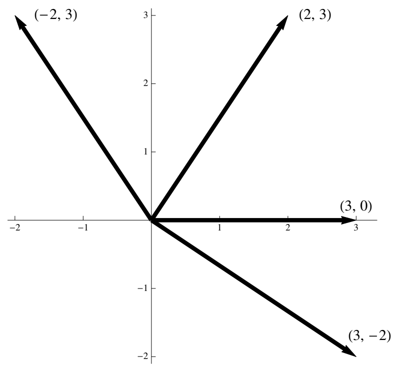
Chapter 1

Vectors

1.1 The Geometry and Algebra of Vectors

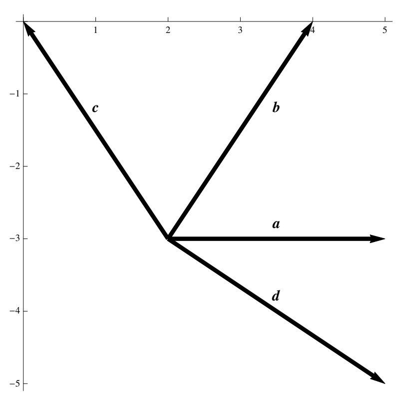
1.



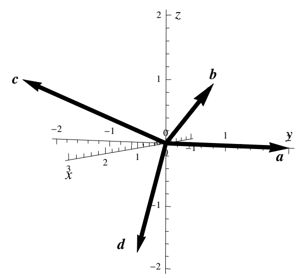
2. Since

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix},$$

plotting those vectors gives



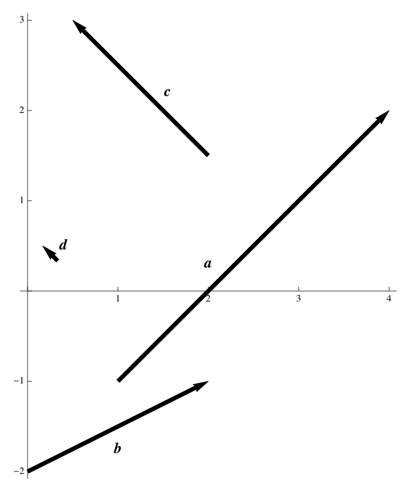
3.



4. Since the heads are all at $(3,2,1)$, the tails are at

$$\begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}0\\2\\0\end{bmatrix}=\begin{bmatrix}3\\0\\1\end{bmatrix},\quad \begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad \begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}2\\4\\0\end{bmatrix},\quad \begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}-1\\-1\\-2\end{bmatrix}=\begin{bmatrix}4\\3\\3\end{bmatrix}.$$

5. The four vectors $\overrightarrow{AB}$ are



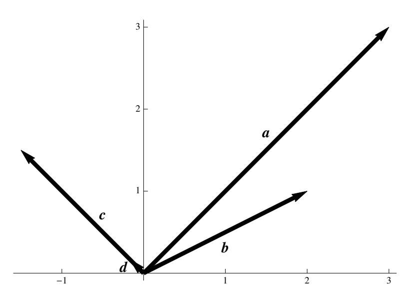
In standard position, the vectors are

(a) $\overrightarrow{AB}=[4-1,2-(-1)]=[3,3]$.

(b) $\overrightarrow{AB}=[2-0,-1-(-2)]=[2,1]$

(c) $\overrightarrow{AB}=\left[\frac{1}{2}-2,3-\frac{3}{2}\right]=\left[-\frac{3}{2},\ \frac{3}{2}\right]$

(d) $\overrightarrow{AB}=\left[\frac{1}{6}-\frac{1}{3},\ \frac{1}{2}-\frac{1}{3}\right]=\left[-\frac{1}{6},\ \frac{1}{6}\right]$.



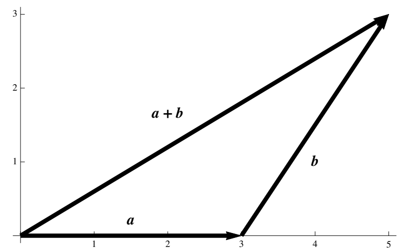
6. Recall the notation that $[a, b]$ denotes a move of a units horizontally and b units vertically. Then during the first part of the walk, the hiker walks 4 km north, so $\mathbf{a} = [0, 4]$. During the second part of the walk, the hiker walks a distance of 5 km northeast. From the components, we get

$$\mathbf{b} = [5\cos 45^\circ,\ 5\sin 45^\circ] = \left[\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right].$$

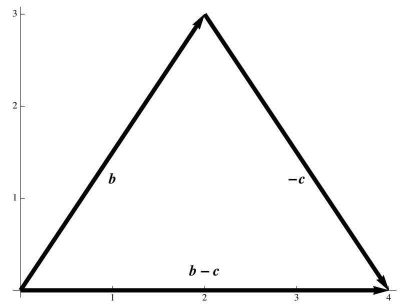
Thus the net displacement vector is

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = \left[\frac{5\sqrt{2}}{2}, 4 + \frac{5\sqrt{2}}{2}\right].$$

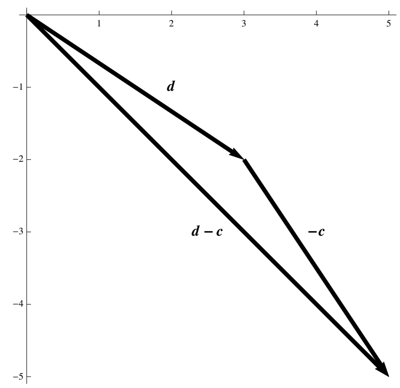
7. $\mathbf{a} + \mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3+2 \\ 0+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$



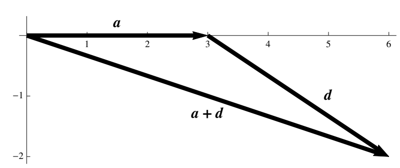
8. $\mathbf{b} - \mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2-(-2) \\ 3-3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$



9. $\mathbf{d} - \mathbf{c} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}.$



10. $\mathbf{a} + \mathbf{d} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+3 \\ 0+(-2) \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$



11. $2\mathbf{a} + 3\mathbf{c} = 2[0, 2, 0] + 3[1, -2, 1] = [2 \cdot 0, 2 \cdot 2, 2 \cdot 0] + [3 \cdot 1, 3 \cdot (-2), 3 \cdot 1] = [3, -2, 3].$

12.

$$\begin{aligned} 3\mathbf{b} - 2\mathbf{c} + \mathbf{d} &= 3[3, 2, 1] - 2[1, -2, 1] + [-1, -1, -2] \\ &= [3 \cdot 3, 3 \cdot 2, 3 \cdot 1] + [-2 \cdot 1, -2 \cdot (-2), -2 \cdot 1] + [-1, -1, -2] \\ &= [6, 9, -1]. \end{aligned}$$

13. $\mathbf{u} = [\cos 60°, \sin 60°] = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$, and $\mathbf{v} = [\cos 210°, \sin 210°] = \left[-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right]$, so that

$$\mathbf{u} + \mathbf{v} = \left[\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} - \frac{1}{2}\right], \quad \mathbf{u} - \mathbf{v} = \left[\frac{1}{2} + \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} + \frac{1}{2}\right].$$

14. **(a)** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

(b) Since $\overrightarrow{OC} = \overrightarrow{AB}$, we have $\overrightarrow{BC} = \overrightarrow{OC} - \mathbf{b} = (\mathbf{b} - \mathbf{a}) - \mathbf{b} = -\mathbf{a}$.

(c) $\overrightarrow{AD} = -2\mathbf{a}$.

(d) $\overrightarrow{CF} = -2\overrightarrow{OC} = -2\overrightarrow{AB} = -2(\mathbf{b} - \mathbf{a}) = 2(\mathbf{a} - \mathbf{b})$.

(e) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = (\mathbf{b} - \mathbf{a}) + (-\mathbf{a}) = \mathbf{b} - 2\mathbf{a}$.

(f) Note that $\overrightarrow{FA}$ and $\overrightarrow{OB}$ are equal, and that $\overrightarrow{DE} = -\overrightarrow{AB}$. Then

$$\overrightarrow{BC} + \overrightarrow{DE} + \overrightarrow{FA} = -\mathbf{a} - \overrightarrow{AB} + \overrightarrow{OB} = -\mathbf{a} - (\mathbf{b} - \mathbf{a}) + \mathbf{b} = \mathbf{0}.$$

15. $2(\mathbf{a} - 3\mathbf{b}) + 3(2\mathbf{b} + \mathbf{a}) \overset{\substack{\text{property e.}\\ \text{distributivity}}}{=} (2\mathbf{a} - 6\mathbf{b}) + (6\mathbf{b} + 3\mathbf{a}) \overset{\substack{\text{property b.}\\ \text{associativity}}}{=} (2\mathbf{a} + 3\mathbf{a}) + (-6\mathbf{b} + 6\mathbf{b}) = 5\mathbf{a}$.

16.

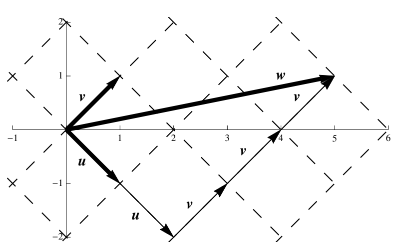
$$\begin{aligned} -3(\mathbf{a} - \mathbf{c}) + 2(\mathbf{a} + 2\mathbf{b}) + 3(\mathbf{c} - \mathbf{b}) &\overset{\substack{\text{property e.}\\ \text{distributivity}}}{=} (-3\mathbf{a} + 3\mathbf{c}) + (2\mathbf{a} + 4\mathbf{b}) + (3\mathbf{c} - 3\mathbf{b}) \\ &\overset{\substack{\text{property b.}\\ \text{associativity}}}{=} (-3\mathbf{a} + 2\mathbf{a}) + (4\mathbf{b} - 3\mathbf{b}) + (3\mathbf{c} + 3\mathbf{c}) \\ &= -\mathbf{a} + \mathbf{b} + 6\mathbf{c}. \end{aligned}$$

17. $\mathbf{x} - \mathbf{a} = 2(\mathbf{x} - 2\mathbf{a}) = 2\mathbf{x} - 4\mathbf{a} \Rightarrow \mathbf{x} - 2\mathbf{x} = \mathbf{a} - 4\mathbf{a} \Rightarrow -\mathbf{x} = -3\mathbf{a} \Rightarrow \mathbf{x} = 3\mathbf{a}$.

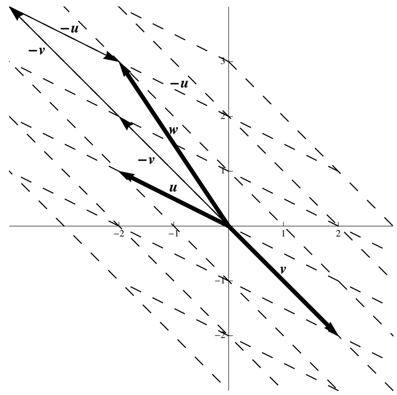
18.

$$\begin{aligned} \mathbf{x} + 2\mathbf{a} - \mathbf{b} = 3(\mathbf{x} + \mathbf{a}) - 2(2\mathbf{a} - \mathbf{b}) = 3\mathbf{x} + 3\mathbf{a} - 4\mathbf{a} + 2\mathbf{b} \quad &\Rightarrow \\ \mathbf{x} - 3\mathbf{x} = -\mathbf{a} - 2\mathbf{a} + 2\mathbf{b} + \mathbf{b} \quad &\Rightarrow \\ -2\mathbf{x} = -3\mathbf{a} + 3\mathbf{b} \quad &\Rightarrow \\ \mathbf{x} = \frac{3}{2}\mathbf{a} - \frac{3}{2}\mathbf{b}. \end{aligned}$$

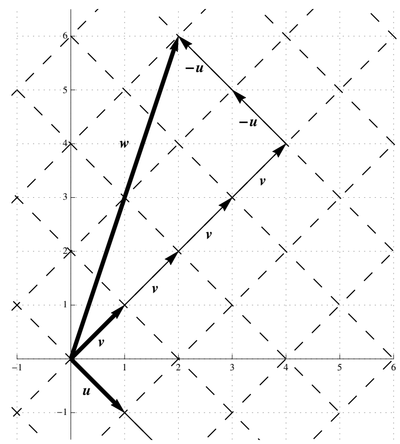
19. We have $2\mathbf{u} + 3\mathbf{v} = 2[1, -1] + 3[1, 1] = [2 \cdot 1 + 3 \cdot 1, 2 \cdot (-1) + 3 \cdot 1] = [5, 1]$. Plots of all three vectors are



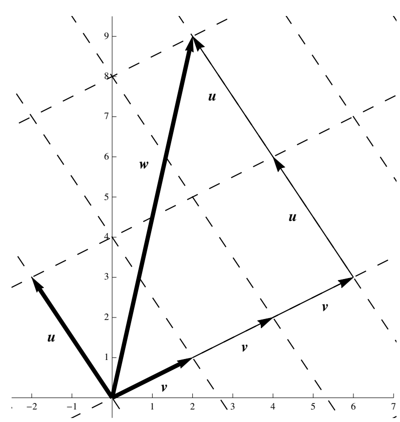
20. We have $-\mathbf{u} - 2\mathbf{v} = -[-2, 1] - 2[2, -2] = [-(-2) - 2 \cdot 2, -1 - 2 \cdot (-2)] = [-2, 3]$. Plots of all three vectors are



21. From the diagram, we see that $\mathbf{w} = -2\mathbf{u} + 4\mathbf{v}$.

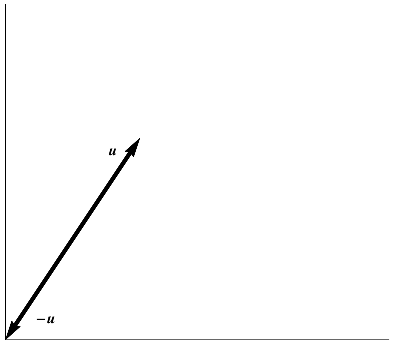


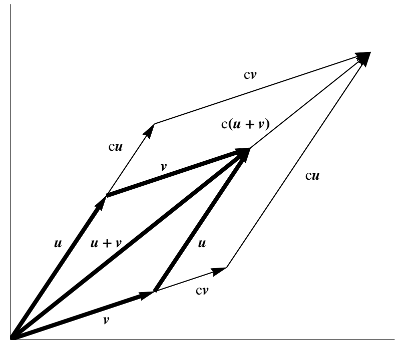
22. From the diagram, we see that $\mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$.



23. Property (d) states that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. The first diagram below shows $\mathbf{u}$ along with $-\mathbf{u}$. Then, as the diagonal of the parallelogram, the resultant vector is $\mathbf{0}$.

Property (e) states that $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$. The second figure illustrates this.





24. Let $\mathbf{u} = [u_1, u_2, \dots, u_n]$ and $\mathbf{v} = [v_1, v_2, \dots, v_n]$, and let c and d be scalars in $\mathbb{R}$.

Property (d):

$$\begin{aligned}\mathbf{u} + (-\mathbf{u}) &= [u_1, u_2, \dots, u_n] + (-1[u_1, u_2, \dots, u_n]) \\ &= [u_1, u_2, \dots, u_n] + [-u_1, -u_2, \dots, -u_n] \\ &= [u_1 - u_1, u_2 - u_2, \dots, u_n - u_n] \\ &= [0, 0, \dots, 0] = \mathbf{0}.\end{aligned}$$

Property (e):

$$\begin{aligned}c(\mathbf{u} + \mathbf{v}) &= c\left([u_1, u_2, \dots, u_n] + [v_1, v_2, \dots, v_n]\right) \\ &= c\left([u_1 + v_1, u_2 + v_2, \dots, u_n + v_n]\right) \\ &= [c(u_1 + v_1), c(u_2 + v_2), \dots, c(u_n + v_n)] \\ &= [cu_1 + cv_1, cu_2 + cv_2, \dots, cu_n + cv_n] \\ &= [cu_1, cu_2, \dots, cu_n] + [cv_1, cv_2, \dots, cv_n] \\ &= c[u_1, u_2, \dots, u_n] + c[v_1, v_2, \dots, v_n] \\ &= c\mathbf{u} + c\mathbf{v}.\end{aligned}$$

Property (f):

$$\begin{aligned}(c + d)\mathbf{u} &= (c + d)[u_1, u_2, \dots, u_n] \\ &= [(c + d)u_1, (c + d)u_2, \dots, (c + d)u_n] \\ &= [cu_1 + du_1, cu_2 + du_2, \dots, cu_n + du_n] \\ &= [cu_1, cu_2, \dots, cu_n] + [du_1, du_2, \dots, du_n] \\ &= c[u_1, u_2, \dots, u_n] + d[u_1, u_2, \dots, u_n] \\ &= c\mathbf{u} + d\mathbf{u}.\end{aligned}$$

Property (g):

$$\begin{aligned}c(d\mathbf{u}) &= c(d[u_1, u_2, \dots, u_n]) \\ &= c[du_1, du_2, \dots, du_n] \\ &= [cdu_1, cdu_2, \dots, cdu_n] \\ &= [(cd)u_1, (cd)u_2, \dots, (cd)u_n] \\ &= (cd)[u_1, u_2, \dots, u_n] \\ &= (cd)\mathbf{u}.\end{aligned}$$

25. $\mathbf{u} + \mathbf{v} = [0, 1] + [1, 1] = [1, 0]$.

26. $\mathbf{u} + \mathbf{v} = [1, 1, 0] + [1, 1, 1] = [0, 0, 1]$.

27. $\mathbf{u}+\mathbf{v} = [1,0,1,1] + [1,1,1,1] = [0,1,0,0]$.

28. $\mathbf{u}+\mathbf{v} = [1,1,0,1,0] + [0,1,1,1,0] = [1,0,1,0,0]$.

29.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

30.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

31. $2+2+2=6=0$ in $\mathbb{Z}_3$.

32. $2\cdot 2\cdot 2 = 3\cdot 2 = 0$ in $\mathbb{Z}_3$.

33. $2(2+1+2) = 2\cdot 2 = 3\cdot 1+1 = 1$ in $\mathbb{Z}_3$.

34. $3+1+2+3 = 4\cdot 2+1 = 1$ in $\mathbb{Z}_4$.

35. $2\cdot 3\cdot 2 = 4\cdot 3+0 = 0$ in $\mathbb{Z}_4$.

36. $3(3+3+2) = 4\cdot 6+0 = 0$ in $\mathbb{Z}_4$.

37. $2+1+2+2+1 = 2$ in $\mathbb{Z}_3$, $2+1+2+2+1 = 0$ in $\mathbb{Z}_4$, $2+1+2+2+1 = 3$ in $\mathbb{Z}_5$.

38. $(3+4)(3+2+4+2) = 2\cdot 1 = 2$ in $\mathbb{Z}_5$.

39. $8(6+4+3) = 8\cdot 4 = 5$ in $\mathbb{Z}_9$.

40. $2^{100} = \left(2^{10}\right)^{10} = (1024)^{10} = 1^{10} = 1$ in $\mathbb{Z}_{11}$.

41. $[2,1,2]+[2,0,1] = [1,1,0]$ in $\mathbb{Z}_3^3$.

42. $2[2,\ 2,\ 1] = [2\cdot 2,\ 2\cdot 2,\ 2\cdot 1] = [1,\ 1,\ 2]$ in $\mathbb{Z}_3^3$.

43. $2([3,1,1,2]+[3,3,2,1]) = 2[2,0,3,3] = [2\cdot 2,\ 2\cdot 0,\ 2\cdot 3,\ 2\cdot 3] = [0,0,2,2]$ in $\mathbb{Z}_4^4$.
$2([3,1,1,2]+[3,3,2,1]) = 2[1,4,3,3] = [2\cdot 1,\ 2\cdot 4,\ 2\cdot 3,\ 2\cdot 3] = [2,3,1,1]$ in $\mathbb{Z}_5^4$.

44. $x = 2+(-3) = 2+2 = 4$ in $\mathbb{Z}_5$.

45. $x = 1+(-5) = 1+1 = 2$ in $\mathbb{Z}_6$

46. $x = 2^{-1} = 2$ in $\mathbb{Z}_3$.

47. No solution. 2 times anything is always even, so cannot leave a remainder of 1 when divided by 4.

48. $x = 2^{-1} = 3$ in $\mathbb{Z}_5$.

49. $x = 3^{-1}4 = 2\cdot 4 = 3$ in $\mathbb{Z}_5$.

50. No solution. 3 times anything is always a multiple of 3, so it cannot leave a remainder of 4 when divided by 6 (which is also a multiple of 3).

51. No solution. 6 times anything is always even, so it cannot leave an odd number as a remainder when divided by 8.

52. $x = 8^{-1}9 = 7 \cdot 9 = 8$ in $\mathbb{Z}_{11}$

53. $x = 2^{-1}(2 + (-3)) = 3(2 + 2) = 2$ in $\mathbb{Z}_5$.

54. No solution. This equation is the same as $4x = 2 - 5 = -3 = 3$ in $\mathbb{Z}_6$. But 4 times anything is even, so it cannot leave a remainder of 3 when divided by 6 (which is also even).

55. Add 5 to both sides to get $6x = 6$, so that $x = 1$ or $x = 5$ (since $6 \cdot 1 = 6$ and $6 \cdot 5 = 30 = 6$ in $\mathbb{Z}_8$).

56. (a) All values. (b) All values. (c) All values.

57. **(a)** All $a \neq 0$ in $\mathbb{Z}_5$ have a solution because 5 is a prime number.

(b) $a = 1$ and $a = 5$ because they have no common factors with 6 other than 1.

(c) a and m can have no common factors other than 1; that is, the *greatest common divisor*, gcd, of a and m is 1.

1.2 Length and Angle: The Dot Product

1. Following Example 1.15, $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (-1) \cdot 3 + 2 \cdot 1 = -3 + 2 = -1$.

2. Following Example 1.15, $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 3 \cdot 4 + (-2) \cdot 6 = 12 - 12 = 0$.

3. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 = 2 + 6 + 3 = 11$.

4. $\mathbf{u} \cdot \mathbf{v} = 3.2 \cdot 1.5 + (-0.6) \cdot 4.1 + (-1.4) \cdot (-0.2) = 4.8 - 2.46 + 0.28 = 2.62$.

5. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -\sqrt{2} \\ 0 \\ -5 \end{bmatrix} = 1 \cdot 4 + \sqrt{2} \cdot (-\sqrt{2}) + \sqrt{3} \cdot 0 + 0 \cdot (-5) = 4 - 2 = 2$.

6. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1.12 \\ -3.25 \\ 2.07 \\ -1.83 \end{bmatrix} \cdot \begin{bmatrix} -2.29 \\ 1.72 \\ 4.33 \\ -1.54 \end{bmatrix} = -1.12 \cdot 2.29 - 3.25 \cdot 1.72 + 2.07 \cdot 4.33 - 1.83 \cdot (-1.54) = 3.6265$.

7. Finding a unit vector $\mathbf{v}$ in the same direction as a given vector $\mathbf{u}$ is called ***normalizing*** the vector $\mathbf{u}$. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5},$$

so a unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

8. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13},$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \end{bmatrix}.$$

9. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14},$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{14}}\\\frac{2}{\sqrt{14}}\\\frac{3}{\sqrt{14}}\end{bmatrix}.$$

10. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{3.2^2 + (-0.6)^2 + (-1.4)^2} = \sqrt{10.24 + 0.36 + 1.96} = \sqrt{12.56} \approx 3.544,$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{3.544}\begin{bmatrix}1.5\\0.4\\-2.1\end{bmatrix} \approx \begin{bmatrix}0.903\\-0.169\\-0.395\end{bmatrix}.$$

11. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{1^2 + \left(\sqrt{2}\right)^2 + \left(\sqrt{3}\right)^2 + 0^2} = \sqrt{6},$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\\sqrt{2}\\\sqrt{3}\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\\frac{\sqrt{2}}{\sqrt{6}}\\\frac{\sqrt{3}}{\sqrt{6}}\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{6}}{6}\\\frac{\sqrt{3}}{3}\\\frac{\sqrt{2}}{2}\\0\end{bmatrix}$$

12. Proceed as in Example 1.19:

$$\begin{aligned}\|\mathbf{u}\| = \sqrt{1.12^2 + (-3.25)^2 + 2.07^2 + (-1.83)^2} &= \sqrt{1.2544 + 10.5625 + 4.2849 + 3.3489}\\ &= \sqrt{19.4507} \approx 4.410,\end{aligned}$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v}\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{4.410}\begin{bmatrix}1.12 & -3.25 & 2.07 & -1.83\end{bmatrix} \approx \begin{bmatrix}0.254 & -0.737 & 0.469 & -0.415\end{bmatrix}.$$

13. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix}-1\\2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}-4\\1\end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-4)^2 + 1^2} = \sqrt{17}.$$

14. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix}3\\-2\end{bmatrix} - \begin{bmatrix}4\\6\end{bmatrix} = \begin{bmatrix}-1\\-8\end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1)^2 + (-8)^2} = \sqrt{65}.$$

15. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\2\end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1)^2 + (-1)^2 + 2^2} = \sqrt{6}.$$

16. Following Example 1.20, we compute: $\mathbf{u}-\mathbf{v} = \begin{bmatrix} 3.2 \\ -0.6 \\ -1.4 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 4.1 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 1.7 \\ -4.7 \\ -1.2 \end{bmatrix}$, so

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{1.7^2+(-4.7)^2+(-1.2)^2} = \sqrt{26.42} \approx 5.14.$$

17. **(a)** $\mathbf{u}\cdot\mathbf{v}$ is a real number, so $\|\mathbf{u}\cdot\mathbf{v}\|$ is the norm of a number, which is not defined.

(b) $\mathbf{u}\cdot\mathbf{v}$ is a scalar, while $\mathbf{w}$ is a vector. Thus $\mathbf{u}\cdot\mathbf{v}+\mathbf{w}$ adds a scalar to a vector, which is not a defined operation.

(c) $\mathbf{u}$ is a vector, while $\mathbf{v}\cdot\mathbf{w}$ is a scalar. Thus $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$ is the dot product of a vector and a scalar, which is not defined.

(d) $c\cdot(\mathbf{u}+\mathbf{v})$ is the dot product of a scalar and a vector, which is not defined.

18. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{3\cdot(-1)+0\cdot 1}{\sqrt{3^2+0^2}\sqrt{(-1)^2+1^2}} = -\frac{3}{3\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Thus $\cos\theta < 0$ (in fact, $\theta = \frac{3\pi}{4}$), so the angle between $\mathbf{u}$ and $\mathbf{v}$ is obtuse.

19. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{2\cdot 1+(-1)\cdot(-2)+1\cdot(-1)}{\sqrt{2^2+(-1)^2+1^2}\sqrt{1^2+(-2)^2+1^2}} = \frac{1}{2}.$$

Thus $\cos\theta > 0$ (in fact, $\theta = \frac{\pi}{3}$), so the angle between $\mathbf{u}$ and $\mathbf{v}$ is acute.

20. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{4\cdot 1+3\cdot(-1)+(-1)\cdot 1}{\sqrt{4^2+3^2+(-1)^2}\sqrt{1^2+(-1)^2+1^2}} = \frac{0}{\sqrt{26}\sqrt{3}} = 0.$$

Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is a right angle.

21. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Note that we can determine whether θ is acute, right, or obtuse by examining the sign of $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$, which is determined by the sign of $\mathbf{u}\cdot\mathbf{v}$. Since

$$\mathbf{u}\cdot\mathbf{v} = 0.9\cdot(-4.5)+2.1\cdot 2.6+1.2\cdot(-0.8) = 0.45 > 0,$$

we have $\cos\theta > 0$ so that θ is acute.

22. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Note that we can determine whether θ is acute, right, or obtuse by examining the sign of $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$, which is determined by the sign of $\mathbf{u}\cdot\mathbf{v}$. Since

$$\mathbf{u}\cdot\mathbf{v} = 1\cdot(-3)+2\cdot 1+3\cdot 2+4\cdot(-2) = -3,$$

we have $\cos\theta < 0$ so that θ is obtuse.

23. Since the components of both $\mathbf{u}$ and $\mathbf{v}$ are positive, it is clear that $\mathbf{u}\cdot\mathbf{v} > 0$, so the angle between them is acute since it has a positive cosine.

24. From Exercise 18, $\cos\theta = -\frac{\sqrt{2}}{2}$, so that $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4} = 135°$.

25. From Exercise 19, $\cos\theta = \frac{1}{2}$, so that $\theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} = 60°$.

26. From Exercise 20, $\cos\theta = 0$, so that $\theta = \frac{\pi}{2} = 90°$ is a right angle.

27. As in Example 1.21, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ and the norms of the two vectors:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 0.9\cdot(-4.5)+2.1\cdot 2.6+1.2\cdot(-0.8)=0.45,\\ \|\mathbf{u}\| &= \sqrt{0.9^2+2.1^2+1.2^2}=\sqrt{6.66},\\ \|\mathbf{v}\| &= \sqrt{(-4.5)^2+2.6^2+(-0.8)^2}=\sqrt{27.65}.\end{aligned}$$

So if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{0.45}{\sqrt{6.66}\sqrt{27.65}}\approx\frac{0.45}{\sqrt{182.817}},$$

so that

$$\theta=\cos^{-1}\left(\frac{0.45}{\sqrt{182.817}}\right)\approx 1.5375\approx 88.09^\circ.$$

Note that it is important to maintain as much precision as possible until the last step, or roundoff errors may build up.

28. As in Example 1.21, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ and the norms of the two vectors:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 1\cdot(-3)+2\cdot 1+3\cdot 2+4\cdot(-2)=-3,\\ \|\mathbf{u}\| &= \sqrt{1^2+2^2+3^2+4^2}=\sqrt{30},\\ \|\mathbf{v}\| &= \sqrt{(-3)^2+1^2+2^2+(-2)^2}=\sqrt{18}.\end{aligned}$$

So if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=-\frac{3}{\sqrt{30}\sqrt{18}}=-\frac{1}{2\sqrt{15}}\quad\text{so that}\quad\theta=\cos^{-1}\left(-\frac{1}{2\sqrt{15}}\right)\approx 1.7\approx 97.42^\circ.$$

29. As in Example 1.21, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ and the norms of the two vectors:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 1\cdot 5+2\cdot 6+3\cdot 7+4\cdot 8=70,\\ \|\mathbf{u}\| &= \sqrt{1^2+2^2+3^2+4^2}=\sqrt{30},\\ \|\mathbf{v}\| &= \sqrt{5^2+6^2+7^2+8^2}=\sqrt{174}.\end{aligned}$$

So if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{70}{\sqrt{30}\sqrt{174}}=\frac{35}{3\sqrt{145}}\quad\text{so that}\quad\theta=\cos^{-1}\left(\frac{35}{3\sqrt{145}}\right)\approx 0.2502\approx 14.34^\circ.$$

30. To show that $\triangle ABC$ is right, we need only show that one pair of its sides meets at a right angle. So let $\mathbf{u}=\overrightarrow{AB}$, $\mathbf{v}=\overrightarrow{BC}$, and $\mathbf{w}=\overrightarrow{AC}$. Then we must show that one of $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{w}$ or $\mathbf{v}\cdot\mathbf{w}$ is zero in order to show that one of these pairs is orthogonal. Then

$$\begin{gathered}\mathbf{u}=\overrightarrow{AB}=[1-(-3),\,0-2]=[4,-2]\,,\quad \mathbf{v}=\overrightarrow{BC}=[4-1,\,6-0]=[3,6]\,,\\ \mathbf{w}=\overrightarrow{AC}=[4-(-3),\,6-2]=[7,4]\,,\end{gathered}$$

and

$$\mathbf{u}\cdot\mathbf{v}=4\cdot 3+(-2)\cdot 6=12-12=0.$$

Since this dot product is zero, these two vectors are orthogonal, so that $\overrightarrow{AB}\perp\overrightarrow{BC}$ and thus $\triangle ABC$ is a right triangle. It is unnecessary to test the remaining pairs of sides.

31. To show that $\triangle ABC$ is right, we need only show that one pair of its sides meets at a right angle. So let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{AC}$. Then we must show that one of $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{w}$ or $\mathbf{v}\cdot\mathbf{w}$ is zero in order to show that one of these pairs is orthogonal. Then

$$\begin{aligned}\mathbf{u} &= \overrightarrow{AB} = [-3-1,\, 2-1,\, (-2)-(-1)] = [-4,1,-1]\,,\\ \mathbf{v} &= \overrightarrow{BC} = [2-(-3),\, 2-2,\, -4-(-2)] = [5,0,-2],\\ \mathbf{w} &= \overrightarrow{AC} = [2-1,\, 2-1,\, -4-(-1)] = [1,1,-3],\end{aligned}$$

and

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= -4\cdot 5 + 1\cdot 0 - 1\cdot(-2) = -18\\ \mathbf{u}\cdot\mathbf{w} &= -4\cdot 1 + 1\cdot 1 - 1\cdot(-3) = 0.\end{aligned}$$

Since this dot product is zero, these two vectors are orthogonal, so that $\overrightarrow{AB}\perp\overrightarrow{AC}$ and thus $\triangle ABC$ is a right triangle. It is unnecessary to test the remaining pair of sides.

32. As in Example 1.22, the dimensions of the cube do not matter, so we work with a cube with side length 1. Since the cube is symmetric, we need only consider one diagonal and adjacent edge. Orient the cube as shown in Figure 1.34; take the diagonal to be $[1,1,1]$ and the adjacent edge to be $[1,0,0]$. Then the angle θ between these two vectors satisfies

$$\cos\theta = \frac{1\cdot 1 + 1\cdot 0 + 1\cdot 0}{\sqrt{3}\sqrt{1}} = \frac{1}{\sqrt{3}}, \qquad \text{so} \qquad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74^\circ.$$

Thus the diagonal and an adjacent edge meet at an angle of 54.74°.

33. As in Example 1.22, the dimensions of the cube do not matter, so we work with a cube with side length 1. Since the cube is symmetric, we need only consider one pair of diagonals. Orient the cube as shown in Figure 1.34; take the diagonals to be $\mathbf{u} = [1,1,1]$ and $\mathbf{v} = [1,1,0] - [0,0,1] = [1,1,-1]$. Then the dot product is

$$\mathbf{u}\cdot\mathbf{v} = 1\cdot 1 + 1\cdot 1 + 1\cdot(-1) = 1 + 1 - 1 = 1 \neq 0.$$

Since the dot product is nonzero, the diagonals are not orthogonal.

34. To show a parallelogram is a rhombus, it suffices to show that its diagonals are perpendicular (Euclid). But

$$\mathbf{d}_1\cdot\mathbf{d}_2 = \begin{bmatrix}2\\2\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\3\end{bmatrix} = 2\cdot 1 + 2\cdot(-1) + 0\cdot 3 = 0.$$

To determine its side length, note that since the diagonals are perpendicular, one half of each diagonal are the legs of a right triangle whose hypotenuse is one side of the rhombus. So we can use the Pythagorean Theorem. Since

$$\|\mathbf{d}_1\|^2 = 2^2 + 2^2 + 0^2 = 8, \qquad \|\mathbf{d}_2\|^2 = 1^2 + (-1)^2 + 3^2 = 11,$$

we have for the side length

$$s^2 = \left(\frac{\|\mathbf{d}_1\|}{2}\right)^2 + \left(\frac{\|\mathbf{d}_2\|}{2}\right)^2 = \frac{8}{4} + \frac{11}{4} = \frac{19}{4},$$

so that $s = \frac{\sqrt{19}}{2} \approx 2.18$.

35. Since $ABCD$ is a rectangle, opposite sides BA and CD are parallel and congruent. So we can use the method of Example 1.1 in Section 1.1 to find the coordinates of vertex D: we compute $\overrightarrow{BA} = [1-3, 2-6, 3-(-2)] = [-2,-4,5]$. If $\overrightarrow{BA}$ is then translated to $\overrightarrow{CD}$, where $C = (0,5,-4)$, then

$$D = (0+(-2),\, 5+(-4),\, -4+5) = (-2,1,1).$$

36. The resultant velocity of the airplane is the sum of the velocity of the airplane and the velocity of the wind:

$$\mathbf{r} = \mathbf{p} + \mathbf{w} = \begin{bmatrix} 200 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -40 \end{bmatrix} = \begin{bmatrix} 200 \\ -40 \end{bmatrix}.$$

37. Let the x direction be east, in the direction of the current, and the y direction be north, across the river. The speed of the boat is 4 mph north, and the current is 3 mph east, so the velocity of the boat is

$$\mathbf{v} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

38. Let the x direction be the direction across the river, and the y direction be downstream. Since $\mathbf{v}t = \mathbf{d}$, use the given information to find $\mathbf{v}$, then solve for t and compute $\mathbf{d}$. Since the speed of the boat is 20 km/h and the speed of the current is 5 km/h, we have $\mathbf{v} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}$. The width of the river is 2 km, and the distance downstream is unknown; call it y. Then $\mathbf{d} = \begin{bmatrix} 2 \\ y \end{bmatrix}$. Thus

$$\mathbf{v}t = \begin{bmatrix} 20 \\ 5 \end{bmatrix} t = \begin{bmatrix} 2 \\ y \end{bmatrix}.$$

Thus $20t = 2$ so that $t = 0.1$, and then $y = 5 \cdot 0.1 = 0.5$. Therefore

(a) Ann lands 0.5 km, or half a kilometer, downstream;

(b) It takes Ann 0.1 hours, or six minutes, to cross the river.

Note that the river flow does not increase the time required to cross the river, since its velocity is perpendicular to the direction of travel.

39. We want to find the angle between Bert's resultant vector, $\mathbf{r}$, and his velocity vector upstream, $\mathbf{v}$. Let the first coordinate of the vector be the direction across the river, and the second be the direction upstream. Bert's velocity vector directly across the river is unknown, say $\mathbf{u} = \begin{bmatrix} x \\ 0 \end{bmatrix}$. His velocity vector upstream compensates for the downstream flow, so $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So the resultant vector is $\mathbf{r} = \mathbf{u} + \mathbf{v} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 1 \end{bmatrix}$. Since Bert's speed is 2 mph, we have $\|\mathbf{r}\| = 2$. Thus

$$x^2 + 1 = \|\mathbf{r}\|^2 = 4, \quad \text{so that} \quad x = \sqrt{3}.$$

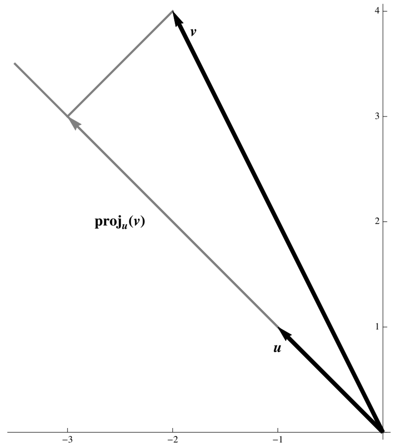
If θ is the angle between $\mathbf{r}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{r} \cdot \mathbf{v}}{\|\mathbf{r}\| \, \|\mathbf{v}\|} = \frac{\sqrt{3}}{2}, \quad \text{so that} \quad \theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ.$$

40. We have

$$\text{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{(-1) \cdot (-2) + 1 \cdot 4}{(-1) \cdot (-1) + 1 \cdot 1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

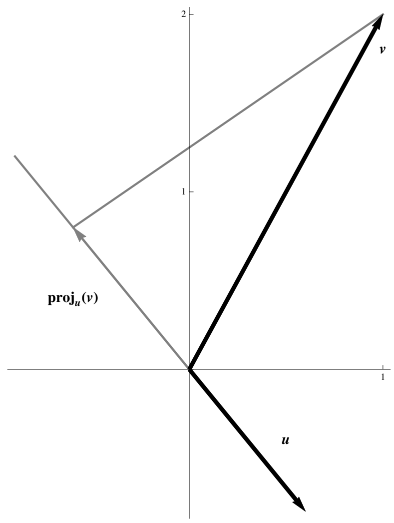
A graph of the situation is (with $\text{proj}_{\mathbf{u}} \mathbf{v}$ in gray, and the perpendicular from $\mathbf{v}$ to the projection also drawn)



41. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{\frac{3}{5}\cdot 1 + \left(-\frac{4}{5}\cdot 2\right)}{\frac{3}{5}\cdot\frac{3}{5} + \left(-\frac{4}{5}\right)\cdot\left(-\frac{4}{5}\right)}\begin{bmatrix}\frac{3}{5}\\ -\frac{4}{5}\end{bmatrix} = -\frac{1}{1}\mathbf{u} = -\mathbf{u}.$$

A graph of the situation is (with $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ in gray, and the perpendicular from $\mathbf{v}$ to the projection also drawn)



42. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{\frac{1}{2}\cdot 2 - \frac{1}{4}\cdot 2 - \frac{1}{2}\cdot(-2)}{\frac{1}{2}\cdot\frac{1}{2} + \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)}\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{4}\\ -\frac{1}{2}\end{bmatrix} = \frac{8}{3}\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{4}\\ -\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{4}{3}\\ -\frac{2}{3}\\ -\frac{4}{3}\end{bmatrix}.$$

43. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{1\cdot 2 + (-1)\cdot(-3) + 1\cdot(-1) + (-1)\cdot(-2)}{1\cdot 1 + (-1)\cdot(-1) + 1\cdot 1 + (-1)\cdot(-1)}\begin{bmatrix}1\\ -1\\ 1\\ -1\end{bmatrix} = \frac{6}{4}\begin{bmatrix}1\\ -1\\ 1\\ -1\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\ -\frac{3}{2}\\ \frac{3}{2}\\ -\frac{3}{2}\end{bmatrix} = \frac{3}{2}\mathbf{u}.$$

44. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{0.5\cdot 2.1 + 1.5\cdot 1.2}{0.5\cdot 0.5 + 1.5\cdot 1.5}\begin{bmatrix}0.5\\ 1.5\end{bmatrix} = \frac{2.85}{2.5}\begin{bmatrix}0.5\\ 1.5\end{bmatrix} = \begin{bmatrix}0.57\\ 1.71\end{bmatrix} = 1.14\mathbf{u}.$$

45. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{3.01\cdot 1.34 - 0.33\cdot 4.25 + 2.52\cdot(-1.66)}{3.01\cdot 3.01 - 0.33\cdot(-0.33) + 2.52\cdot 2.52}\begin{bmatrix}3.01\\-0.33\\2.52\end{bmatrix}$$

$$= -\frac{1.5523}{15.5194}\begin{bmatrix}3.01\\-0.33\\2.52\end{bmatrix} \approx \begin{bmatrix}-0.301\\0.033\\-0.252\end{bmatrix} \approx -\frac{1}{10}\mathbf{u}.$$

46. Let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix}2-1\\2-(-1)\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix}4-1\\0-(-1)\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$.

(a) To compute the projection, we need

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix} = 6, \qquad \mathbf{u}\cdot\mathbf{u} = \begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix} = 10.$$

Thus

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{3}{5}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}\frac{3}{5}\\\frac{9}{5}\end{bmatrix},$$

so that

$$\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v} = \begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}\frac{3}{5}\\\frac{9}{5}\end{bmatrix} = \begin{bmatrix}\frac{12}{5}\\-\frac{4}{5}\end{bmatrix}.$$

Then

$$\|\mathbf{u}\| = \sqrt{1^2+3^2} = \sqrt{10}, \quad \|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{4\sqrt{10}}{5},$$

so that finally

$$\mathcal{A} = \frac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \frac{1}{2}\sqrt{10}\cdot\frac{4\sqrt{10}}{5} = 4.$$

(b) We already know $\mathbf{u}\cdot\mathbf{v} = 6$ and $\|\mathbf{u}\| = \sqrt{10}$ from part (a). Also, $\|\mathbf{v}\| = \sqrt{3^2+1^2} = \sqrt{10}$. So

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{6}{\sqrt{10}\sqrt{10}} = \frac{3}{5},$$

so that

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}.$$

Thus

$$\mathcal{A} = \frac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta = \frac{1}{2}\sqrt{10}\sqrt{10}\cdot\frac{4}{5} = 4.$$

47. Let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix}4-3\\-2-(-1)\\6-4\end{bmatrix} = \begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix}5-3\\0-(-1)\\2-4\end{bmatrix} = \begin{bmatrix}2\\1\\-2\end{bmatrix}$.

(a) To compute the projection, we need

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}1\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix} = -3, \qquad \mathbf{u}\cdot\mathbf{u} = \begin{bmatrix}1\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix} = 6.$$

Thus

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = -\frac{3}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\-1\end{bmatrix},$$

so that

$$\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

Then

$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}, \quad \|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2} = \frac{\sqrt{30}}{2},$$

so that finally

$$\mathcal{A} = \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \frac{1}{2}\sqrt{6} \cdot \frac{\sqrt{30}}{2} = \frac{3\sqrt{5}}{2}.$$

(b) We already know $\mathbf{u} \cdot \mathbf{v} = -3$ and $\|\mathbf{u}\| = \sqrt{6}$ from part (a). Also, $\|\mathbf{v}\| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$. So

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-3}{3\sqrt{6}} = -\frac{\sqrt{6}}{6},$$

so that

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{\sqrt{6}}{6}\right)^2} = \frac{\sqrt{30}}{6}.$$

Thus

$$\mathcal{A} = \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta = \frac{1}{2}\sqrt{6} \cdot 3 \cdot \frac{\sqrt{30}}{6} = \frac{3\sqrt{5}}{2}.$$

48. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their dot product $\mathbf{u} \cdot \mathbf{v} = 0$. So we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for k:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} k+1 \\ k-1 \end{bmatrix} = 0 \;\Rightarrow\; 2(k+1) + 3(k-1) = 0 \;\Rightarrow\; 5k - 1 = 0 \;\Rightarrow k = \frac{1}{5}.$$

Substituting into the formula for $\mathbf{v}$ gives

$$\mathbf{v} = \begin{bmatrix} \frac{1}{5} + 1 \\ \frac{1}{5} - 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \end{bmatrix}.$$

As a check, we compute

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \end{bmatrix} = \frac{12}{5} - \frac{12}{5} = 0,$$

and the vectors are indeed orthogonal.

49. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their dot product $\mathbf{u} \cdot \mathbf{v} = 0$. So we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for k:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix} = 0 \;\Rightarrow\; k^2 - k - 6 = 0 \;\Rightarrow\; (k+2)(k-3) = 0 \;\Rightarrow k = 2, -3.$$

Substituting into the formula for $\mathbf{v}$ gives

$$k = 2: \;\mathbf{v}_1 = \begin{bmatrix} (-2)^2 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}, \qquad k = -3: \;\mathbf{v}_2 = \begin{bmatrix} 3^2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix}.$$

As a check, we compute

$$\mathbf{u} \cdot \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix} = 1 \cdot 4 - 1 \cdot (-2) + 2 \cdot (-3) = 0, \; \mathbf{u} \cdot \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix} = 1 \cdot 9 - 1 \cdot 3 + 2 \cdot (-3) = 0$$

and the vectors are indeed orthogonal.

50. Two vectors **u** and **v** are orthogonal if and only if their dot product $\mathbf{u}\cdot\mathbf{v}=0$. So we set $\mathbf{u}\cdot\mathbf{v}=0$ and solve for y in terms of x:

$$\mathbf{u}\cdot\mathbf{v}=\begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=0 \;\Rightarrow\; 3x+y=0 \;\Rightarrow\; y=-3x.$$

Substituting $y=-3x$ back into the formula for **v** gives

$$\mathbf{v}=\begin{bmatrix}x\\-3x\end{bmatrix}=x\begin{bmatrix}1\\-3\end{bmatrix}.$$

Thus any vector orthogonal to $\begin{bmatrix}3\\1\end{bmatrix}$ is a multiple of $\begin{bmatrix}1\\-3\end{bmatrix}$. As a check,

$$\mathbf{u}\cdot\mathbf{v}=\begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}x\\-3x\end{bmatrix}=3x-3x=0 \text{ for any value of } x,$$

so that the vectors are indeed orthogonal.

51. As noted in the remarks just prior to Example 1.16, the zero vector **0** is orthogonal to all vectors in $\mathbb{R}^2$. So if $\begin{bmatrix}a\\b\end{bmatrix}=\mathbf{0}$, any vector $\begin{bmatrix}x\\y\end{bmatrix}$ will do. Now assume that $\begin{bmatrix}a\\b\end{bmatrix}\neq\mathbf{0}$; that is, that either a or b is nonzero. Two vectors **u** and **v** are orthogonal if and only if their dot product $\mathbf{u}\cdot\mathbf{v}=0$. So we set $\mathbf{u}\cdot\mathbf{v}=0$ and solve for y in terms of x:

$$\mathbf{u}\cdot\mathbf{v}=\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=0 \;\Rightarrow\; ax+by=0.$$

First assume $b\neq 0$. Then $y=-\frac{a}{b}x$, so substituting back into the expression for **v** we get

$$\mathbf{v}=\begin{bmatrix}x\\-\frac{a}{b}x\end{bmatrix}=x\begin{bmatrix}1\\-\frac{a}{b}\end{bmatrix}=\frac{x}{b}\begin{bmatrix}b\\-a\end{bmatrix}.$$

Next, if $b=0$, then $a\neq 0$, so that $x=-\frac{b}{a}y$, and substituting back into the expression for **v** gives

$$\mathbf{v}=\begin{bmatrix}-\frac{b}{a}y\\y\end{bmatrix}=y\begin{bmatrix}-\frac{b}{a}\\1\end{bmatrix}=-\frac{y}{a}\begin{bmatrix}b\\-a\end{bmatrix}.$$

So in either case, a vector orthogonal to $\begin{bmatrix}a\\b\end{bmatrix}$, if it is not the zero vector, is a multiple of $\begin{bmatrix}b\\-a\end{bmatrix}$. As a check, note that

$$\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}rb\\-ra\end{bmatrix}=rab-rab=0 \text{ for all values of } r.$$

52. (a) The geometry of the vectors in Figure 1.26 suggests that if $\|\mathbf{u}+\mathbf{v}\|=\|\mathbf{u}\|+\|\mathbf{v}\|$, then **u** and **v** point in the same direction. This means that the angle between them must be 0. So we first prove

Lemma 1. *For all vectors* **u** *and* **v** *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$, $\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\,\|\mathbf{v}\|$ *if and only if the vectors point in the same direction.*

Proof. Let θ be the angle between **u** and **v**. Then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},$$

so that $\cos\theta=1$ if and only if $\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\,\|\mathbf{v}\|$. But $\cos\theta=1$ if and only if $\theta=0$, which means that **u** and **v** point in the same direction. □

We can now show

Theorem 2. *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$, $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *point in the same direction.*

Proof. First assume that $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. Then $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|$, and thus

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} && \text{By Example 1.9}\\
&= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 && \text{Since } \mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2 \text{ for any vector } \mathbf{w}\\
&= \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \text{By the lemma}\\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.
\end{aligned}$$

Since $\|\mathbf{u}+\mathbf{v}\|$ and $\|\mathbf{u}\|+\|\mathbf{v}\|$ are both nonnegative, taking square roots gives $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\|+\|\mathbf{v}\|$. For the other direction, if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$, then their squares are equal, so that

$$\begin{aligned}
(\|\mathbf{u}\| + \|\mathbf{v}\|)^2 &= \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \quad \text{and}\\
\|\mathbf{u}+\mathbf{v}\|^2 &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}
\end{aligned}$$

are equal. But $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u}$ and similarly for $\mathbf{v}$, so that canceling those terms gives $2\mathbf{u}\cdot\mathbf{v} = 2\|\mathbf{u}\|\,\|\mathbf{v}\|$ and thus $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|$. Using the lemma again shows that $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. □

(b) The geometry of the vectors in Figure 1.26 suggests that if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$, then $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions. In addition, since $\|\mathbf{u}+\mathbf{v}\| \ge 0$, we must also have $\|\mathbf{u}\| \ge \|\mathbf{v}\|$. If they point in opposite directions, the angle between them must be π. This entire proof is exactly analogous to the proof in part (a). We first prove

Lemma 3. *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$, $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\,\|\mathbf{v}\|$ *if and only if the vectors point in opposite directions.*

Proof. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},$$

so that $\cos\theta = -1$ if and only if $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\,\|\mathbf{v}\|$. But $\cos\theta = -1$ if and only if $\theta = \pi$, which means that $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions. □

We can now show

Theorem 4. *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$, $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *point in opposite directions and* $\|\mathbf{u}\| \ge \|\mathbf{v}\|$.

Proof. First assume that $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions and $\|\mathbf{u}\| \ge \|\mathbf{v}\|$. Then $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\,\|\mathbf{v}\|$, and thus

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} && \text{By Example 1.9}\\
&= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 && \text{Since } \mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2 \text{ for any vector } \mathbf{w}\\
&= \|\mathbf{u}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \text{By the lemma}\\
&= (\|\mathbf{u}\| - \|\mathbf{v}\|)^2.
\end{aligned}$$

Now, since $\|\mathbf{u}\| \ge \|\mathbf{v}\|$ by assumption, we see that both $\|\mathbf{u}+\mathbf{v}\|$ and $\|\mathbf{u}\| - \|\mathbf{v}\|$ are nonnegative, so that taking square roots gives $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$. For the other direction, if $\|\mathbf{u}+\mathbf{v}\| =$

$\|\mathbf{u}\| - \|\mathbf{v}\|$, then first of all, since the left-hand side is nonnegative, the right-hand side must be as well, so that $\|\mathbf{u}\| \geq \|\mathbf{v}\|$. Next, we can square both sides of the equality, so that

$$(\|\mathbf{u}\| - \|\mathbf{v}\|)^2 = \|\mathbf{u}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \text{ and}$$
$$\|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}$$

are equal. But $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u}$ and similarly for $\mathbf{v}$, so that canceling those terms gives $2\mathbf{u}\cdot\mathbf{v} = -2\|\mathbf{u}\|\|\mathbf{v}\|$ and thus $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\|\mathbf{v}\|$. Using the lemma again shows that $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions. □

53. Prove Theorem 1.2(b) by applying the definition of the dot product:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_1(v_1+w_1) + u_2(v_2+w_2) + \cdots + u_n(v_n+w_n)\\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \cdots + u_nv_n + u_nw_n\\ &= (u_1v_1 + u_2v_2 + \cdots + u_nv_n) + (u_1w_1 + u_2w_2 + \cdots + u_nw_n)\\ &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}.\end{aligned}$$

54. Prove the three parts of Theorem 1.2(d) by applying the definition of the dot product and various properties of real numbers:

***Part 1*:** For any vector $\mathbf{u}$, we must show $\mathbf{u}\cdot\mathbf{u} \geq 0$. But

$$\mathbf{u}\cdot\mathbf{u} = u_1u_1 + u_2u_2 + \cdots + u_nu_n = u_1^2 + u_2^2 + \cdots + u_n^2.$$

Since for any real number x we know that $x^2 \geq 0$, it follows that this sum is also nonnegative, so that $\mathbf{u}\cdot\mathbf{u} \geq 0$.

***Part 2*:** We must show that if $\mathbf{u} = \mathbf{0}$ then $\mathbf{u}\cdot\mathbf{u} = 0$. But $\mathbf{u} = 0$ means that $u_i = 0$ for all i, so that

$$\mathbf{u}\cdot\mathbf{u} = 0\cdot 0 + 0\cdot 0 + \cdots + 0\cdot 0 = 0.$$

***Part 3*:** We must show that if $\mathbf{u}\cdot\mathbf{u} = 0$, then $\mathbf{u} = \mathbf{0}$. From part 1, we know that

$$\mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2 + \cdots + u_n^2,$$

and that $u_i^2 \geq 0$ for all i. So if the dot product is to be zero, each u_i^2 must be zero, which means that $u_i = 0$ for all i and thus $\mathbf{u} = \mathbf{0}$.

55. We must show $\mathrm{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|\mathbf{v}-\mathbf{u}\| = \mathrm{d}(\mathbf{v},\mathbf{u})$. By definition, $\mathrm{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$. Then by Theorem 1.3(b) with $c = -1$, we have $\|-\mathbf{w}\| = \|\mathbf{w}\|$ for any vector $\mathbf{w}$; applying this to the vector $\mathbf{u}-\mathbf{v}$ gives

$$\|\mathbf{u}-\mathbf{v}\| = \|-(\mathbf{u}-\mathbf{v})\| = \|\mathbf{v}-\mathbf{u}\|,$$

which is by definition equal to $\mathrm{d}(\mathbf{v},\mathbf{u})$.

56. We must show that for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ that $\mathrm{d}(\mathbf{u},\mathbf{w}) \leq \mathrm{d}(\mathbf{u},\mathbf{v}) + \mathrm{d}(\mathbf{v},\mathbf{w})$. This is equivalent to showing that $\|\mathbf{u}-\mathbf{w}\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\|$. Now substitute $\mathbf{u}-\mathbf{v}$ for x and $\mathbf{v}-\mathbf{w}$ for y in Theorem 1.5, giving

$$\|\mathbf{u}-\mathbf{w}\| = \|(\mathbf{u}-\mathbf{v}) + (\mathbf{v}-\mathbf{w})\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\|.$$

57. We must show that $\mathrm{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = 0$ if and only if $\mathbf{u} = \mathbf{v}$. This follows immediately from Theorem 1.3(a), $\|\mathbf{w}\| = 0$ if and only if $\mathbf{w} = \mathbf{0}$, upon setting $\mathbf{w} = \mathbf{u}-\mathbf{v}$.

58. Apply the definitions:

$$\begin{aligned}\mathbf{u}\cdot c\mathbf{v} &= [u_1, u_2, \ldots, u_n]\cdot[cv_1, cv_2, \ldots, cv_n]\\ &= u_1cv_1 + u_2cv_2 + \cdots + u_ncv_n\\ &= cu_1v_1 + cu_2v_2 + \cdots + cu_nv_n\\ &= c(u_1v_1 + u_2v_2 + \cdots + u_nv_n)\\ &= c(\mathbf{u}\cdot\mathbf{v}).\end{aligned}$$

59. We want to show that $\|\mathbf{u}-\mathbf{v}\| \geq \|\mathbf{u}\| - \|\mathbf{v}\|$. This is equivalent to showing that $\|\mathbf{u}\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}\|$. This follows immediately upon setting $\mathbf{x} = \mathbf{u}-\mathbf{v}$ and $\mathbf{y} = \mathbf{v}$ in Theorem 1.5.

60. If $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$, it does *not* follow that $\mathbf{v} = \mathbf{w}$. For example, since $\mathbf{0}\cdot\mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathbb{R}^n$, the zero vector is orthogonal to every vector $\mathbf{v}$. So if $\mathbf{u} = \mathbf{0}$ in the above equality, we know nothing about $\mathbf{v}$ and $\mathbf{w}$. (as an example, $\mathbf{0}\cdot[1,2] = \mathbf{0}\cdot[-17,12]$). Note, however, that $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$ implies that $\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{w} = \mathbf{u}(\mathbf{v}-\mathbf{w}) = \mathbf{0}$, so that $\mathbf{u}$ is orthogonal to $\mathbf{v}-\mathbf{w}$.

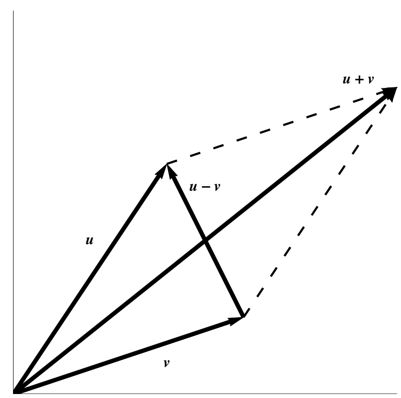
61. We must show that $(\mathbf{u}+\mathbf{v})(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for all vectors in $\mathbb{R}^n$. Recall that for any $\mathbf{w}$ in $\mathbb{R}^n$ that $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2$, and also that $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$. Then

$$(\mathbf{u}+\mathbf{v})(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 + \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2.$$

62. **(a)** Let $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. Then

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v}) \\ &= \left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) + 2\mathbf{u}\cdot\mathbf{v} + \left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) - 2\mathbf{u}\cdot\mathbf{v} \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.\end{aligned}$$

(b) Part (a) tells us that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its four sides.



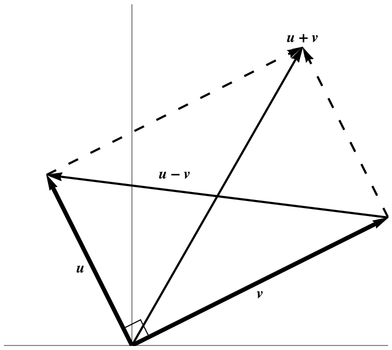
63. Let $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. Then

$$\begin{aligned}\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 &= \frac{1}{4}\left[(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) - ((\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\right] \\ &= \frac{1}{4}\left[(\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v})\right] \\ &= \frac{1}{4}\left[\left(\|\mathbf{u}\|^2 - \|\mathbf{u}\|^2\right) + \left(\|\mathbf{v}\|^2 - \|\mathbf{v}\|^2\right) + 4\mathbf{u}\cdot\mathbf{v}\right] \\ &= \mathbf{u}\cdot\mathbf{v}.\end{aligned}$$

64. **(a)** Let $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. Then using the previous exercise,

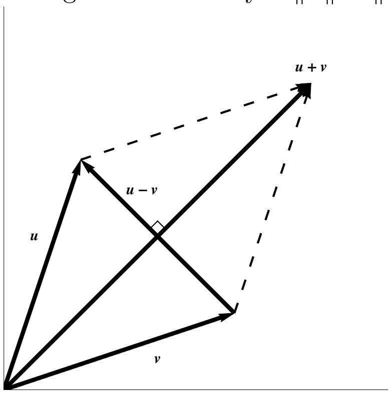
$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\| &\Leftrightarrow \|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2 \\ &\Leftrightarrow \|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = 0 \\ &\Leftrightarrow \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 = 0 \\ &\Leftrightarrow \mathbf{u}\cdot\mathbf{v} = 0 \\ &\Leftrightarrow \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}\end{aligned}$$

(b) Part (a) tells us that a parallelogram is a rectangle if and only if the lengths of its diagonals are equal.



65. **(a)** By Exercise 55, $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$. Thus $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = 0$ if and only if $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$. It follows immediately that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

(b) Part (a) tells us that the diagonals of a parallelogram are perpendicular if and only if the lengths of its sides are equal, i.e., if and only if it is a rhombus.



66. From Example 1.9 and the fact that $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2$, we have $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$. Taking the square root of both sides yields $\|\mathbf{u}+\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2}$. Now substitute in the given values $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = \sqrt{3}$, and $\mathbf{u}\cdot\mathbf{v} = 1$, giving

$$\|\mathbf{u}+\mathbf{v}\| = \sqrt{2^2 + 2\cdot 1 + \left(\sqrt{3}\right)^2} = \sqrt{4+2+3} = \sqrt{9} = 3.$$

67. From Theorem 1.4 (the Cauchy-Schwarcz inequality), we have $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$. If $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 2$, then $|\mathbf{u}\cdot\mathbf{v}| \le 2$, so we cannot have $\mathbf{u}\cdot\mathbf{v} = 3$.

68. **(a)** If $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$. Then

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} = 0 + 0 = 0,$$

so that $\mathbf{u}$ is orthogonal to $\mathbf{v}+\mathbf{w}$.

(b) If $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$. Then

$$\mathbf{u}\cdot(s\mathbf{v}+t\mathbf{w}) = \mathbf{u}\cdot(s\mathbf{v}) + \mathbf{u}\cdot(t\mathbf{w}) = s(\mathbf{u}\cdot\mathbf{v}) + t(\mathbf{u}\cdot\mathbf{w}) = s\cdot 0 + t\cdot 0 = 0,$$

so that $\mathbf{u}$ is orthogonal to $s\mathbf{v}+t\mathbf{w}$.

69. We have

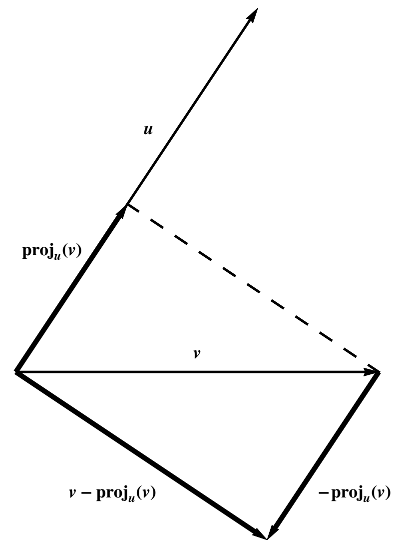
$$\begin{aligned}
\mathbf{u}\cdot(\mathbf{v} - \operatorname{proj}_{\mathbf{u}}\mathbf{v}) &= \mathbf{u}\cdot\left(\mathbf{v} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right) \\
&= \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} \\
&= \mathbf{u}\cdot\mathbf{v} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)(\mathbf{u}\cdot\mathbf{u}) \\
&= \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} \\
&= 0.
\end{aligned}$$

70. **(a)** $\text{proj}_{\mathbf{u}}(\text{proj}_{\mathbf{u}}\,\mathbf{v}) = \text{proj}_{\mathbf{u}}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\,\text{proj}_{\mathbf{u}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \text{proj}_{\mathbf{u}}\,\mathbf{v}$.

(b) Using part (a),

$$\begin{aligned}\text{proj}_{\mathbf{u}}\left(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}\right) = \text{proj}_{\mathbf{u}}\left(\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\text{proj}_{\mathbf{u}}\,\mathbf{u} \\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \mathbf{0}.\end{aligned}$$

(c) From the diagram, we see that $\text{proj}_{\mathbf{u}}\,\mathbf{v} \| \mathbf{u}$, so that $\text{proj}_{\mathbf{u}}\left(\text{proj}_{\mathbf{u}}\,\mathbf{v}\right) = \text{proj}_{\mathbf{u}}\,\mathbf{v}$. Also, $\left(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}\right) \perp \mathbf{u}$, so that $\text{proj}_{\mathbf{u}}\left(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}\right) = \mathbf{0}$.



71. **(a)** We have

$$\begin{aligned}(u_1^2+u_2^2)(v_1^2+v_2^2) - (u_1v_1+u_2v_2)^2 &= u_1^2v_1^2 + u_1^2v_2^2 + u_2^2v_1^2 + u_2^2v_2^2 - u_1^2v_1^2 - 2u_1v_1u_2v_2 - u_2^2v_2^2 \\ &= u_1^2v_2^2 + u_2^2v_1^2 - 2u_1u_2v_1v_2 \\ &= (u_1v_2 - u_2v_1)^2.\end{aligned}$$

But the final expression is nonnegative since it is a square. Thus the original expression is as well, showing that $(u_1^2+u_2^2)(v_1^2+v_2^2) - (u_1v_1+u_2v_2)^2 \ge 0$.

(b) We have

$$\begin{aligned}&(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2 \\ &\quad= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \\ &\qquad\quad - u_1^2v_1^2 - 2u_1v_1u_2v_2 - u_2^2v_2^2 - 2u_1v_1u_3v_3 - u_3^2v_3^2 - 2u_2v_2u_3v_3 \\ &\quad= u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 \\ &\qquad\quad - 2u_1u_2v_1v_2 - 2u_1v_1u_3v_3 - 2u_2v_2u_3v_3 \\ &\quad= (u_1v_2 - u_2v_1)^2 + (u_1v_3 - u_3v_1)^2 + (u_3v_2 - u_2v_3)^2.\end{aligned}$$

But the final expression is nonnegative since it is the sum of three squares. Thus the original expression is as well, showing that $(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2 \ge 0$.

72. **(a)** Since $\text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$, we have

$$\begin{aligned}\text{proj}_{\mathbf{u}}\,\mathbf{v}\cdot(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}) &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\cdot\left(\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) \\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\left(\mathbf{u}\cdot\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\cdot\mathbf{u}\right) \\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}(\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v}) \\ &= 0,\end{aligned}$$

so that $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ is orthogonal to $\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}$. Since their vector sum is $\mathbf{v}$, those three vectors form a right triangle with hypotenuse $\mathbf{v}$, so by Pythagoras' Theorem,

$$\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|^2 \le \|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|^2 + \|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\|^2 = \|\mathbf{v}\|^2 .$$

Since norms are always nonnegative, taking square roots gives $\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\| \le \|\mathbf{v}\|$.

(b)

$$\begin{aligned}
\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\| \le \|\mathbf{v}\| &\iff \left\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right\| \le \|\mathbf{v}\| \\
&\iff \left\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\right)\mathbf{u}\right\| \le \|\mathbf{v}\| \\
&\iff \left|\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\right| \|\mathbf{u}\| \le \|\mathbf{v}\| \\
&\iff \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|} \le \|\mathbf{v}\| \\
&\iff |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|,
\end{aligned}$$

which is the Cauchy-Schwarcz inequality.

73. Suppose $\operatorname{proj}_{\mathbf{u}} \mathbf{v} = c\mathbf{u}$. From the figure, we see that $\cos\theta = \frac{c\|\mathbf{u}\|}{\|\mathbf{v}\|}$. But also $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. Thus these two expressions are equal, i.e.,

$$\frac{c\|\mathbf{u}\|}{\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \Rightarrow c\|\mathbf{u}\| = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|} \Rightarrow c = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{u}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}.$$

74. The basis for induction is the cases $n = 1$ and $n = 2$. The $n = 1$ case is the assertion that $\|\mathbf{v}_1\| \le \|\mathbf{v}_2\|$, which is obviously true. The $n = 2$ case is the Triangle Inequality, which is also true.

Now assume the statement holds for $n = k \ge 2$; that is, for any $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$,

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k\| \le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k\| .$$

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$ be any vectors. Then

$$\begin{aligned}
\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k + \mathbf{v}_{k+1}\| &= \|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + (\mathbf{v}_k + \mathbf{v}_{k+1})\| \\
&\le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k + \mathbf{v}_{k+1}\|
\end{aligned}$$

using the inductive hypothesis. But then using the Triangle Inequality (or the case $n = 2$ in this theorem), $\|\mathbf{v}_k + \mathbf{v}_{k+1}\| \le \|\mathbf{v}_k\| + \|\mathbf{v}_{k+1}\|$. Substituting into the above gives

$$\begin{aligned}
\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k + \mathbf{v}_{k+1}\| &\le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k + \mathbf{v}_{k+1}\| \\
&\le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k\| + \|\mathbf{v}_{k+1}\|,
\end{aligned}$$

which is what we were trying to prove.

Exploration: Vectors and Geometry

1. As in Example 1.25, let $\mathbf{p} = \overrightarrow{OP}$. Then $\mathbf{p} - \mathbf{a} = \overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$, so that $\mathbf{p} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{1}{3}(2\mathbf{a} + \mathbf{b})$. More generally, if P is the point $\frac{1}{n}$ of the way from A to B along $\overrightarrow{AB}$, then $\mathbf{p} - \mathbf{a} = \overrightarrow{AP} = \frac{1}{n}\overrightarrow{AB} = \frac{1}{n}(\mathbf{b} - \mathbf{a})$, so that $\mathbf{p} = \mathbf{a} + \frac{1}{n}(\mathbf{b} - \mathbf{a}) = \frac{1}{n}((n-1)\mathbf{a} + \mathbf{b})$.

2. Use the notation that the vector $\overrightarrow{OX}$ is written $\mathbf{x}$. Then from exercise 1, we have $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$ and $\mathbf{q} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, so that

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\overrightarrow{AB}.$$

3. Draw $\overrightarrow{AC}$. Then from exercise 2, we have $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB} = \overrightarrow{SR}$. Also draw $\overrightarrow{BD}$. Again from exercise 2, we have $\overrightarrow{PS} = \frac{1}{2}\overrightarrow{BD} = \overrightarrow{QR}$. Thus opposite sides of the quadrilateral $PQRS$ are equal. They are also parallel: indeed, $\triangle BPQ$ and $\triangle BAC$ are similar, since they share an angle and $BP : BA = BQ : BC$. Thus $\angle BPQ = \angle BAC$; since these angles are equal, $PQ \| AC$. Similarly, $SR \| AC$ so that $PQ \| SR$. In a like manner, we see that $PS \| RQ$. Thus $PQRS$ is a parallelogram.

4. Following the hint, we find $\mathbf{m}$, the point that is two-thirds of the distance from A to P. From exercise 1, we have

$$\mathbf{p} = \frac{1}{2}(\mathbf{b}+\mathbf{c}), \text{ so that } \mathbf{m} = \frac{1}{3}(2\mathbf{p}+\mathbf{a}) = \frac{1}{3}\left(2\cdot\frac{1}{2}(\mathbf{b}+\mathbf{c})+\mathbf{a}\right) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Next we find $\mathbf{m}'$, the point that is two-thirds of the distance from B to Q. Again from exercise 1, we have

$$\mathbf{q} = \frac{1}{2}(\mathbf{a}+\mathbf{c}), \text{ so that } \mathbf{m}' = \frac{1}{3}(2\mathbf{q}+\mathbf{b}) = \frac{1}{3}\left(2\cdot\frac{1}{2}(\mathbf{a}+\mathbf{c})+\mathbf{b}\right) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Finally we find $\mathbf{m}''$, the point that is two-thirds of the distance from C to R. Again from exercise 1, we have

$$\mathbf{r} = \frac{1}{2}(\mathbf{a}+\mathbf{b}), \text{ so that } \mathbf{m}'' = \frac{1}{3}(2\mathbf{r}+\mathbf{c}) = \frac{1}{3}\left(2\cdot\frac{1}{2}(\mathbf{a}+\mathbf{b})+\mathbf{c}\right) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Since $\mathbf{m} = \mathbf{m}' = \mathbf{m}''$, all three medians intersect at the centroid, G.

5. With notation as in the figure, we know that $\overrightarrow{AH}$ is orthogonal to $\overrightarrow{BC}$; that is, $\overrightarrow{AH}\cdot\overrightarrow{BC} = 0$. Also $\overrightarrow{BH}$ is orthogonal to $\overrightarrow{AC}$; that is, $\overrightarrow{BH}\cdot\overrightarrow{AC} = 0$. We must show that $\overrightarrow{CH}\cdot\overrightarrow{AB} = 0$. But

$$\overrightarrow{AH}\cdot\overrightarrow{BC} = 0 \Rightarrow (\mathbf{h}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c}) = 0 \Rightarrow \mathbf{h}\cdot\mathbf{b}-\mathbf{h}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} = 0$$
$$\overrightarrow{BH}\cdot\overrightarrow{AC} = 0 \Rightarrow (\mathbf{h}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a}) = 0 \Rightarrow \mathbf{h}\cdot\mathbf{c}-\mathbf{h}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a} = 0.$$

Adding these two equations together and canceling like terms gives

$$0 = \mathbf{h}\cdot\mathbf{b}-\mathbf{h}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} = (\mathbf{h}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{a}) = \overrightarrow{CH}\cdot\overrightarrow{AB},$$

so that these two are orthogonal. Thus all the altitudes intersect at the orthocenter H.

6. We are given that $\overrightarrow{QK}$ is orthogonal to $\overrightarrow{AC}$ and that $\overrightarrow{PK}$ is orthogonal to $\overrightarrow{CB}$, and must show that $\overrightarrow{RK}$ is orthogonal to $\overrightarrow{AB}$. By exercise 1, we have $\mathbf{q} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$, $\mathbf{p} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$, and $\mathbf{r} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$. Thus

$$\overrightarrow{QK}\cdot\overrightarrow{AC} = 0 \Rightarrow (\mathbf{k}-\mathbf{q})\cdot(\mathbf{c}-\mathbf{a}) = 0 \Rightarrow \left(\mathbf{k}-\frac{1}{2}(\mathbf{a}+\mathbf{c})\right)\cdot(\mathbf{c}-\mathbf{a}) = 0$$
$$\overrightarrow{PK}\cdot\overrightarrow{CB} = 0 \Rightarrow (\mathbf{k}-\mathbf{p})\cdot(\mathbf{b}-\mathbf{c}) = 0 \Rightarrow \left(\mathbf{k}-\frac{1}{2}(\mathbf{b}+\mathbf{c})\right)\cdot(\mathbf{b}-\mathbf{c}) = 0.$$

Expanding the two dot products gives

$$\mathbf{k}\cdot\mathbf{c}-\mathbf{k}\cdot\mathbf{a}-\frac{1}{2}\mathbf{a}\cdot\mathbf{c}+\frac{1}{2}\mathbf{a}\cdot\mathbf{a}-\frac{1}{2}\mathbf{c}\cdot\mathbf{c}+\frac{1}{2}\mathbf{a}\cdot\mathbf{c} = 0$$
$$\mathbf{k}\cdot\mathbf{b}-\mathbf{k}\cdot\mathbf{c}-\frac{1}{2}\mathbf{b}\cdot\mathbf{b}+\frac{1}{2}\mathbf{b}\cdot\mathbf{c}-\frac{1}{2}\mathbf{c}\cdot\mathbf{b}+\frac{1}{2}\mathbf{c}\cdot\mathbf{c} = 0.$$

Add these two together and cancel like terms to get

$$0 = \mathbf{k}\cdot\mathbf{b}-\mathbf{k}\cdot\mathbf{a}-\frac{1}{2}\mathbf{b}\cdot\mathbf{b}+\frac{1}{2}\mathbf{a}\cdot\mathbf{a} = \left(\mathbf{k}-\frac{1}{2}(\mathbf{b}+\mathbf{a})\right)\cdot(\mathbf{b}-\mathbf{a}) = (\mathbf{k}-\mathbf{r})\cdot(\mathbf{b}-\mathbf{a}) = \overrightarrow{RK}\cdot\overrightarrow{AB}.$$

Thus $\overrightarrow{RK}$ and $\overrightarrow{AB}$ are indeed orthogonal, so all the perpendicular bisectors intersect at the circumcenter.

7. Let O, the center of the circle, be the origin. Then $\mathbf{b} = -\mathbf{a}$ and $\|\mathbf{a}\|^2 = \|\mathbf{c}\|^2 = r^2$ where r is the radius of the circle. We want to show that $\overrightarrow{AC}$ is orthogonal to $\overrightarrow{BC}$. But

$$\begin{aligned}\overrightarrow{AC} \cdot \overrightarrow{BC} &= (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{b}) \\ &= (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} + \mathbf{a}) \\ &= \|\mathbf{c}\|^2 + \mathbf{c} \cdot \mathbf{a} - \|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{c} \\ &= (\mathbf{a} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{a}) + (r^2 - r^2) = 0.\end{aligned}$$

Thus the two are orthogonal, so that $\angle ACB$ is a right angle.

8. As in exercise 5, we first find $\mathbf{m}$, the point that is halfway from P to R. We have $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\mathbf{r} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$, so that

$$\mathbf{m} = \frac{1}{2}(\mathbf{p} + \mathbf{r}) = \frac{1}{2}\left(\frac{1}{2}(\mathbf{a} + \mathbf{b}) + \frac{1}{2}(\mathbf{c} + \mathbf{d})\right) = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}).$$

Similarly, we find $\mathbf{m}'$, the point that is halfway from Q to S. We have $\mathbf{q} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$ and $\mathbf{s} = \frac{1}{2}(\mathbf{a} + \mathbf{d})$, so that

$$\mathbf{m}' = \frac{1}{2}(\mathbf{q} + \mathbf{s}) = \frac{1}{2}\left(\frac{1}{2}(\mathbf{b} + \mathbf{c}) + \frac{1}{2}(\mathbf{a} + \mathbf{d})\right) = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}).$$

Thus $\mathbf{m} = \mathbf{m}'$, so that $\overrightarrow{PR}$ and $\overrightarrow{QS}$ intersect at their mutual midpoints; thus, they bisect each other.

1.3 Lines and Planes

1. **(a)** The normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, or $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \left(\mathbf{x} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = 0$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, we get

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 3x + 2y = 0.$$

The general form is $3x + 2y = 0$.

2. **(a)** The normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, or $\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \left(\mathbf{x} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 0$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, we get

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix} = 3(x - 1) - 4(y - 2) = 0.$$

Expanding and simplifying gives the general form $3x - 4y = -5$.

3. **(a)** In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and expanding the vector form from part (a) gives $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 - t \\ 3t \end{bmatrix}$, which yields the parametric form $x = 1 - t$, $y = 3t$.

4. **(a)** In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and expanding the vector form from part (a) gives the parametric form $x = -4 + t$, $y = 4 + t$.