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2

Coordinating Coordinates

2.1 Starting with $\vec{r} = \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j}$:

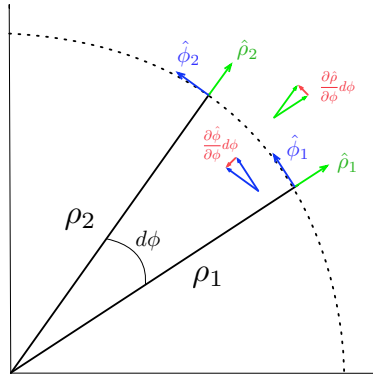
$$\hat{\rho} = \frac{\partial \vec{r} / \partial \rho}{|\partial \vec{r} / \partial \rho|} = \frac{\cos \phi \hat{i} + \sin \phi \hat{j}}{\sqrt{\cos^2 \phi + \sin^2 \phi}} = \cos \phi \hat{i} + \sin \phi \hat{j} \quad \checkmark$$

and

$$\hat{\phi} = \frac{\partial \vec{r} / \partial \phi}{|\partial \vec{r} / \partial \phi|} = \frac{-\rho \sin \phi \hat{i} + \rho \cos \phi \hat{j}}{\sqrt{\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi}} = -\sin \phi \hat{i} + \cos \phi \hat{j} \quad \checkmark$$

Similar manipulations in spherical coordinates verify Eqn. (2.16).

2.2



2.3 Cylindrical: $d\vec{r} = \frac{\partial \vec{r}}{\partial \rho} d\rho + \frac{\partial \vec{r}}{\partial \phi} d\phi + \frac{\partial \vec{r}}{\partial z} dz$:

$$\begin{aligned} \frac{\partial \vec{r}}{\partial \rho} &= \hat{\rho} \left| \frac{\partial \vec{r}}{\partial \rho} \right| = \hat{\rho} |\cos \phi \hat{i} + \sin \phi \hat{j}| = \hat{\rho} \\ \frac{\partial \vec{r}}{\partial \phi} &= \hat{\phi} \left| \frac{\partial \vec{r}}{\partial \phi} \right| = \hat{\phi} |-\rho \sin \phi \hat{i} + \rho \cos \phi \hat{j}| = \hat{\phi} \rho \\ \frac{\partial \vec{r}}{\partial z} &= \hat{k} \left| \frac{\partial \vec{r}}{\partial z} \right| = \hat{k} \end{aligned}$$

Spherical: $d\vec{r} = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial \phi} d\phi$:

$$\begin{aligned} \frac{\partial \vec{r}}{\partial r} &= \hat{r} \left| \frac{\partial \vec{r}}{\partial r} \right| = \hat{r} |\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}| = \hat{r} \\ \frac{\partial \vec{r}}{\partial \theta} &= \hat{\theta} \left| \frac{\partial \vec{r}}{\partial \theta} \right| = \hat{\theta} |r (\cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k})| = \hat{\theta} r \\ \frac{\partial \vec{r}}{\partial \phi} &= \hat{\phi} \left| \frac{\partial \vec{r}}{\partial \phi} \right| = \hat{\phi} |r (-\sin \theta \sin \phi \hat{i} + \sin \theta \cos \phi \hat{j})| = \hat{\phi} r \sin \theta \end{aligned}$$

2.4 From Eqn. (2.18), the matrix mapping $\{\hat{i}, \hat{j}, \hat{k}\}$ to $\{\hat{\rho}, \hat{\phi}, \hat{k}\}$ is $M = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Similarly, Eqn. (2.19) gives the matrix N mapping $\{\hat{i}, \hat{j}, \hat{k}\}$ to $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$. So the matrix mapping spherical coordinates into cylindrical coordinates is MN^{-1} . Since these are rotations, we can save a lot of work invoking $N^{-1} = N^T$. Then that the mapping from $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$ to $\{\hat{\rho}, \hat{\phi}, \hat{k}\}$ multiplies out to be

$$MN^T = \begin{pmatrix} \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \\ \cos \theta & -\sin \theta & 0 \end{pmatrix}.$$

The inverse transformation is NM^T — which is just the transpose of MN^T .

2.5 Writing out the matrix equation

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix},$$

it's straightforward to verify that $\hat{r} \cdot \hat{r} = \hat{\theta} \cdot \hat{\theta} = \hat{\phi} \cdot \hat{\phi} = 1$ and that $\hat{r} \times \hat{\theta} = \hat{\phi}$, etc.

2.6 (a) cartesian: $x^2 + y^2 + z^2 = 1$
 cylindrical: $\rho^2 + z^2 = 1$
 spherical: $r = 1$

(b) cartesian: $x^2 + y^2 = 1$
 cylindrical: $\rho = 1$
 spherical: $r \sin \theta = 1$

2.7 The direction cosines are the cartesian components of a unit vector from the origin making angles α, β, γ with the axes. Thus, given two different unit vectors $\vec{A}$ and $\vec{B}$, we see $\vec{A} \cdot \vec{B} = \cos \theta$ gives the identity in (b); part (a) is just a special case of this result.

2.8 Decomposing the vectors into cartesian components, but using spherical coordinates,

$$\begin{aligned} \vec{r} &= r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k} \\ \vec{r}' &= r' \sin \theta' \cos \phi' \hat{i} + r' \sin \theta' \sin \phi' \hat{j} + r' \cos \theta' \hat{k} \end{aligned}$$

Then

$$\begin{aligned} \vec{r} \cdot \vec{r}' &\equiv rr' \cos \gamma \\ &= rr' \sin \theta \sin \theta' \cos \phi \cos \phi' + rr' \sin \theta \sin \theta' \sin \phi \sin \phi' + rr' \cos \theta \cos \theta'. \end{aligned}$$

Solving:

$$\begin{aligned} \cos \gamma &= \sin \theta \sin \theta' (\cos \phi \cos \phi' + \sin \phi \sin \phi') + \cos \theta \cos \theta' \\ &= \sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta'. \end{aligned}$$

2.9 Executing the steps outlined in Example 2.2:

$$\begin{aligned}
 \vec{a} &= \frac{d}{dt} (\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}) \\
 &= \ddot{\rho}\hat{\rho} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\dot{\phi}\hat{\phi} \\
 &= \ddot{\rho}\hat{\rho} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} - \rho\dot{\phi}\dot{\phi}\hat{\rho} \\
 &= \ddot{\rho}\hat{\rho} + 2\dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} - \rho\dot{\phi}\dot{\phi}\hat{\rho} \\
 &= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi} \\
 &= (\ddot{\rho} - \rho\omega^2)\hat{\rho} + (\rho\alpha + 2\dot{\rho}\omega)\hat{\phi}.
 \end{aligned}$$

2.10 First, $|J|^2 = |J||J| = |J^T||J| = |J^T J|$. Then

$$\begin{aligned}
 |J|^2 &= \left| \begin{pmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{pmatrix} \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} \right| \\
 &= \left| \begin{pmatrix} (\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2 & \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial v} \frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} \frac{\partial y}{\partial u} & (\frac{\partial x}{\partial v})^2 + (\frac{\partial y}{\partial v})^2 \end{pmatrix} \right|
 \end{aligned}$$

Before trying to calculate this horrific determinant, note that since $\hat{u} \sim \partial \vec{r}/\partial u$ and $\hat{v} \sim \partial \vec{r}/\partial v$, the off-diagonal terms are just $\hat{u} \cdot \hat{v}$ — which vanishes for an orthogonal system. Moreover, the diagonal terms are just the scale factors h_u^2 and h_v^2 . Thus

$$|J|^2 = \begin{vmatrix} h_u^2 & 0 \\ 0 & h_v^2 \end{vmatrix} = h_u^2 h_v^2. \quad \checkmark$$

2.11 Since $\hat{\theta}$ and $\hat{\phi}$ span the tangent plane to the sphere, then using little more than $\hat{i} \times \hat{j} = \hat{k}$ does the trick:

$$\begin{aligned}
 \hat{n} &\equiv \hat{\theta} \times \hat{\phi} = (\hat{i} \cos \theta \cos \phi + \hat{j} \cos \theta \sin \phi - \hat{k} \sin \theta) \times (-\hat{i} \sin \phi + \hat{j} \cos \phi) \\
 &= \hat{k} \cos \theta \cos^2 \phi + \hat{k} \cos \theta \sin^2 \phi + \hat{j} \sin \theta \sin \phi + \hat{i} \sin \theta \cos \phi \\
 &= \hat{i} \sin \theta \cos \phi + \hat{j} \sin \theta \sin \phi + \hat{k} \cos \theta \equiv \hat{r} \quad \checkmark
 \end{aligned}$$

2.12 Area elements

(a) In cylindrical coordinates, the scale factors are $h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$. Then

- i. on surface of constant ρ , $d\vec{a} = \hat{\rho} \rho d\phi dz$
- ii. on surface of constant ϕ , $d\vec{a} = \hat{\phi} d\rho dz$
- iii. on surface of constant z , $d\vec{a} = \hat{k} \rho d\rho d\phi$

(b) In spherical coordinates, the scale factors are $h_r = 1$, $h_\theta = r$, $h_\phi = r \sin \theta$. Then

- i. on surface of constant r , $d\vec{a} = \hat{r} r^2 \sin \theta d\theta d\phi = \hat{r} r^2 d\Omega$
- ii. on surface of constant θ , $d\vec{a} = \hat{\theta} r \sin \theta dr d\phi$
- iii. on surface of constant ϕ , $d\vec{a} = \hat{\phi} r dr d\theta$

2.13 (a) Directly leveraging Eqn. (2.12) immediately yields

$$\begin{aligned}
 \text{radial equation: } m\ddot{r} - m r \dot{\phi}^2 &= -\frac{k}{r^2} \\
 \text{angular equation: } r\ddot{\phi} + 2\dot{r}\dot{\phi} &= 0
 \end{aligned}$$

(b) Simplifying the angular equation as $\frac{1}{r} \frac{d}{dt} (r^2 \dot{\phi}) = \frac{1}{mr} \frac{d}{dt} (mr^2 \dot{\phi}) = 0$ reveals angular momentum conservation: $\ell \equiv mr^2 \dot{\phi} = \text{constant}$

(c) Using Eqn. (2.9) to express the kinetic energy in polar coordinates gives

$$\frac{1}{2}mv^2 = \frac{1}{2}m \left(\frac{d\vec{r}}{dt} \right) \cdot \left(\frac{d\vec{r}}{dt} \right) = \frac{1}{2}m (\dot{r}^2 + r^2\dot{\phi}^2) .$$

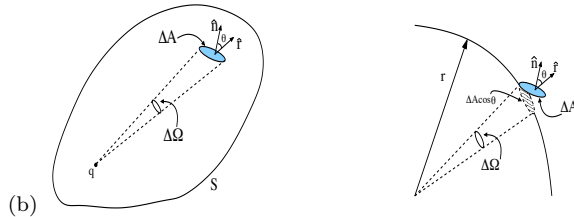
Using angular momentum conservation in part (b), the kinetic energy is

$$KE = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} .$$

For a circular orbit $\dot{r} = 0$, so the entire kinetic energy is due to angular motion.

2.14 (a) A point charge q has electric field $\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}$, so that

$$d\Phi = \vec{E} \cdot \hat{n} dA = \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \cdot \hat{n} dA = \frac{q}{4\pi\epsilon_0 r^2} \cos\theta dA$$



(b)

(c) The component of the vector-valued area $d\vec{A} \equiv dA \hat{n}$ perpendicular to the radial line is $dA \cos\theta$, and subtends the same solid angle as dA ; using the definition of solid angle (which requires r perpendicular to the subtended area),

$$d\Omega = \frac{dA \cos\theta}{r^2}$$

we have

$$d\Phi = \frac{q}{4\pi\epsilon_0} d\Omega .$$

Since the total solid angle is 4π , the net flux is

$$\Phi = \oint_S \vec{E} \cdot \hat{n} dA = \frac{q}{4\pi\epsilon_0} \oint_S d\Omega = \frac{q}{\epsilon_0} ,$$

where no explicit integration was necessary!

2.15 Starting with

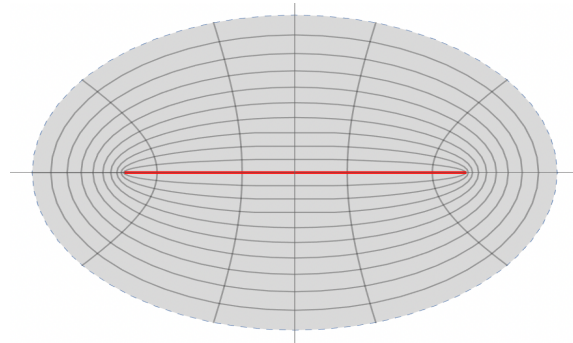
$$x = r \sin\theta \cos\phi , \quad y = r \sin\theta \sin\phi , \quad z = r \cos\theta$$

we get

$$\begin{aligned} |J| &\equiv \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial \phi \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial \phi \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial \phi \end{vmatrix} \\ &= \begin{vmatrix} \sin\theta \cos\phi & r \cos\theta \cos\phi & -r \sin\theta \sin\phi \\ \sin\theta \sin\phi & r \cos\theta \sin\phi & r \sin\theta \cos\phi \\ \cos\theta & -r \sin\theta & 0 \end{vmatrix} = \dots = r^2 \sin\theta . \end{aligned}$$

- 2.16 (a) $\frac{x^2}{a^2 \cosh^2 u} + \frac{y^2}{a^2 \sinh^2 u} = \cos^2 \phi + \sin^2 \phi = 1$. For constant u , this describes an ellipse with semi-major and minor axes $a \cosh u$, $a \sinh u$ and foci at $c = \pm \sqrt{a^2 \cosh^2 u - a^2 \sinh^2 u} = \pm a$.

```
elliptic[u_, φ_] := {a Cosh[u] Cos[φ], a Sinh[u] Sin[φ]};
a = 1;
uPlots = ParametricPlot[elliptic[u, φ], {u, 0, 1}, {φ, 0, 2π},
    BoundaryStyle → Dashed, Mesh → 9, Frame → False,
    PlotStyle → Gray, Ticks → None, PlotRange → All];
zeroPlot = ParametricPlot[elliptic[0, φ], {φ, 0, 2π},
    PlotStyle → {Red, Thick}, Ticks → None];
Show[uPlots, zeroPlot]
```



- (b) The easiest approach is via Eqn. (2.27b):

$$\begin{aligned} h_u^2 &= \left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 \\ &= (a \sinh u \cos \phi)^2 + (a \cosh u \sin \phi)^2 \\ &= a^2 [\sinh^2 u (1 - \sin^2 \phi) + (1 + \sinh^2 u) \sin^2 \phi] = a^2 (\sinh^2 u + \sin^2 \phi) \end{aligned}$$

Similarly,

$$\begin{aligned} h_\phi^2 &= \left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 \\ &= (-a \cosh u \sin \phi)^2 + (a \sinh u \cos \phi)^2 \\ &= a^2 [(1 + \sinh^2 u) \sin^2 \phi + \sinh^2 u (1 - \sin^2 \phi)] = a^2 (\sinh^2 u + \sin^2 \phi) \end{aligned}$$

- 2.17 (a) $-\frac{x^2+y^2}{a^2 \cosh^2 u} + \frac{z^2}{a^2 \sinh^2 u} = \sin^2 \theta + \cos^2 \theta = 1$. For constant u , this is the equation of an ellipsoidal surface. Note that the surface intersects the xy -plane in circles of radius $a \cosh u$, whereas ellipses in the xz - and yz -planes have semi-major axis $a \cosh u$, and semi-minor axis $a \sinh u$.
- $-\frac{x^2+y^2}{a^2 \sin^2 \theta} - \frac{z^2}{a^2 \cos^2 \theta} = \cosh^2 u - \sinh^2 u = 1$. For constant θ , this is the equation of a hyperbolic surface. Once again, the curves in the xy -plane are circles (this time of radius $a \sin \theta$), but those in the xz - and yz -planes are hyperbolas.

- (b) Starting with $x = a \cosh u \sin \theta \cos \phi$, $y = a \cosh u \sin \theta \sin \phi$, $z = a \sinh u \cos \theta$, the easiest

approach is via Eqn. (2.27b):

$$\begin{aligned} h_u^2 &= \left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial u} \right)^2 \\ &= (a \sinh u \sin \theta \cos \phi)^2 + (a \sinh u \sin \theta \sin \phi)^2 + (a \cosh u \cos \theta)^2 \\ &= a^2 (\sinh^2 u \sin^2 \theta + \cosh^2 u \cos^2 \theta) \\ &= a^2 [\sinh^2 u (1 - \cos^2 \theta) + (1 + \sinh^2 u) \cos^2 \theta] = a^2 (\sinh^2 u + \cos^2 \theta) \quad \checkmark \end{aligned}$$

It's easy to see that h_θ gives the same. As for h_ϕ :

$$h_\phi^2 = \left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 + \left(\frac{\partial z}{\partial \phi} \right)^2 = (-a \cosh u \sin \theta \sin \phi)^2 + (a \cosh u \sin \theta \cos \phi)^2 = a^2 \cosh^2 u \sin^2 \theta \quad \checkmark$$

(c) Easy: $|J| = h_u h_\theta h_\phi = a^3 \cosh u \sin \theta (\sinh^2 u + \cos^2 \theta)$. But if you prefer the long way:

$$\begin{aligned} |J| &\equiv \begin{vmatrix} \partial x / \partial u & \partial x / \partial \theta & \partial x / \partial \phi \\ \partial y / \partial u & \partial y / \partial \theta & \partial y / \partial \phi \\ \partial z / \partial u & \partial z / \partial \theta & \partial z / \partial \phi \end{vmatrix} \\ &= a^3 \begin{vmatrix} \sinh u \sin \theta \cos \phi & \cosh u \cos \theta \cos \phi & -\cosh u \sin \theta \sin \phi \\ \sinh u \sin \theta \sin \phi & \cosh u \cos \theta \sin \phi & \cosh u \sin \theta \cos \phi \\ \cosh u \cos \theta & -\sinh u \sin \theta & 0 \end{vmatrix} \\ &= \dots = a^3 \cosh u \sin \theta (\sinh^2 u + \cos^2 \theta) . \end{aligned}$$

(d) Surfaces of constant u have

$$da = h_\theta h_\phi d\theta d\phi = a^2 \cosh u \sin \theta \sqrt{\sinh^2 u + \cos^2 \theta} d\theta d\phi .$$

Since $h_u = h_\theta$, surfaces of constant θ have the almost-identical

$$da = h_u h_\phi du d\phi = a^2 \cosh u \sin \theta \sqrt{\sinh^2 u + \cos^2 \theta} du d\phi .$$

2.18 Using the h_i for Problem 2.17 in Eqns. (2.27), and with

$$\vec{r} = a \cosh u \sin \theta \cos \phi \hat{i} + a \cosh u \sin \theta \sin \phi \hat{j} + a \sinh u \cos \theta \hat{k} ,$$

we find

$$\hat{e}_u = \frac{1}{h_u} \frac{\partial \vec{r}}{\partial u} = \frac{1}{\sqrt{\sinh^2 u + \cos^2 \theta}} (\sinh u \sin \theta \cos \phi \hat{i} + \sinh u \sin \theta \sin \phi \hat{j} + a \cosh u \cos \theta \hat{k})$$

$$\hat{e}_\theta = \frac{1}{h_\theta} \frac{\partial \vec{r}}{\partial \theta} = \frac{1}{\sqrt{\sinh^2 u + \cos^2 \theta}} (\cosh u \cos \theta \cos \phi \hat{i} + \cosh u \cos \theta \sin \phi \hat{j} - \sinh u \sin \theta \hat{k})$$

and

$$\hat{e}_\phi = \frac{1}{h_\phi} \frac{\partial \vec{r}}{\partial \phi} = \frac{1}{\cosh u \sin \theta} (-\cosh u \sin \theta \sin \phi \hat{i} + \cosh u \sin \theta \cos \phi \hat{j}) .$$

Multiple use of the identities $\cos^2 + \sin^2 = 1$ and $\cosh^2 - \sinh^2 = 1$ readily demonstrates that these vectors form an orthonormal set.

2.19 From Eqns. (2.27), we have $h_i \hat{e}_i = \partial \vec{r} / \partial u_i$:

$$\begin{aligned} h_r \hat{r} &= (\sin \psi \sin \theta \cos \phi, \sin \psi \sin \theta \sin \phi, \sin \psi \cos \theta, \cos \psi) \\ h_\psi \hat{\psi} &= r (\cos \psi \sin \theta \cos \phi, \cos \psi \sin \theta \sin \phi, \cos \psi \cos \theta, -\sin \psi) \\ h_\theta \hat{\theta} &= r (\sin \psi \cos \theta \cos \phi, \sin \psi \cos \theta \sin \phi, -\sin \psi \sin \theta, 0) \\ h_\phi \hat{\phi} &= r (-\sin \psi \sin \theta \sin \phi, \sin \psi \sin \theta \cos \phi, 0, 0) \end{aligned}$$

Each is indeed orthogonal to the other three, and the magnitude of each gives the scale factor:

$$h_r = 1 \quad h_\psi = r \quad h_\theta = r \sin \psi \quad h_\phi = r \sin \psi \sin \theta ,$$

which together produce the required line element in $\mathbb{R}^4$. Finally, the product of the four gives the Jacobian, $|J| = r^3 \sin^2 \psi \sin \theta$.

2.20 For hyperspherical coordinates

$$\begin{aligned} x_4 &= r \cos \psi \\ x_3 &= r \sin \psi \cos \theta \\ x_2 &= r \sin \psi \sin \theta \cos \phi \\ x_1 &= r \sin \psi \sin \theta \sin \phi , \end{aligned}$$

the Jacobian matrix is (in reverse order, as suggested)

$$\begin{aligned} |J| &= \left| \frac{\partial(x_4, x_3, x_2, x_1)}{\partial(r, \psi, \theta, \phi)} \right| \\ &= \begin{vmatrix} \cos \psi & -r \sin \psi & 0 & 0 \\ \sin \psi \cos \theta & r \cos \psi \cos \theta & -r \sin \psi \sin \theta & 0 \\ \sin \psi \sin \theta \cos \phi & r \cos \psi \sin \theta \cos \phi & r \sin \psi \cos \theta \cos \phi & -r \sin \psi \sin \theta \sin \phi \\ \sin \psi \sin \theta \sin \phi & r \cos \psi \sin \theta \sin \phi & r \sin \psi \cos \theta \sin \phi & r \sin \psi \sin \theta \cos \phi \end{vmatrix} \\ &= \cos \psi \begin{vmatrix} r \cos \psi \cos \theta & -r \sin \psi \sin \theta & 0 \\ r \cos \psi \sin \theta \cos \phi & r \sin \psi \cos \theta \cos \phi & -r \sin \psi \sin \theta \sin \phi \\ r \cos \psi \sin \theta \sin \phi & r \sin \psi \cos \theta \sin \phi & r \sin \psi \sin \theta \cos \phi \end{vmatrix} \\ &\quad + r \sin \psi \begin{vmatrix} \sin \psi \cos \theta & -r \sin \psi \sin \theta & 0 \\ \sin \psi \sin \theta \cos \phi & r \sin \psi \cos \theta \cos \phi & -r \sin \psi \sin \theta \sin \phi \\ \sin \psi \sin \theta \sin \phi & r \sin \psi \cos \theta \sin \phi & r \sin \psi \sin \theta \cos \phi \end{vmatrix} . \end{aligned}$$

Now each additive element of a determinant has exactly one contribution from each row and column. So a common multiplicative factor of a column or row can be moved outside the determinant. We can do just this with all the factors of $\sin \psi$ and $\cos \psi$, giving

$$|J| = (r \cos^2 \psi \sin^2 \psi + r \sin^4 \psi) \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \end{vmatrix} .$$

Of course, we could also have taken out two more factors of r , but leaving them helps us recognize the remaining 3×3 determinant as the Jacobian $r^2 \sin \theta$ of standard spherical coordinates in $\mathbb{R}^3$. Thus

$$|J| = (r \cos^2 \psi \sin^2 \psi + r \sin^4 \psi) r^2 \sin \theta = r^3 \sin^2 \psi \sin \theta .$$

3

Complex Numbers

- 3.1 (a) $|e^{-i\pi}|^2 = e^{i\pi}e^{-i\pi} = 1 \checkmark$
 (b) $\pi^{ie} = e^{ie \ln \pi} \checkmark$
 (c) $(-\pi)^{ie} = (e^{i\pi}e^{\ln \pi})^{ie} = e^{-e\pi}e^{ie \ln \pi} \times$
 (d) $i^{e\pi} = (e^{i\pi/2})^{e\pi} = e^{ie\pi^2/2} \checkmark$
 (e) $(-i)^{e\pi^i} = (e^{i3\pi/2})^{e^{(1+i \ln \pi)}} = e^{i3e\pi/2}e^{-3\pi \ln(\pi)/2} \times$

3.2 In the cartesian representation $z = a + ib$,

$$|z|^2 = zz^* = (a + ib)(a - ib) = a^2 + b^2$$

$$|z|^2 = |(a + ib)^2| = |(a^2 - b^2) + i(2ab)| = \sqrt{(a^2 - b^2)^2 + (2ab)^2} = \sqrt{(a^2 + b^2)^2} = a^2 + b^2$$

In the polar representation $z = re^{i\varphi}$ it's even easier:

$$|z|^2 = zz^* = re^{i\varphi} re^{-i\varphi} = r^2$$

$$|z^2| = |(re^{i\varphi})^2| = |r^2 e^{2i\varphi}| = r^2$$

In general, $|z^n| = |z|^n = r^n$.

3.3 Starting with $z = \cos \theta + i \sin \theta$, we have

$$dz = (-\sin \theta + i \cos \theta)d\theta$$

$$= i(\cos \theta + i \sin \theta)d\theta = izd\theta .$$

Thus

$$\int \frac{dz}{z} = i \int d\theta \implies z = e^{i\theta} ,$$

where the constant of integration is determined from $z(\theta = 0) = 1$.

3.4 Compute the square in two equivalent ways:

$$1) |(a + ib)(c + id)|^2 = |(ac - bd) + i(bc + ad)|^2 = (ac - bd)^2 + (bc + ad)^2 \equiv p^2 + q^2$$

$$2) |(a + ib)(c + id)|^2 = (a^2 + b^2)(c^2 + d^2) \equiv MN .$$

Thus $MN = p^2 + q^2$. For example,

$$\left. \begin{array}{l} M = 13 = 2^2 + 3^2 \\ N = 25 = 3^2 + 4^2 \end{array} \right\} MN = 325 = (2 \cdot 3 - 3 \cdot 4)^2 + (3 \cdot 3 + 2 \cdot 4)^2 = 6^2 + 17^2$$

- 3.5 (a) $z_1 = i + \sqrt{3} : |z_1| = \sqrt{1+3} = 2, \arg(z_1) = \arctan(1/\sqrt{3}) = \frac{\pi}{6} \implies z_1 = 2e^{i\pi/6}$
 $z_2 = i - \sqrt{3} : |z_2| = 2, \arg(z_2) = \arctan(-1/\sqrt{3}) = \frac{5\pi}{6} \implies z_2 = 2e^{i5\pi/6} = 2e^{-i\pi/6}$
 (b) $2i = 2e^{i\pi/2} \rightsquigarrow \sqrt{2}i = \pm \sqrt{2}e^{i\pi/4} = \pm \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = \pm(1 + i)$
 $2 + 2i\sqrt{3} = 4e^{i\pi/3} \rightsquigarrow \sqrt{2 + 2i\sqrt{3}} = \pm 2e^{i\pi/6} = \pm 2 \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) = \pm(\sqrt{3} + i)$