1 Lorentz and Poincaré Invariance

1.1 Problem 1

Problem:

A muon is a more massive version of an electron and has a mass of 105.7 MeV/c^2. The dominant decay mode of the muon is to an electron, an electron antineutrino and a muon neutrino, $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$. If we have $N(t)$ unstable particles at time t then the fraction of particles decaying per unit time is a constant, i.e., we have $dN/N = -(1/\tau)dt$ for some constant τ. This gives $dN/dt = -(1/\tau)N$, which has the solution $N(t) = N_0 e^{-t/\tau}$ where we have N_0 unstable particles at $t = 0$. The fraction of particles decaying in the interval t to $t+dt$ is $-dN/N_0 = (-dN/dt)dt/N_0 = (1/\tau)e^{-t/\tau}dt$. So the *mean lifetime* (or *lifetime*) is $\int_0^\infty t\,(-dN/dt)\,dt/N_0 = \tau\int_0^\infty xe^{-x}dx = \tau$, where $x = t/\tau$. The *half-life*, $t_{1/2}$, is the time taken for half the particles to decay, $e^{-t_{1/2}/\tau} = \frac{1}{2}$, which means that $t_{1/2} = \tau \ln 2$. The *decay rate*, Γ, is defined as the probability per unit time that a particle will decay, i.e., $\Gamma = (-dN/dt)/N = 1/\tau$ is the inverse mean lifetime. A muon at rest has a lifetime of $\tau = 2.197 \times 10^{-6}$ s. *Cosmic rays* are high-energy particles that have traveled enormous distances from outside our solar system. *Primary cosmic rays* are particles that have been accelerated by some extreme astrophysical event and *secondary cosmic rays* are those resulting from collisions of primary cosmic rays with interstellar gas or with our atmosphere. Most cosmic rays reaching our atmosphere will be stable particles such as photons, neutrinos, electrons, protons and stable atomic nuclei (mostly helium nuclei). Muon cosmic rays therefore are secondary cosmic rays produced when primary or secondary cosmic rays collide with out atmosphere. A typical height in the atmosphere for the production of cosmic ray muons is ~ 15 km. What is the minimum velocity that this muon be produced with in order that it have a 50% chance of reaching the surface of the Earth before decaying?

Solution:

A stationary observer on Earth will see the time experienced by a muon traveling at speed v to be dilated by a factor of $\gamma = (1 - v^2/c^2)^{-1/2}$ compared to their own. So according to the observer, the half-life of the muon will be $\gamma\, t_{1/2}$. If the muon travels at a speed that allows it to traverse the $L = 15$ km of the atmosphere in this time, it will therefore have a 50% chance of reaching the Earth's surface. That is, the speed of the muon must satisfy

$$v = \frac{L}{\gamma t_{1/2}} = \frac{\sqrt{1-\frac{v^2}{c^2}}\,L}{t_{1/2}} \tag{1.1}$$

to have a 50% chance of reaching the surface before decaying. This can be rearranged to give

$$v = \frac{L}{\sqrt{t_{1/2}^2 + \frac{L^2}{c^2}}}. \tag{1.2}$$

Now from the given information, the half-life of the muon is

$$t_{1/2} = \tau \ln 2 = (2.197 \times 10^{-6}\ \mathrm{s}) \ln 2 = 1.523 \times 10^{-6}\ \mathrm{s}, \tag{1.3}$$

so the necessary speed is

$$\begin{aligned} v &= \frac{15 \times 10^3\ \mathrm{m}}{\sqrt{(1.523 \times 10^{-6}\ \mathrm{s})^2 + \frac{(15\times 10^3\ \mathrm{m})^2}{(3\times 10^8\ \mathrm{ms}^{-1})^2}}} \\ &= 299,653,700\ \mathrm{ms}^{-1} \\ &= 0.9995c. \end{aligned} \tag{1.4}$$

An inertial observer traveling with the muon will see the height of the atmosphere contracted by a factor of γ, which at this speed gives a height of

$$\begin{aligned} \frac{L}{\gamma} &= \sqrt{1-\frac{v^2}{c^2}}L \\ &= \sqrt{1-0.9995^2}\,(15 \times 10^3\ \mathrm{m}) \\ &= 456\ \mathrm{m}. \end{aligned} \tag{1.5}$$

In other words, from the muon's point of view, it only needs to travel 456 m to reach the Earth's surface.

1.2 Problem 2

Problem:

The diameter of our Milky Way spiral galaxy is approximately 100,000 - 180,000 light years and our solar system is approximately 25,000 light years from the center of our galaxy. Recalling the effects of time dilation, approximately how fast would you have to travel to reach the center of the galaxy in your lifetime? Estimate how much energy would it take to accelerate your body to this speed.

Solution:

From the solution to Problem 1.1, the speed needed for an observer to travel a proper distance L in a time T is

$$v = \frac{L}{\sqrt{T^2 + \frac{L^2}{c^2}}}. \tag{1.6}$$

So for a human with a lifetime of, say, $T = 80$ years, to travel to the center of the Milky Way $L = 25{,}000$ light years $= 25{,}000c$ years away, a speed of

$$\begin{aligned} v &= \frac{25{,}000c \text{ years}}{\sqrt{(80 \text{ years})^2 + (25{,}000 \text{ years})^2}} \qquad (1.7)\\ &= 0.999995c \qquad (1.8) \end{aligned}$$

would be required. The kinetic energy possessed by a human of mass $m = 70$ kg at this speed is

$$\begin{aligned} E_{\text{kin}} &= (\gamma - 1)mc^2 \\ &= \left(\frac{1}{\sqrt{1 - 0.999995^2}} - 1\right)(70 \text{ kg})(3 \times 10^8 \text{ ms}^{-1}) \\ &= 2 \times 10^{21} \text{ J}. \qquad (1.9) \end{aligned}$$

For reference, it would take humanity slightly more than 3 years to consume this much energy at current rates.

1.3 Problem 3

Problem:
Consider two events that occur at the same spatial point in the frame of some inertial observer $\mathcal{O}$. Explain why the two events occur in the same temporal order in every inertial frame connected to it by a Lorentz transformation that does not invert time. Show that the time separation between the two events is a minimum in the frame of $\mathcal{O}$. (Hint: Consider Figs. 1.2 and 1.5.)

Solution:
Since the two events E_1 and E_2 occur at the same spatial point in frame $\mathcal{O}$, the displacement vector $E_2 - E_1$ between them will point entirely along the time axis in this frame, in the positive direction, say. Under any given Lorentz transformation, displacement vectors will be moved along hyperbolae in a spacetime diagram, as illustrated in Fig. 1.1. Further, Lorentz transformations that do not invert time will only move displacement vectors along *branches* of those hyperbolae. Our vector of interest $E_2 - E_1$ lies on a positive-time branch, so under an orthochronus Lorentz transformation it will remain on that positive-time branch. That is, the temporal order of the events remains the same. And the point on such a positive-time branch that has the smallest time component is the one that lies on the time axis, as $E_2 - E_1$

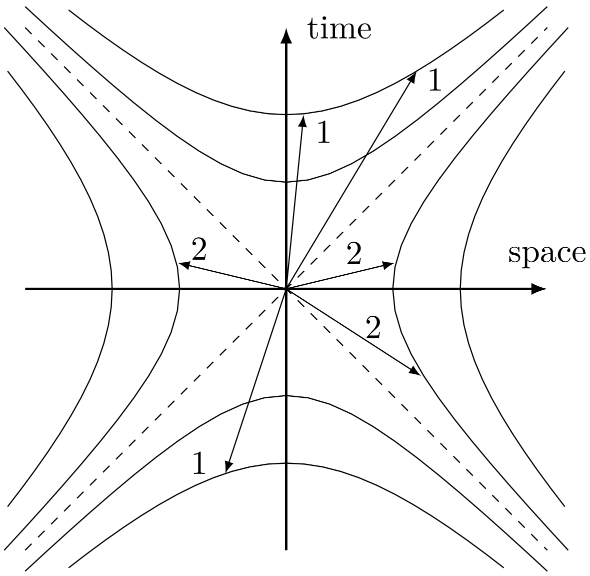

Figure 1.1 A two-dimensional representation of spacetime displacement with displacements represented by vectors. Vectors labeled by the same number (1 or 2) are related by Lorentz transformations.

does in frame $\mathcal{O}$. So under an orthochronus Lorentz transformation, $E_2 - E_1$ will be moved to a point on the spacetime diagram with a time component at least as large as that in frame $\mathcal{O}$, showing that the time separation between the events is minimized in this frame.

To see this algebraically, label the two events in frame $\mathcal{O}$ as

$$E_1 = (ct_1, \mathbf{x})^T, \tag{1.10}$$
$$E_2 = (ct_2, \mathbf{x})^T, \tag{1.11}$$

where $t_2 > t_1$. Then under a Lorentz transformation Λ, the events go to

$$E_1 \to E_1' = \Lambda E_1 = \Lambda(ct_1, \mathbf{x})^T, \tag{1.12}$$
$$E_2 \to E_2' = \Lambda E_2 = \Lambda(ct_2, \mathbf{x})^T. \tag{1.13}$$

The difference between these events is

$$E_2' - E_1' = \Lambda \left(c(t_2 - t_1), \mathbf{0}\right)^T. \tag{1.14}$$

In particular, the temporal component of this difference is $\Lambda^0{}_0\, c(t_2 - t_1)$, which is at least as great as $c(t_2 - t_1)$, because Lorentz transformations that do not reverse time satisfy $\Lambda^0{}_0 \geq 1$. Again we conclude the time separation is minimized in frame $\mathcal{O}$.

1.4 Problem 4

Problem:

Consider any two events that occur at the same time in the frame of an inertial observer $\mathcal{O}$. Show that by considering any Lorentz transformation there is no limit to the possible time separation of the two events and that the smallest spatial separation of the two events occurs in the frame of $\mathcal{O}$.

Solution:

Label the two events in frame $\mathcal{O}$ as

$$E_1 = (ct, \mathbf{x}_1)^T, \tag{1.15}$$
$$E_2 = (ct, \mathbf{x}_2)^T. \tag{1.16}$$

Assume without loss of generality that the events are separated in the x-direction by an amount $|\Delta x| \neq 0$, and consider a boost in the x-direction with speed v. The difference in the boosted temporal coordinates will be given by Eq. (1.2.89),

$$c\Delta t' = \gamma\left(c\Delta t - \frac{v}{c}\Delta x\right) = -\frac{v}{c}\gamma\Delta x, \tag{1.17}$$

and so the magnitude of the time separation in the boosted coordinates will be

$$|\Delta t'| = \frac{v}{c^2\sqrt{1-\frac{v^2}{c^2}}}|\Delta x|. \tag{1.18}$$

The factor $|\Delta x|$ is a non-zero constant, so this expression approaches infinity as $v \to c$. Hence, there is no limit to the possible time separation between the two events.

Next, the spatial separations between the events along each direction in the boosted frame,

$$\Delta x' = \gamma\left(\Delta x - v\Delta t\right) = \gamma\Delta x, \tag{1.19}$$
$$\Delta y' = \Delta y, \tag{1.20}$$
$$\Delta z' = \Delta z, \tag{1.21}$$

imply a total spatial separation of

$$|\Delta \mathbf{x}'| = \sqrt{(\gamma\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \tag{1.22}$$
$$\geq \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = |\Delta \mathbf{x}|. \tag{1.23}$$

Hence, the smallest spatial separation between the two events occurs in frame $\mathcal{O}$.

1.5 Problem 5

Problem:

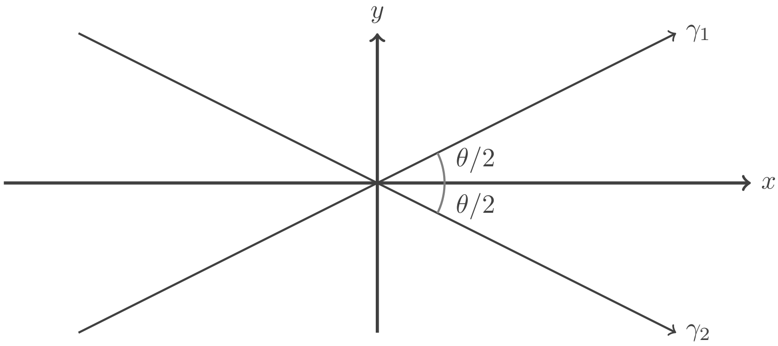

Figure 1.2 Coordinate system being used for Problem 1.5, where γ_1 and γ_2 label the two beams of light.

Two narrow light beams intersect at angle θ, where θ is the angle between the outgoing beams. The beams intersect head on when $\theta = 180°$. Using the addition of velocities formula in Eq. (1.2.126) show that for any angle θ there is always an inertial frame in which the beams intersect head on.

Solution:

Set up the coordinate system such that the beams of light lie in the $x-y$ plane, with the origin at their intersection and the x axis bisecting the angle θ made by the beams, as in Fig. 1.2. Then the velocity vectors of the beams will be

$$\mathbf{u}_1 = \left(c\cos\left(\frac{\theta}{2}\right), c\sin\left(\frac{\theta}{2}\right), 0\right)^T, \tag{1.24}$$

$$\mathbf{u}_2 = \left(c\cos\left(\frac{\theta}{2}\right), -c\sin\left(\frac{\theta}{2}\right), 0\right)^T. \tag{1.25}$$

We will boost with speed v in the $+x$-direction, hoping to find a speed such that the beams are directed entirely in the y'-direction and moving oppositely. The velocities in the x'-direction, given by Eq. (1.2.126) in the text, must therefore vanish:

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = 0 \tag{1.26}$$

$$\Longrightarrow \quad v = u_x = c\cos\left(\frac{\theta}{2}\right). \tag{1.27}$$

If $\cos(\theta/2) = 1$ then the beams intersect head on in the original frame, and it is not necessary to perform a boost. If not, then Eq. 1.27 shows that $v < c$ and therefore our proposed boost is valid. The velocities in the y'-direction are

$$u'_y = \frac{1}{\gamma}\frac{u_y}{1 - \frac{u_x v}{c^2}}$$

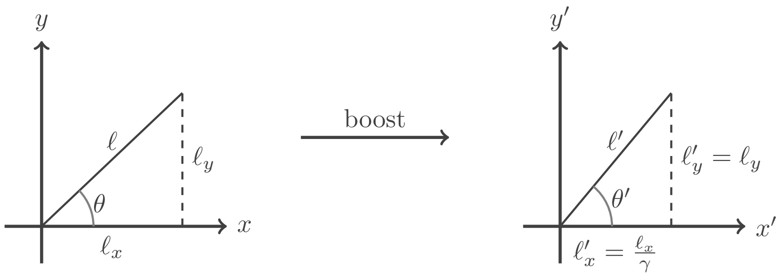

Figure 1.3 Diagram showing the length contraction of the ruler in a boosted frame.

$$
\begin{aligned}
&= \frac{u_y}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= \frac{\pm c \sin\left(\frac{\theta}{2}\right)}{\sqrt{1-\cos^2\left(\frac{\theta}{2}\right)}} \\
&= \pm c
\end{aligned}
\tag{1.28}
$$

where the $\pm$ refers to the first and the second beam, respectively. We see that the beams are oppositely directed in the y'-direction and therefore intersect head on in this frame.

1.6 Problem 6

Problem:

A ruler of rest length ℓ is at rest in the frame of inertial observer $\mathcal{O}$ and is at an angle θ with respect to the $+x$-direction. Now consider an inertial observer $\mathcal{O}'$ in an identical inertial frame except that it has been boosted by speed v in the $+x$-direction. What is the length ℓ' of the ruler and what is the angle θ' with respect to the $+x'$-direction that will be measured by $\mathcal{O}'$?

Solution:

As a result of Eq. (1.2.89), the lengths measured by observer $\mathcal{O}'$ will be contracted compared to the lengths in frame $\mathcal{O}$ by a factor of γ in the x'-direction, and remain the same in the perpendicular directions, as illustrated in Fig. 1.3. If the ruler lies in the $x-y$ plane, $\mathcal{O}'$ will therefore measure the lengths in the x' and y' directions to be

$$\ell'_x = \frac{\ell_x}{\gamma} = \sqrt{1-\frac{v^2}{c^2}}\,\ell\cos\theta, \tag{1.29}$$

$$\ell'_y = \ell_y = \ell\sin\theta, \tag{1.30}$$

and therefore the total length to be

$$\ell' = \sqrt{(\ell'_x)^2 + (\ell'_y)^2} = \ell\sqrt{1 - \frac{v^2}{c^2}\cos^2\theta}. \tag{1.31}$$

The angle θ' in the boosted frame will satisfy

$$\tan\theta' \equiv \frac{\ell'_y}{\ell'_x} = \gamma\frac{\ell_y}{\ell_x} \equiv \gamma\tan\theta \qquad \Longrightarrow \quad \theta' = \tan^{-1}\left(\frac{\tan\theta}{\sqrt{1-\frac{v^2}{c^2}}}\right). \tag{1.32}$$

1.7 Problem 7

Problem:

If a spaceship approaches earth at 1.5×10^8 ms^{-1} and emits a microwave frequency of 10 GHz, what frequency will an observer on Earth detect?

Solution:

The ratio of the frequencies in the source frame and observer frame is given by Eq. (1.2.123) in the text,

$$\frac{f_s}{f_o} = \sqrt{\frac{1+\beta}{1-\beta}}. \tag{1.33}$$

Since the ship is moving *towards* Earth at a speed of 1.5×10^8 ms^{-1} $= c/2$, we write $v = -c/2$ and so $\beta = v/c = -1/2$. An observer on Earth will then detect a frequency of

$$f_o = \sqrt{\frac{1-\beta}{1+\beta}}f_s \tag{1.34}$$

$$= \sqrt{\frac{1+\frac{1}{2}}{1-\frac{1}{2}}}(10\text{ GHz}) \tag{1.35}$$

$$= 17.3\text{ GHz}. \tag{1.36}$$

Note that $f_o > f_s$, so the beam has been blueshifted as expected when the source is moving towards the observer.

1.8 Problem 8

Problem:

An inertial observer observes two spaceship moving directly toward one another. She measures one to be traveling at 0.7 c in her inertial frame and the other at 0.9

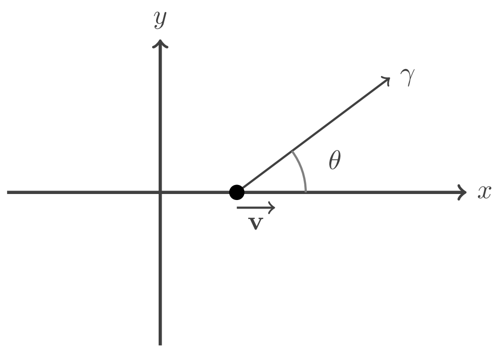

Figure 1.4 A particular ray of light emitted by the moving light source in frame $\mathcal{O}$.

c in the opposite direction. What is the magnitude of the relative velocity that each spaceship measures the other to have?

Solution:

Take the $+x$-direction to be the direction of motion of the spaceship moving at $0.9c$ in the frame of the inertial observer. Boost by $v = 0.9c$ in this direction to go to the rest frame of the spaceship. In this frame, the other spaceship will have a speed

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}, \tag{1.37}$$

where $u_x = -0.7c$ is the speed of that spaceship in the original frame along the $+x$-direction, by Eq. (1.2.126). Substituting in the given values, we find

$$\begin{aligned} u'_x &= \frac{-0.7c - 0.9c}{1 - \frac{(-0.7c)(0.9c)}{c^2}} \\ &= -0.9816c. \end{aligned} \tag{1.38}$$

That is, each spaceship measures the other to be moving towards them at $0.98c$.

1.9 Problem 9

Problem:

A light source moves with constant velocity $\boldsymbol{v}$ in the frame of inertial observer $\mathcal{O}$. The source radiates isotropically in its rest frame. Show that in the inertial frame of $\mathcal{O}$ the light is concentrated in the direction of motion of the source, where half of the photons lie in a cone of semi-angle θ, where $\cos\theta = v/c$.

Solution:

In frame $\mathcal{O}$, assume without loss of generality the source of light is moving in the $+x$-direction at speed v. Consider the ray of light emitted by the source in the $x-y$ plane making an angle θ with the x-axis, shown in Fig. 1.4. It will have velocity vector

$$\mathbf{u} = (c\cos\theta,\ c\sin\theta,\ 0)^T. \tag{1.39}$$

Boost by speed v in the $+x$-direction to move to the rest frame of the light source. Here, the x'-component of the ray's velocity vector will be given by Eq. (1.2.126):

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \\ &= \frac{c\cos\theta - v}{1 - \frac{v}{c}\cos\theta} \\ &\equiv c\cos\theta', \end{aligned} \tag{1.40}$$

which implies

$$\cos\theta = \frac{v}{c} + \left(1 - \frac{v}{c}\cos\theta\right)\cos\theta'. \tag{1.41}$$

Now the source emits photons uniformly in all directions in this boosted frame, so half of the photons it emits will lie in the region $-90° < \theta' < +90°$. The boundary of this region satisfies $\cos\theta' = 0$, which corresponds in the original frame to $\cos\theta = v/c$ by Eq. 1.41. Since the boost transformation is continuous, the interior of the original region (satisfying $\cos\theta' > 0$) will be mapped to the interior of the boosted region (satisfying $\cos\theta > v/c$). That is, half of the emitted photons will lie in a cone of semi-angle $\theta = \cos^{-1}(v/c) < 90°$, showing that the light is concentrated in the direction of motion of the source.

1.10 Problem 10

Problem:

Write down the Lorentz transformation rule for an arbitrary $(3,2)$ tensor $A^{\mu\nu\rho}{}_{\sigma\tau}$. Hence show that any double contraction of this tensor leads to a contravariant vector, i.e., to a $(1,0)$ tensor.

Solution:

Under a Lorentz transformation Λ, a $(3,2)$ tensor $A^{\mu\nu\rho}{}_{\sigma\tau}$ will transform as

$$A^{\mu\nu\rho}{}_{\sigma\tau} \to A'^{\mu\nu\rho}{}_{\sigma\tau} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}\Lambda^{\rho}{}_{\rho'}A^{\mu'\nu'\rho'}{}_{\sigma'\tau'}(\Lambda^{-1})^{\sigma'}{}_{\sigma}(\Lambda^{-1})^{\tau'}{}_{\tau}. \tag{1.42}$$

Define $T^\mu = A^{\mu\nu\rho}{}_{\nu\rho}$. Then using Eq. 1.42, T^μ will transform as

$$\begin{aligned} T^\mu \to T'^\mu &= A'^{\mu\nu\rho}{}_{\nu\rho} \\ &= \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}\Lambda^{\rho}{}_{\rho'}A^{\mu'\nu'\rho'}{}_{\sigma'\tau'}(\Lambda^{-1})^{\sigma'}{}_{\nu}(\Lambda^{-1})^{\tau'}{}_{\rho} \end{aligned}$$