hypotenuse *should* be $\sqrt{c^2 + \frac{c^2}{10^4}}$, so $\tan\theta = \frac{c/100}{c} \simeq \theta$, so $\theta \simeq \frac{1}{100}$, the same either way to three significant figures. ■

★★ **Problem 1.5** (a) Snow is falling vertically toward the ground at speed v. (a) A bus driver is driving through the snowstorm on a horizontal road at speed $v/3$. At what angle to the vertical are the snowflakes falling as seen by the driver? (b) Suppose that the large windshield in the flat, vertical front of the bus has been knocked out, leaving a hole of area A in the vertical plane. Given that N is the number of falling snowflakes per unit horizontal area per unit time, if the bus moves at constant speed $v/3$ to reach a destination at distance d, how many snowflakes fall into the bus before the destination is reached? (c) To minimize the total number of snowflakes that fall in, the driver considers driving faster or slower. What would be the best speed to take?

Solution

(a) From the point of view of the ground, snowflakes fall straight down at speed v, so from the point of view of the bus the snowflakes fall at an angle of $\theta = \tan^{-1}\frac{V/3}{V} = \tan^{-1}(\frac{1}{3})$.

(b) In a time t the volume swept into the bus is $V = A(\frac{V}{3}t) = Ad$, so the number of snowflakes entering is NAd, regardless of speed.

(c) The speed of the bus doesn't matter. If the bus has higher velocity, more snowflakes come in per unit time, but the time to travel the distance d is less. The number of snowflakes entering the bus is the same whether the bus moves fast or slow. ■

★★ **Problem 1.6** The jet stream is flowing due east at velocity v_J relative to the ground. An aircraft is traveling at velocity v_C in the northeast direction relative to the air. (a) Relative to the ground, find the speed of the aircraft and the angle of its motion relative to the east. (b) Keeping the same speed v_C relative to the air, at what angle would the plane have to move through the air relative to the east so that it would travel northeast relative to the ground?

Solution

(a) Note that

$$v_{\text{net,horizontal}} = v_J + v_C\cos 45^\circ \quad \text{and} \quad v_{\text{net,vertical}} = v_C \sin 45^\circ$$

$$\text{Therefore, since } \cos 45^\circ = \sin 45^\circ = 1/\sqrt{2},$$

it follows that

$$\vec{v}_{net} = \left(v_J + v_C/\sqrt{2}\right)\hat{x} + (v_C/\sqrt{2})\hat{y}$$

($\hat{x} + \hat{y}$ are unit vectors).

$$\theta = \tan^{-1}\frac{v_C/\sqrt{2}}{v_J + v_C/\sqrt{2}}$$

$$v_{net} = \sqrt{(v_J + v_C/\sqrt{2})^2 + (v_0/\sqrt{2})^2}$$

(b) Note that

$$v_{\text{net}}\sin 45^\circ = v_C \sin(\pi - \theta) = v_C \sin\theta.$$

Also

$$v_J = v_{net}\cos 45 + v_C\cos(\pi - \theta) = v_{\text{net}}\cos 45 - v_C\cos\theta$$

$$v_J = v_c\sin\theta - v_C\cos\theta = v_C(\sin\theta - \cos\theta) = v_C\sqrt{2}\sin(\theta - (\pi/4))$$

since

$$\sin(\theta - (\pi/4)) = \sin\theta\cos(\pi/4) - \cos\theta\sin(\pi/4) = \frac{1}{\sqrt{2}}(\sin\theta - \cos\theta).$$

Thus

$$\sin(\theta - (\pi/4)) = \frac{v_J}{\sqrt{2}v_C}$$

$$\theta - (\pi/4) = \sin^{-1}\frac{v_J}{\sqrt{2}v_C}$$

$$\theta = \frac{\pi}{4} + \sin^{-1}\frac{v_J}{\sqrt{2}v_C}.$$

For example, suppose $v_J = v_C \cos 45 = \sqrt{2}v_C$; then

$$\theta = \frac{\pi}{4} + \sin^{-1}1 = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4},$$

which is correct. ■

⋆ **Problem 1.7** The earth orbits the sun once/year in a nearly circular orbit of radius 150×10^6 km. The speed of light is $c = 3 \times 10^5$ km/s. Looking through a telescope, we observe that a particular star is directly overhead. If the earth were quickly stopped and made to move in the opposite direction at the same speed, at what angle to the vertical would the same star now be observed?

Solution

The speed of the earth's orbit is found from

$$F = ma: \quad \frac{-GM_{\text{sun}}m}{r_e^2} = -\frac{mv_e^2}{r_e}$$

$$\Rightarrow v_e = \sqrt{\frac{GM_{\text{sun}}}{r_e}}$$

Here $G = 6.67 \times 10^{-11}\frac{m^3}{\text{kg s}^2}$, $M_{\text{sun}} = 2.0 \times 10^{30}$ kg, $r_e = 1.5 \times 10^{11}$ m. Therefore

$$v_e = \sqrt{\frac{(2/3)\cdot 10^{-10}\cdot 2\cdot 10^{30}}{1.5\cdot 10^{11}}} = \sqrt{8.89 \times 10^8} \text{ m/s} = 2.98 \times 10^4 \text{ m/s}$$

The speed of light is $c = 3 \times 10^8$ m/s, so since $v_e/c \ll 1$, $\theta \simeq \frac{v_e}{c} = \frac{2.98\times10^4}{3.0\times10^8} \approx 10^{-4}$ radians.

Moving in the original direction the star (apparently overhead) is actually 10^{-4} radians in the forward direction. So if the earth were moving in the opposite direction the star would appear to be $\theta \simeq 2 \times 10^{-4}$ radians from the vertical. ■

★ **Problem 1.8** A long chain is tied tightly between two trees and a horizontal force F_0 is applied at right angles to the chain at its midpoint. The chain comes to equilibrium so that each half of the chain is at angle θ from the straight line between the chain endpoints. Neglecting gravity, what is the tension in the chain?

Solution

Balancing forces perpendicular to the chain, $F_0 = 2T\sin\theta \Rightarrow T = F_0/2\sin\theta$, where T is the tension. ■

★★ **Problem 1.9** An object of mass m is subject to a drag force $F = -kv^n$, where v is its velocity in the medium, and k and n are constants. If the object begins with velocity v_0 at time $t = 0$, find its subsequent velocity as a function of time.

Solution

$F = -kv^n = mdv/dt$ by Newton's Second Law. Therefore

$$\int_0^t dt = -\frac{m}{k}\int_{v_0}^{v} v^{-n}dv \Rightarrow t = -\left(\frac{m}{k}\right)\frac{v^{-n+1}}{-n+1}\Big|_{v_0}^{v}$$

$$-\frac{kt}{m} = \frac{v^{-n+1} - v_0^{-n+1}}{-n+1} \Rightarrow v^{-n+1} = v_0^{-n+1} - \frac{kt}{m}(-n+1)$$

$$\Rightarrow v \equiv v^{\frac{-n+1}{-n+1}} = \left[v_0^{-n+1} - (-n+1)\frac{kt}{m}\right]^{\frac{1}{-n+1}}.$$

■

★★ **Problem 1.10** A small spherical ball of mass m and radius R is dropped from rest into a liquid of high viscosity η, such as honey, tar, or molasses. The only appreciable forces on it are gravity mg and a linear drag force given by Stokes's law, $F_{\text{Stokes}} = -6\pi\eta R v$, where v is the ball's velocity, and the minus sign indicates that the drag force is opposite to the direction of v. (a) Find the velocity of the ball as a function of time. Then show that your answer makes sense for (b) small times; (c) large times.

Solution

Let $\alpha = 6\pi\eta R$, so $F_{\text{Stokes}} = -\alpha v$. Then $F_{net} = mg - \alpha v = ma = mdv/dt$.

(a) It follows from $F = m\frac{dv}{dt}$ that $dt = \frac{m\,dv}{mg-\alpha v}$, where v is positive downward. Then

$$t = \int dt = \int_0^v \frac{m\,dv}{mg - \alpha v}.$$

Let $u \equiv mg - \alpha v$, so $du = -\alpha dv$. Therefore

$$t = \int_{mg}^{mg-\alpha v} \frac{m(1/-\alpha)du}{u} = -\frac{n}{\alpha}\ln u\Big|_{mg}^{mg-\alpha v} = \frac{m}{\alpha}\ln\left(\frac{mg}{mg-\alpha v}\right).$$

Therefore $e^{\frac{\alpha t}{m}} = \frac{mg}{mg-\alpha v}$, so $(mg - \alpha v) = mge^{-\alpha t/m}$. Then

$$\alpha v = mg\left[1 - e^{-\alpha t/m}\right] \quad \text{so} \quad v(t) = \left(\frac{mg}{\alpha}\right)(1 - e^{-\alpha t/m}).$$

(b) For small times $e^{-\alpha t/m} \cong 1 - \frac{\alpha t}{m} = 1 - (\frac{6\pi\eta R}{m})t$. (series expansion $e^x = 1 + x + x^2/2! + \cdots$). Therefore

$$v(t) \cong \frac{mg}{\alpha}\left(1 - \left(1 - \frac{6\pi\eta R}{m}t\right)\right) + \cdots \cong \frac{mg}{6\pi\eta R}\left(\frac{6\pi\eta R}{m}t\right) = gt,$$

which is correct, because for very small times the drag force is negligible.

(c) For large times $e^{-\alpha t/m} \to 0$, so $v(t) \cong mg/6\pi\eta R$, $mg = 6\pi\eta R v$. In this case the forces balance, with no additional acceleration. The ball is approaching its terminal velocity. ■

⋆⋆⋆ **Problem 1.11** We showed in Example 1.2 that the distance a ball falls as a function of time, starting from rest and subject to both gravity g downward and a quadratic drag force upward, is

$$y = (v_T^2/g)\ln(\cosh(gt/v_T)),$$

where v_T is its terminal velocity. (a) Invert this equation to find how long it takes the ball to reach the ground in terms of its initial height h. (b) Check your result in the limits of *small* h and *large* h. (For part (b) it is useful to know the infinite series expansions of the functions e^x, $(1 + x)^n$, and $\ln(1 + x)$ for small x.)

Solution

(a) From the given equation, it follows that $\frac{gy}{v_T^2} = \ln(\cosh(gt/v_T))$, so

$$\cosh\frac{gt}{v_T} = e^{gy/v_T^2} \equiv \frac{e^{gt/v_T} + e^{-gt/v_T}}{2}$$

Multiply by e^{gt/v_T}: $(e^{gt/v_T})^2 - 2e^{gy/v_T}(e^{gt/v_T}) + 1 = 0$, which is a quadratic equation in e^{gt/v_T}, with solutions

$$e^{gt/v_T} = e^{\frac{gy}{v_T^2}}\left[1 \pm \sqrt{1 - e^{-2gh/v_T^2}}\right]$$

using the quadratic equation, and where now h is the initial height and t is the time to reach the ground. Which sign is correct?

Note that as $h \to \infty$, $e^{-2gh/v_T^2} \to 0$ and $(1 - e^{-2gh/v_T^2})^{1/2} \to 1 - \frac{1}{2}e^{-2gh/v_T^2}$ by the binomial approximation. So with the lower sign,

$$e^{gt/v_T} \simeq e^{gh/v_T^2}\left[\frac{1}{2}e^{-2gh/v_T^2}\right] = \frac{1}{2}e^{-gh\sqrt{v_T^2}}$$

which is incorrect, because it implies that t decreases as h increases. So using the upper sign,

$$e^{gt/v_T} = e^{gh/v_T^2}\left[1 + \sqrt{1 - e^{-2gh/v_T^2}}\right].$$

Take the natural log of both sides, giving

$$\frac{gt}{v_T} = \frac{gh}{v_T^2} + \ln\left[1 + \sqrt{1 - e^{-2gh/v_T^2}}\right]$$

so

$$t = \frac{h}{v_T} + \frac{v_T}{g}\ln\left[1 + \sqrt{1 - e^{-2gh/v_T^2}}\right].$$

(b) Check the result:
For small h, $e^{-2gh/v_T^2} \simeq 1 - (2gh/v_T^2)$, since $e^x = 1 + x + x^2/2! + \cdots$ for small x, and so

$$\sqrt{1 - e^{-2gh/v_T^2}} \simeq \sqrt{2gh/v_T^2}$$

Therefore

$$\ln\left[1 + \sqrt{2gh/v_T^2}\right] \simeq \sqrt{2gh/v_T^2}$$

since $\ln(1 + x) \simeq x$ for $x \ll 1$. Thus

$$t \simeq \frac{h}{v_T} + \frac{v_T}{g}\sqrt{2gh/v_T^2} \simeq \sqrt{\frac{2h}{g}}$$

for small h. Therefore $t = \sqrt{\frac{2h}{g}}$, uniformly accelerated motion for small times, valid before the drag force becomes appreciable. For large h,

$$(1 - e^{-2gh/v_T^2})^{1/2} \simeq 1 - \frac{1}{2}e^{-2gh/v_T^2}$$

so

$$t \simeq \frac{h}{v_T} + \frac{v_T}{g}\ln 2 \simeq \frac{h}{v_T}$$

which is also correct, since then most of the trip is essentially at the terminal velocity v_T. ■

⋆ **Problem 1.12** For objects with linear size between a few millimeters and a few meters moving through air near the ground, and with speed less than a few hundred meters per second, the drag force is close to a quadratic function of velocity, $F_D = (1/2)C_D A\rho v^2$, where ρ is the mass density of air near the ground, A is the cross-sectional area of the object, and C_D is the drag coefficient, which depends upon the shape of the object. A rule of thumb is that in air near the ground (where $\rho = 1.2\ \text{kg/m}^3$), then $F_D \simeq \frac{1}{4}Av^2$. (a) Estimate the terminal velocity v_T of a skydiver of mass m and cross-sectional area A. (b) Find v_T for a skydiver with $A = 0.75\ \text{m}^2$ and mass 75 kg. (The result is large, but a few lucky people have survived a fall without a parachute. An example is 21-year old Nicholas Alkemade, a British Royal Air Force tail gunner during World War II. On March 24, 1944 his plane caught fire over Germany and his parachute was destroyed. He had the choice of burning to death or jumping out. He jumped and fell about 6 km, slowed at the end by falling though pine trees and landing in soft snow, ending up with nothing but a sprained leg. He was captured by the Gestapo, who at first did not believe his story, but when they found his plane they changed their minds. He was imprisoned, and at the end of the war set free, with a certificate signed by the Germans corroborating his story.)

Solution

At terminal velocity, $F_D \simeq \frac{1}{4}Av^2 \simeq mg$ in SI units, so $v_T \simeq \sqrt{\frac{4mg}{A}}$. Then $v_T \cong \sqrt{\frac{4mg}{A}} = 2\sqrt{\frac{(75)(9.8)}{0.75}} = 62.6$ m/s $\cong 225$ km/hr $\cong 140$ mi/hr. ■

★ **Problem 1.13** A damped oscillator consists of a mass m attached to a spring k, with frictional damping forces. If the mass is released from rest with amplitude A, and after 100 oscillations the amplitude is $A/2$, what is the total work done by friction during the 100 oscillations?

Solution

We can simply see how much energy is lost. The initial amplitude is A, so the initial energy is all potential energy $\frac{1}{2}kA^2$. After 100 oscillations the amplitude is $A/2$, so the energy is $\frac{1}{2}k(A/2)^2 = \frac{1}{8}kA^2$. The energy lost is $\frac{1}{2}kA^2 - \frac{1}{8}kA^2 = \frac{3}{8}kA^2$, so the work done by friction according to the work-energy theorem is $-\frac{3}{8}kA^2$. ■

★ **Problem 1.14** The solution of the underdamped harmonic oscillator is $x(t) = Ae^{-\beta t}\cos(\omega_1 t + \varphi)$, where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. Find the arbitrary constants A and φ in terms of the initial position x_0 and initial velocity v_0.

Solution

Given $x(t) = Ae^{-\beta t}\cos(\omega_1 t + \varphi)$, where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ and A and ϕ are arbitrary constants that can be found in terms of the initial conditions $x(0) = x_0$ and $v(0) = v_0$. So $x_0 = A\cos\phi$ and $v_0 = A[-\beta\cos\phi - \omega_1\sin\phi]$. Note $\sin\phi = \sqrt{1-\cos\phi^2}$ (using the plus sign), $\sin\phi = \sqrt{1-(x_0/A)^2}$. Therefore

$$v_0 = A\left[-\beta x_0/A - \omega_1\sqrt{1-(x_0/A)^2}\right] = -\beta x_0 - \omega_1\sqrt{A^2 - x_0^2}$$

It follows that

$$A^2 - x_0^2 = \frac{(v_0 + \beta x_0)^2}{\omega_1^2} \quad \text{so} \quad A = \frac{\sqrt{v_0^2 + 2\beta x_0 v_0 + x_0^2(\beta^2 + \omega_1^2)}}{\omega_1}$$

and

$$\phi = \sin^{-1}\left[1 - (\frac{x_0}{A})^2\right] = \sin^{-1}\left[1 - \frac{(x_0\omega_1)^2}{v_0^2 + 2\beta x_0 v_0 + x_0^2(\beta^2 + \omega_1^2)}\right]$$

$$\phi = \sin^{-1}\left[\frac{v_0^2 + 2\beta x_0 v_0 + x_0^2\beta^2}{v_0^2 + 2\beta x_0 v_0 + x_0^2(\beta^2 + \omega_1^2)}\right].$$

■

★★ **Problem 1.15** An overdamped oscillator is released at location $x = x_0$ with initial velocity v_0. What is the maximum number of times the oscillator can subsequently pass through $x = 0$?

Solution

The overdamped solution is

$$x(t) = A_1 e^{\gamma_1 t} + A_2 e^{\gamma_2 t} ,$$

where

$$\gamma_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

with $\beta > \omega_0$.

$$v(t) = x(t) = A_1 \gamma_1 e^{\gamma_1 t} + A_2 \gamma_2 e^{\gamma_2 t} .$$

At $t = 0$, $x = x_0$ and $x(0) = v(0) = v_0$. Therefore $x_0 = A_1 + A_2$ and

$$v_0 = A_1(-\beta + \sqrt{\beta^2 - \omega_0^2}) + A_2(-\beta - \sqrt{\beta^2 - \omega_0^2}) .$$

Eliminate A_2, using $A_2 = x_0 - A_1$, so

$$v_0 = A_1(-\beta + \sqrt{\beta^2 - \omega_0^2}) + (x_0 - A_1)(-\beta - \sqrt{\beta^2 - \omega_0^2}) = x_0 \left[-\beta - \sqrt{\beta^2 - \omega_0^2}\right]$$
$$+ A_1 \left[-\beta + \sqrt{\beta^2 - \omega_0^2} + \beta + \sqrt{\beta^2 - \omega_0^2}\right] = -x_0(\beta + \sqrt{\beta^2 - \omega_0^2}) + A_1 2\sqrt{\beta^2 - \omega_0^2} .$$

Therefore

$$A_1 = \frac{v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{2\sqrt{\beta^2 - \omega_0^2}}$$

$$A_2 = x_0 - A_1 = \frac{x_0 2\sqrt{\beta^2 - \omega_0^2} - v_0 - x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{2\sqrt{\beta^2 - \omega_0^2}}.$$

$$A_2 = \frac{x_0\sqrt{\beta^2 - \omega_0^2} - v_0 - \beta x_0}{2\sqrt{\beta^2 - \omega_0^2}} = \frac{-v_0 + x_0(\sqrt{\beta^2 - \omega_0^2} - \beta)}{2\sqrt{\beta^2 - \omega_0^2}}$$

$$x(t) = \frac{v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{2\sqrt{\beta^2 - \omega_0^2}} e^{\gamma_1 t} + \frac{-v_0 - x_0(\beta - \sqrt{\beta^2 - \omega_0^2})}{2\sqrt{\beta^2 - \omega_0^2}} e^{\gamma_2 t}.$$

$$x(t) = \frac{\left[v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})\right] e^{\gamma_1 t} - \left[v_0 + x_0(\beta - \sqrt{\beta^2 - \omega_0^2})\right] e^{\gamma_2 t}}{2\sqrt{\beta^2 - \omega_0^2}} .$$

Without loss of generality, we can assume $x_0 > 0$. Then if the mass reaches $x = 0$ we must have

$$\left[v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})\right] e^{\gamma_1 t} = \left[v_0 + x_0(\beta - \sqrt{\beta^2 - \omega_0^2})\right] e^{\gamma_2 t}$$

or

$$e^{(\gamma_2 - \gamma_1)t} = e^{-2\sqrt{\beta^2 - \omega_0^2}t} = \frac{v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{v_0 + x_0(\beta - \sqrt{\beta^2 - \omega_0^2})}\,.$$

Now for $t > 0$,

$$e^{-2\sqrt{\beta^2 - \omega_0^2}t} < 1$$

But for $x_0 > 0, \beta > 0$, this is only possible if $v_0 < 0$, in fact, $v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0^2}) < 0$. So also $v_0 + x_0(\beta - \sqrt{\beta^2 - \omega_0^2}) < 0$ as well.

$$e^{-2\sqrt{\beta^2 - \omega_0^2}t} = \frac{-|v_0| + x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{-|v_0| + x_0(\beta - \sqrt{\beta^2 - \omega_0^2})} = \frac{|v_0| - x_0(\beta + \sqrt{\beta^2 - \omega_0^2})}{|v_0| - x_0(\beta - \sqrt{\beta^2 - \omega_0^2})} < 1\,.$$

(Note $v_0 < 0$). There can be only a single time t_0 when the mass passes through $x = 0$. Plotting $x(t)$ for a strongly negative v_0 shows that $x(t)$ can pass from positive to negative values one time, but then approaches $x = 0$ asymptotically from below. ■

⋆ **Problem 1.16** There are thought to be three types of the particles called *neutrinos*: electron-type (ν_e), muon type (ν_μ), and tau-type (ν_τ). If they were all massless they could not spontaneously convert from one type into a different type. But if there is a mass difference between two types, call them types ν_1 and ν_2, the probability that a neutrino starting out as a ν_1 becomes a ν_2 is given by the oscillating probability $P = S_{12}\sin^2(L/\lambda)$, where S_{12} is called the *mixing strength parameter*, which we take to be constant, L is the distance traveled by the neutrino, and λ is a characteristic length, given in kilometers by

$$\lambda = \frac{E}{1.27\Delta(m)^2}$$

where E is the energy of the neutrino in units of GeV (1 GeV = 10^9 eV) and $\Delta(m)^2$ is the difference in the *squares* of the two masses in units $(\text{eV})^2$). Neutrinos are formed in earth's atmosphere by the collision of cosmic-ray protons from outer space with atomic nuclei in the atmosphere. The giant detector *Super Kamiokande*, located deep underground in a mine west of Tokyo, saw equal numbers of electron-type neutrinos coming (1) from the atmosphere above the detector (2) from the atmosphere on the other side of the earth, which pass through our planet on their way to the detector. However, Super K saw more muon-type neutrinos coming down from above than those coming up from above. This was strong evidence that muon-type neutrinos oscillated into tau-type neutrinos (which Super

K could not detect) as they penetrated the earth, since it requires more time to go 13,000 km through the earth than 20 km through the atmosphere above the mine. (a) Suppose $(\Delta m)^2 = 0.01\ \text{eV}^2$ between ν_μ and ν_τ type neutrinos, and that the neutrino energy is $E = 5$ GeV. What is λ? How would this explain the fewer number of muon neutrinos seen from below than from above? (b) The best experimental fit is $(\Delta m)^2 = 0.0022\ \text{eV}^2$. Again assuming $E = 5$ GeV, what is λ? Make a crude estimate of the ratio one might expect for the number of muon neutrinos from below and from above.

Solution

(a) $\lambda = \frac{E}{1.27\Delta(m)^2} = \frac{5}{1.27(0.01)} = 394$ km. Therefore since the atmosphere has a thickness of only about 20 km, few of the muon-type neutrinos would have had time to convert to τ-type neutrinos, but there would have been several oscillations coming through the 13, 000 km of the earth.

(b) $\lambda = 5/[1.27(0.0022)] = 1790$ km, so very few of the muon neutrinos coming through the atmosphere only will convert. In penetrating the earth the probability of conversion is approximately

$$P = S_{12}\sin^2(L/\lambda) = S_{12}\sin^2\frac{13,000}{1790} = S_{12}\sin^2 7.26 \simeq 0.687 S_{12}$$

so the probability of remaining a muon-type neutriino is $S_{12} \cdot 100\%$ if penetrating the atmosphere only, and $S_{12} \cdot 31.3\%$ if penetrating the earth. So a crude estimate of the number of muon neutrinos from below compared with the number of muon neutrinos from above is roughly 0.31. This is very rough, because some neutrinos will pass through only a portion of the earth. ■

★★ **Problem 1.17** The "quality factor" Q of an underdamped oscillator can be defined as

$$Q = 2\pi\frac{E}{|\Delta E|}$$

where at some time E is the total energy of the oscillator and $|\Delta E|$ is the energy loss in one cycle. (a) Show that $Q \simeq \pi/\beta P$, where β is the damping constant and P is the period of oscillation. Therefore if the damping increases, Q decreases. (b) What is Q for a simple pendulum that loses 1% of its energy during each cycle? (c) The quality factor also describes the sharpness of the resonance curve of a driven, lightly-damped oscillator. Show that to a good approximation $Q \simeq \omega/(\Delta\omega)$, where $\Delta\omega$ is the angular frequency difference between the two locations on the amplitude resonance curve for which the amplitude is $1/\sqrt{2}$ that at peak resonance.

Solution

(a) The oscillator follows the solution $x(t) = Ae^{-\beta t}\cos(\omega t + \varphi)$ where the energy of the oscillator is proportional to x^2, so $E \propto A^2 e^{-2\beta t}$. One cycle corresponds to a period of $P = 2\pi/\omega$, so

$$\Delta E = A^2 e^{-2\beta t} - A^2 e^{-2\beta(t+P)} = A^2 e^{-2\beta t}(1 - e^{-2\beta P})$$
$$\simeq A^2 e^{-2\beta t}\left[1 - (1 - 2\beta P)\right] = A^2 e^{-2\beta t}(2\beta P)$$

(using $e^x = 1 + x + x^2/2! + \cdots$). Thus

$$Q = 2\pi \frac{E}{|\Delta E|} = \frac{2\pi A^2 e^{-2\beta t}}{A^2 e^{-2\beta t}(2\beta P)} = \frac{\pi}{\beta P}$$

(b) $Q = 2\pi E/0.01E \cong 628$

(c) The resonance curve is (for the amplitude of oscillation):

$$C(\omega) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}\,.$$

The resonance peak is at

$$C_R = \frac{f_0}{2\beta\omega_1} \simeq \frac{f_0}{2\beta\omega_0}$$

for light damping. Suppose

$$C(\omega) = \frac{1}{\sqrt{2}} C_R = \frac{1}{\sqrt{2}}\left(\frac{f_0}{2\beta\omega_0}\right) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}\,.$$

Again, $4\beta^2\omega^2 = 4\beta^2\omega_0^2$ for a narrow resonance curve, so

$$\sqrt{2}(2\beta\omega_0) \simeq \sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \quad \text{so} \quad (\omega_0^2 - \omega^2)^2 \simeq 4\beta^2\omega_0^2.$$

Therefore

$$\omega = \omega_0\left[1 \pm \frac{2\beta}{\omega_0}\right]^{1/2} \Rightarrow \omega_+ = \omega_0\left(1 + \frac{\beta}{\omega_0}\right) \quad \text{and} \quad \omega_- = \omega_0\left(1 - \frac{\beta}{\omega_0}\right)$$

by the binomial approximation. Therefore, the difference is

$$\Delta\omega \equiv \omega_+ - \omega_- \equiv 2\beta.$$

Thus

$$\frac{\omega}{\Delta\omega} \cong \frac{\omega_0}{2\beta} = \frac{(2\pi/P)}{2\beta}\,.$$

$$\frac{\omega}{\Delta\omega} = \frac{\pi}{\beta P} = Q\,.$$

■

⋆ **Problem 1.18** Consider the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{r}}$, and $\hat{\boldsymbol{\theta}}$ in a plane. (a) Find $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in terms of any or all of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, x, and y.(b) Find $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in terms of any or all of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, r, and θ.

Solution

By drawing a picture in the x, y plane, it is easy to show that (a)

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}} \tag{1.1}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}} \tag{1.2}$$

where $\sin\theta = y/r = y/\sqrt{x^2+y^2}$ and $\cos\theta = x/\sqrt{x^2+y^2}$. (b) Multiply 6.1 by $\cos\theta$ and 6.1 by $\sin\theta$;

$$\hat{\mathbf{r}}\cos\theta = \cos^2\theta\hat{\mathbf{x}} + \sin\theta\cos\theta\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\theta}}\sin\theta = -\sin^2\theta\hat{\mathbf{x}} + \sin\theta\cos\theta\hat{\mathbf{y}}$$

Subtract these equations: $\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta = \hat{\mathbf{x}}$, $\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\theta - \hat{\theta}\sin\theta$.

Then multiply 6.1 by $\sin\theta$, 6.1 by $\cos\theta$.

$$\hat{\mathbf{r}}\sin\theta = \sin\theta\cos\theta\hat{\mathbf{x}} + \sin^2\theta\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\theta}}\cos\theta = -\sin\theta\cos\theta\hat{\mathbf{x}} + \cos^2\theta\hat{\mathbf{y}}$$

add these to find $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$. ■

⋆ **Problem 1.19** The mass and mean radius of the moon are $m = 7.35 \times 10^{22}$ kg and $R = 1.74 \times 10^6$ m. (a) From these parameters, along with Newton's constant of gravity $G = 6.674 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}, find the moon's escape velocity in m/s. (b) For a slingshot boom of length 50 m, what must be the minimum rotation frequency ω to sling material off the moon, as described in Example 1.3? Take into account both the radial and tangential components of the payload velocity when it comes off the end of the boom. Assume payloads are initially set upon the boom at radius $r = 3$ meters and with $\dot{r} = 0$.

Solution

(a) At escape velocity $E = \frac{1}{2}mv_{esc}^2 - GMm/r = 0$, so

$$v_{esc} = \sqrt{\frac{2GM}{r}} = 2.37 \times 10^3 \text{m/s} = 2.37 \text{ km/s}$$

(b) $r = r_0 \cosh\omega t$ as shown in the chapter, so $\cosh\omega t = r/r_0 = 50\text{ m}/3\text{ m} = 16.7$. The velocity of the payload is

$$\begin{aligned}\mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = r_0\omega(\sinh\omega t\hat{\mathbf{r}} + \cosh\omega t\hat{\boldsymbol{\theta}}) \Rightarrow v^2 = \mathbf{v}\cdot\mathbf{v} \\ &= r_0^2\omega^2(\sinh^2\omega t + \cosh^2\omega t) = r_0^2\omega^2(2\cosh^2\omega t - 1)\end{aligned}$$

(since $\cosh^2 - \sinh^2 = 1$). Therefore,

$$\omega = \frac{v_{esc}/r_0}{\sqrt{2\cosh^2\omega t - 1}} = \frac{(2.37 \times 10^3 \text{ m/s})/3 \text{ m}}{\sqrt{2(16.7)^2 - 1}} = \frac{0.79 \text{ s}^{-1} \times 10^3}{23.6} = 33.5 \text{ s}^{-1},$$

so is swinging around very very fast for a 50 m boom. ■

⋆ **Problem 1.20** Ninety percent of the initial mass of a rocket is in the form of fuel. If the rocket starts from rest and then moves in gravity-free empty space, find its final velocity v if the speed u of its exhaust is (a) 3.0 km/s (typical chemical burning), (b) 1000 km/s, (c) $c/10$, where c is the speed of light. (d) If the exhaust velocity is 3.0 km/s, for how long can the rocket maintain the acceleration $a = 10$ m/s^2?

Solution

The rocket equation is $v = v_0 + u\ln\frac{m_0}{m} = 0 + u\ln\frac{m_0}{0.1m_0} = u\ln 10 = 2.30u$.

(a) $v = 2.30(3.0 \text{ km/s}) = 6.9$ km/s.

(b) $v = 2.30(1000 \text{ km/s}) = 2300$ km/s.

(c) $v = 2.30(3 \times 10^7 \text{ m/s}) = 6.9 \times 10^4$ km/s.

(d) $a = \frac{dv}{dt} = u\frac{d}{dt}(\ln m_0 - \ln m) = -um^{-1}\frac{dm}{dt}$. $10 \text{ m/s}^2 = -(3.0\text{km/s})\frac{1}{m}(\frac{dm}{dt})$. $\frac{1}{m}\frac{dm}{dt} = -\frac{10 \text{ m/s}^2}{3000 \text{ m/s}} = -\frac{1}{3}10^{-2} \text{ s}^{-1}$ =constant. so

$$\int \frac{dm}{m} = -\frac{1}{300}\text{s}^{-1}t \Rightarrow \int_{m_0}^{m} \frac{dm}{m} = \ln\frac{m}{m_0} = -\frac{t}{300}\text{ s}^{-1} \Rightarrow$$
$$t = 300\ln\frac{m_0}{m}s = 300\ln\frac{m_0}{0.1m_0}\text{s} = 300\ln 10 \text{ s} = 690 \text{ s} = 11.5 \text{ minutes.}$$

■

⋆ **Problem 1.21** A space traveler pushes off from his coasting spaceship with relative speed v_0; he and his spacesuit together have mass M, and he is carrying a wrench of mass m. Twenty minutes later he decides to return, but his thruster doesn't work. In another forty minutes his oxygen supply will run out, so he immediately throws the wrench away from the ship direction at speed v_w relative to himself prior to the throw. (a) What then is his speed relative to the ship? (b) In terms of given parameters, what is the minimum value of v_w required so he will return in time?

Solution

(a) Conserving momentum of the traveler and wrench,

$$(M+m)v_0 = Mv_f + m(v_0 + v_w) \Rightarrow Mv_f = (M+m)v_0 - m(v_0+v_w)$$
$$= Mv_0 - mv_w \Rightarrow v_f = \frac{Mv_0 - mv_w}{M}$$

(> 0 if he is moving away from the ship). His velocity must be $-\frac{v_0}{2}$ to make it back in 40 minutes, since he has twice as long.

(b) $v_f = -\frac{v_0}{2} = v_0 - \frac{m}{M}v_w \Rightarrow v_w = (\frac{M}{m})\frac{3}{2}v_0$. ■

⋆⋆ **Problem 1.22** An astronaut of mass M, initially at rest in some inertial frame in gravity-free empty space, holds n wrenches, each of mass $M/2n$. (a) Calculate her recoil velocity v_1 if she throws all the wrenches at once in the same direction with speed u relative to her original inertial frame. (b) Find her final velocity v_2 if she first throws half of the wrenches with speed u relative to her original inertial frame, and then the other half with speed u relative to the frame she reached after the first throw. Compare v_2 with v_1 from part (a).

(c) Then find her total recoil velocity v_n if she throws all n wrenches, one at a time and in the same direction, and each with speed u relative to her instantaneous inertial frame just before she throws it. (d) Find her total recoil velocity in the limit $n \to \infty$, and compare with the rocket equation.

Solution

(a) $v_{recoil} = u/2$

(b) Throw half the wrenches: $v_{\text{recoil}} = u/5$. Throw the second half: $v_{\text{recoil}} = u/4$. Total recoil velocity $\frac{u}{5} + \frac{u}{4} = \frac{9}{20}u$.

(c) Throw 1/3 at a time: first throw gives $v_{\text{recoil}} = u/8$; second throw $v_{\text{recoil}} = u/7$; third throw $v_{\text{recoil}} = u/6$. So throwing $1/3$ at a time gives a total recoil

$$u\left[\frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] = u(\frac{78}{168}).$$

One throw: $\frac{u}{2}$; two: $\frac{u}{4} + \frac{u}{5}$; three: $\frac{u}{6} + \frac{u}{7} + \frac{u}{8}$; four: $\frac{u}{8} + \frac{u}{9} + \frac{u}{10} + \frac{u}{11}$; five: $\frac{u}{10} + \frac{u}{11} + \frac{u}{12} + \frac{u}{13} + \frac{u}{14}$.

(c) In general,

$$(\frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} + \cdots \frac{1}{2n+n-1})u = \sum_{n=0}^{n-1} \frac{dk}{2n+k} \to$$

let $x = 2n + k, dx = dk$.

$$u\int_{2n}^{3n-1} \frac{dx}{x} = u \ln \frac{3n-1}{2n} \to \left(\ln \frac{3}{2}\right) u$$

as $n \to \infty$. The rocket equation gives

$$v = u \ln \frac{m_0}{m} = u \ln \frac{3/2\ m}{m} = u \ln \frac{3}{2}$$

which agrees in the limit $n \to \infty$. ■

★★ **Problem 1.23** We are planning to travel in a rocket for 6 months with acceleration 10 m/s^2, and with a final payload mass 1000 tonnes (1 tonne = 1000 kg). (a) Using a chemically-fueled rocket with exhaust speed 3160 m/s, what must be the original ship mass m_0? Compare m_0 with the mass of the observed universe. (Including so-called "dark matter", the mass density is approximately 6×10^{-30} g/cm^3 and the observed radius is of order 10^{10} light years.) (b) Redo part (a) if instead we use a fuel that can be ejected at 3.16×10^7 m/s, about 10 percent the speed of light. (c) How fast would this ship be moving at the end of 6 months? (d) How far will the ship have gone by this time? Compare this distance with the distance to the star Alpha Centauri, about 4 light-years away.

Solution

The rocket equation is $v = u\ln(m_0/m)$, so the acceleration of the rocket is

$$a = dv/dt = u\frac{d/dt(m_0/m)}{m_0/m} = \frac{um_0(-\frac{dm/dt}{m^2})}{m_0/m} = -\frac{u}{m}\frac{dm}{dt} = 10 \text{ m/s}^2.$$

Therefore

$$\left|\frac{dm}{dt}\right| = \frac{a\,dt}{u} \quad \text{resulting in } m = m_0 e^{-at/u}, \quad \text{where} \quad = 10 \text{ m/s}^2.$$

(a) If $u = 3160$ m/s and $t = \frac{1}{2}$ year $= \frac{1}{2}(3.16 \times 10^7 \text{s})$, then $m/m_0 = e^{-10\cdot\frac{1}{2}\cdot 3/16\times 10^7/3.16\times 10^3} = e^{-6\times 10^4} \cdot \log_{10}(m/m_0) = -5 \times 10^4 \log_{10} e$.

Then $m = 10^6$ kg so $6 = \log m_0 - 5 \times 10^4$, and so $\log m_0 = 6 + 5 \times 10^4(0.434) = 6 + 21700 = 21706$. This gives $m_0 \simeq 10^{21{,}700}$ kg.

The mass of the observed universe is

$$\begin{aligned}
&\sim \frac{4}{3}\pi R^3 \rho \sim \frac{4}{3}\pi (10^{10} c \text{ yrs})^3 (6 \times 10^{-30} \text{ g/cm}^3) \\
&\sim \frac{4\pi}{3}(10^{10}\ 3 \times 10^8 \text{ m/s } 3.16 \times 10^7 \text{ s})^3\ 6 \times 10^{-30} \text{ g/cm}^3 \frac{\text{kg}}{1000\text{g}} \left(\frac{100 \text{ cm}}{1 \text{ m}}\right)^3 \\
&\sim 4(9.5 \times 10^{25} \text{ m})^3\ 6 \times 10^{-30} \text{ kg/m}^3 \times 10^3 \\
&\sim 20{,}000 \times 10^{48} \text{ kg} \sim 2 \times 10^{52} \text{ kg}
\end{aligned}$$

The mass of the ship would be hypothetically much much larger.

(b) If instead $u = 3.16 \times 10^7$ m/s, then

$$m/m_0 = e^{-(10/2)\frac{3.16\times 10^7}{3.16\times 10^7}} = e^{-5}$$

$\log_{10}(m/m_0) = -5 \log_{10} e$.

$$m = 10^6\text{kg} \Rightarrow 6 = \log m_0 - 5(.484), \quad \log m_0 = 8.17, \quad m_0 = 10^{8.17}\text{kg},$$

which is more reasonable.

(c) At the end of six months

$$\begin{aligned}
v &= u \ln(m_0/m) = 3.16 \times 10^7 \text{ m/s} \ln\left[\frac{10^{8.17}}{10^6}\right] = 3.16 \times 10^7 \text{ m/s} \ln 10^{2.17} \\
&= (3.16)(2.17) \ln(10) 10^7 \text{ m/s} = (6.86)(2.30) \times 10^7 \text{ m/s} = 15.8 \times 10^7 \text{ m/s} \\
&= 1.58 \times 10^8 \text{ m/s}
\end{aligned}$$

about half the speed of light. (This is a relativistic speed, so it would be prudent to redo the problem using equations for relativistic rockets. See Chapter 2 problems.)

(d) At uniform acceleration $d = \frac{1}{2}at^2 = \frac{1}{2}(10 \text{ m/s}^2)(\frac{3.16\times 10^7 \text{ s}}{2})^2 = 12.5 \times 10^{14}$ m $= 1.25 \times 10^{15}$ m. One light-year $= 3 \times 10^8$ m/s $\cdot\ 3.16 \times 10^7$ s $\cong 9.5 \times 10^{15}$ m, so

$$1.25 \times 10^{15} \text{ m} = 1.25 \times 10^{15} \text{ m}\left(\frac{1c \cdot yr}{9.5 \times 10^{15} \text{ m}}\right) \cong 0.13 \text{ light-year}$$

So in 6 months, the ship would get only a small fraction of the distance to α Centauri. ■

⋆⋆⋆ **Problem 1.24** A single-stage rocket rises vertically from its launchpad by burning liquid fuel in its combustion chamber; the gases escape with a net momentum downward, while the rocket, in reaction, accelerates upward. The gravitational field is g. (a) Pretending that air resistance is negligible, show that the rocket's equation of motion is

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - mg$$

where m is the instantaneous mass of the rocket at time t, v is its upward velocity, and u is the speed of the exhaust relative to the rocket. (b) Assume that g and u remain constant while the fuel is burning, and that fuel is burned at a constant rate $|dm/dt| = \alpha$. Integrate the rocket equation to find $v(m)$. (c) Suppose that $u = 4.4$ km/s and that all the fuel is burned up in one minute. If the rocket achieves the escape velocity from earth of 11.2 km/s, what percentage of the original launchpad mass was fuel?

Solution

(a) At time t the rocket has mass m and is moving vertically upward at velocity v. At time $t + \Delta t$ the rocket has mass $m + \Delta m$ (with $\Delta m < 0$), and is moving upward at velocity $v + \Delta v$. There is also a bit of exhaust $-\Delta m \equiv |\Delta m|$ moving downward with velocity $u - v$. From Newton's second law, the change in total momentum is $\Delta p = p(t + \Delta t) - p(t) = F\Delta t = -mg\Delta t$, where the positive direction is upward. Here

$$\Delta p = (m + \Delta m)(v + \Delta v) - |\Delta m|(u - v) - mv.$$

Cancelling some terms and neglecting the second-order product $\Delta m\, \Delta v$, we find $m\Delta v = -\Delta m u - mg\Delta t$. Dividing by Δt and taking the limit $\Delta t \to 0$, we find the differential equation given in the problem statement.

(b) Given that $dm/dt = -\alpha$, where α is a positive constant, it follows that $m = m_0 - \alpha t$. Also, using the chain rule,

$$\frac{dv}{dt} = \frac{u}{m}\alpha - g = \frac{dv}{dm}\frac{dm}{dt} = -\alpha\frac{dv}{dm}$$

Dividing by $-\alpha$ and integrating over m, we find the velocity as a function of mass during fuel burning,

$$v = v_0 + u\ln\frac{m_0}{m} - \frac{g}{\alpha}(m_0 - m).$$

(c) Alternatively, we can write the velocity as a function of time during fuel burning,

$$v = v_0 + u\ln\frac{m_0}{m_0 - \alpha t} - gt.$$

Here $\alpha = m_{fuel}/60$ seconds. We find $\ln(1 - m_{fuel}/m_0) = -(v + gt)/u = -2.68$, from which we find that the initial percentage in fuel is 93.2%. ■

★★ **Problem 1.25** A rocket in gravity-free empty space has fueled mass M_0 and exhaust velocity u equal to that of a first-stage Saturn V rocket (as used in sending men to the moon): $M_0 = 3100$ tons $= 28 \times 10^6$ kg and $u = 2500$ m/s. The ship's acceleration is kept constant at 10 m/s^2. (a) Find the initial rate of fuel ejection $|dM/dt|_{t=0}$. (b) After how many minutes will the ship mass be reduced to $1/e$ of its initial value? (c) Suppose the ship accelerates as described for 20 minutes. What percent of its initial mass is left? How many kilograms is this? What is the ship's velocity at this time?

Solution

(a) The rocket equation is $v = u\ln\frac{m_0}{m}$, so for constant acceleration we have

$$a = \frac{dv}{dt} = u\frac{d}{dt}(\ln m_0 - \ln m) = -u\frac{dm/dt}{m} = u\frac{|dm/dt|}{m}$$

Therefore

$$|dm/dt|_0 = \frac{am_0}{u} = \frac{10\ \text{m/s}^2 \times 28 \times 10^6\ \text{kg}}{2500\ \text{m/s}} = 1.1 \times 10^5\ \text{kg/s}.$$

(b)

$$a = u\frac{|dm/dt|}{m} = -u\frac{dm/dt}{m}, \quad \text{so} \quad -\int \frac{adt}{u} = \int \frac{dm}{m}.$$

Therefore

$$m = m_0 e^{-\frac{a}{u}t}, \quad \frac{gt}{u} = 1 \Rightarrow t = \frac{u}{g} = \frac{2500\ \text{m/s}}{90\ \text{m/s}^2} = 250\ \text{s} = 4.17\ \text{minutes}.$$

(c)

$$m = m_0 e^{-\frac{g}{u}(20\ \text{min})}, \left(\frac{m}{m_0}\right) = e^{-\frac{20\ \text{min}}{4.17\ \text{min}}} = e^{-4.80} = 8.23 \times 10^{-3},$$

so 0.832% of the mass is left.

$$m = .00823\left[23 \times 10^8\text{kg}\right] = 0.23 \times 10^6\text{kg} = 230,000\ \text{kg} = 230\ \text{tons}$$

$$v = 2500\ \text{m/s}\ \ln\left(\frac{3100}{230}\right) = 2500\ \text{m/s} \cdot \ln(13.5) = 6500\ \text{m/s} = 6.50\ \text{km/s}$$

■

★★ **Problem 1.26** Beginning at time $t = 0$, astronauts in a landing module are descending toward the surface of an airless moon with a downward initial velocity $-|v_0|$ and altitude $y = h$ above the surface. The gravitational field g is essentially constant throughout this descent. An onboard retrorocket can provide a fixed downward exhaust velocity u. The astronauts need to select a fixed exhaust rate $\lambda = |dm/dt|$ in order to provide a soft landing with velocity $v = 0$ when they reach the surface at $y = 0$. (a) Explain briefly why Newton's second law for the module during its descent has the form

$$m(t)\frac{dv}{dt} = u\left|\frac{dm}{dt}\right| - m(t)g$$

(b) Find the velocity v of the module as a function of time, in terms of $|v_0|, u, m_0, \lambda$, and g. (c) During the descent its velocity is $v = dy/dt$, negative because it is downward. Find an expression for $y(t)$ in terms of $|v_0|, g, u, \lambda, m_0$, and h.

Solution

(a) The thrust $u|dm/dt|$ behaves like an upward force, while the gravitational force mg is downward. So Newton's second law becomes $ma = mdv/dt = u|dm/dt| - mg$ where a is positive upward.

(b) Given that $\frac{dm}{dt} = -\lambda$, where λ is a positive constant, we have $\frac{dv}{dt} = \frac{u}{m}\lambda - g = \frac{dv}{dm}\frac{dm}{d4t}$ by the chain rule, so $-\lambda\frac{dv}{dm} = \lambda\frac{u}{m} - g$. Divide by $(-\lambda)$ and integrate over m:

$$\int_{v_0}^{v} dv = -u\int_{m_0}^{m}\frac{dm}{m} + \frac{g}{\lambda}\int_{m_0}^{m} dm$$

so

$$v = v_0 - gt + u\ln\left(\frac{m_0}{m_0 - \lambda t}\right)$$

where $v_0 = -|v_0|$.

(c) Integrating once again, starting at $y = h$,

$$y - h = \int_0^t v\,dt = -|v_0|t - \frac{1}{2}gt^2 - u\int_0^t dt\,\ln\left(\frac{m_0 - \lambda t}{m_0}\right)$$

Let $q \equiv \frac{m_0 - \lambda t}{m_0}$, so then

$$\int dt\,\ln q = -(m_0/\lambda)\int dq\,\ln q = -\frac{m_0}{\lambda}[q(\ln q - 1)]$$

so

$$\begin{aligned} y &= h - |v_0|t - \frac{1}{2}gt^2 + \frac{u}{\lambda}\left[(m_0 - \lambda t)\left(\ln\frac{m_0 - \lambda t}{m_0} - 1\right) + m_0\right] \\ &= h - (|v_0| - u)t - \frac{1}{2}gt^2 + \frac{u}{\lambda}(m_0 - \lambda t)\ln\left(\frac{m_0 - \lambda t}{m_0}\right). \end{aligned}$$

■

★★★ **Problem 1.27** A spaceprobe of mass M is propelled by light fired continuously from a bank of lasers on the moon. A mirror covers the rear of the probe; light from the lasers strikes the mirrors and bounces directly back. In the rest-frame of the lasers, n_γ photons are fired per second, each with momentum $p_\gamma = h\nu_\gamma/c$, where h is Planck's constant, c is the speed of light, and ν is the photon's frequency. (a) Show that in a short time interval Δt the change in the probe's momentum is $2n'_\gamma p'_\gamma \Delta t$, where n'_γ is the number of photons striking the mirror per second, and p'_γ is the momentum of each photon, both in the *probe's* frame of reference. (b) The photons are Doppler-shifted in the probe's frame, so their frequency is only $\nu' \approx \nu(1-v/c)$, where v is the velocity of the probe. Show also that $n'_\gamma = n_\gamma(1-v/c)$, and then show that the ship's acceleration has the form $a = \alpha(1 - v/c)^2$ where α is a constant. Express α in terms of M, n_γ, and p_γ. (c) Find an expression for the probe's velocity as a function of time. Briefly discuss the nature of this result as the probe travels faster and faster.

Solution

(a) The change in momentum of one photon in the instantaneous rest-frame of the probe is $2p'_\gamma$, so that is also the change in the probe's momentum for each photon. During a short time interval Δt the number of photons striking the probe is $n'_\gamma\Delta t$, so the overall change of momentum of the probe is $2p'_\gamma(n'_\gamma\Delta t)$.

(b) The frequency of a photon in the probe's frame is $\nu' \approx \nu(1 - v/c)$ where ν is the frequency in the moon's frame. Therefore $p'_\gamma = p_\gamma(1 - v/c)$ since for each photon there is a Doppler shift. Also $n'_\gamma = n_\gamma(1 - v/c)$, where n_γ is the number per second in the moon's frame, and n'_γ is the number per second in the probe's frame. This can be seen by picturing a tube of radiation which, in the frame of the moon, has a length of one light-second. This radiation is directed towards the right, aimed at the probe. Then all the photons in this tube will pass through the right end of the tube within a time of one second. The probe is also moving toward the right, so in one second it will move a distance $v\times$ 1 second. Therefore there are some photons in the tube that will not be able to reach the probe in 1 second: namely, those within a length $v\times$ 1 second at the left end of the tube, which comprise a fraction v/c of all the photons in the tube. Those reaching the probe in 1 second are therefore a fraction $(1 - v/c)$ of the total.

Now the overall change of momentum of the probe in time Δt is $\frac{\Delta P}{\Delta t} = M\frac{\Delta v}{\Delta t} = Ma$, so its acceleration is

$$\frac{\Delta P/\Delta t}{M} = \frac{2n'_\gamma p'_\gamma}{M} = \frac{2n_\gamma p_\gamma}{M}(1 - v/c)^2 \equiv \alpha(1 - v/c)^2$$

where $\alpha = 2n_\gamma p_\gamma/M$.

(c)

$$\frac{dv}{dt} = \alpha(1 - v/c)^2 \Rightarrow \int \frac{dv}{(1 - v/c)^2} = \alpha \int dt\,.$$

Let

$$u \equiv 1 - v/c \Rightarrow du = -dv/c, -\int \frac{du\, c}{u^2} = \alpha t\,,$$

$$t = -\frac{c}{\alpha}\int \frac{du}{u^2} = \frac{c}{\alpha u}\Big|_{u_0}^{u}, \frac{\alpha t}{c} = \frac{1}{u} - \frac{1}{u_0}$$
$$= \frac{1}{1 - v/c} - 1 \Rightarrow 1 - v/c = (1 + \frac{\alpha t}{c})^{-1} \Rightarrow v/c = 1 - \frac{1}{1 + \alpha t/c}\,.$$

$$\frac{v}{c} = \frac{\alpha t/c}{1 + \alpha t/c}\,.$$

At first the probe accelerates quickly, with acceleration α. The acceleration falls off with time, because each photon has been Doppler-shifted to the red, and also fewer photons per second strike the probe as the probe moves faster and faster. ■

★★ **Problem 1.28** A proposed interstellar ram-jet would sweep up deuterons in space, burn them in an onboard fusion reactor, and expel the reaction products out the tail of the ship. In a reference frame instantaneously at rest relative to the ship, deuterons, each of mass m, approach the ship at relative velocity v. They are burned, and the burn products, with essentially the same total mass, are ejected from the rear of the ship at velocity $v + u$. The ship mass M stays constant, the cross-sectional area of the ship is A, and the number of deuterons per unit volume is n. (a) Find dN/dt, the number of deuterons swept up per unit