

# Instructor's Manual to Accompany



## Chapter 1

1.  $S = \{(R, R), (R, G), (R, B), (G, R), (G, G), (G, B), (B, R), (B, G), (B, B)\}$   
 The probability of each point in  $S$  is  $1/9$ .
2.  $S = \{(R, G), (R, B), (G, R), (G, B), (B, R), (B, G)\}$
3.  $S = \{(e_1, e_2, \dots, e_n), n \geq 2\}$  where  $e_i \in \{\text{heads}, \text{tails}\}$ . In addition,  $e_n = e_{n-1} = \text{heads}$  and for  $i = 1, \dots, n-2$  if  $e_i = \text{heads}$ , then  $e_{i+1} = \text{tails}$ .

$$\begin{aligned} P\{4 \text{ tosses}\} &= P\{(t, t, h, h)\} + P\{(h, t, h, h)\} \\ &= 2 \left[ \frac{1}{2} \right]^4 = \frac{1}{8} \end{aligned}$$

4. (a)  $F(E \cup G)^c = FE^cG^c$   
 (b)  $EFG^c$   
 (c)  $E \cup F \cup G$   
 (d)  $EF \cup EG \cup FG$   
 (e)  $EFG$   
 (f)  $(E \cup F \cup G)^c = E^cF^cG^c$   
 (g)  $(EF)^c(EG)^c(FG)^c$   
 (h)  $(EFG)^c$
5.  $\frac{3}{4}$ . If he wins, he only wins \$1, while if he loses, he loses \$3.
6. If  $E(F \cup G)$  occurs, then  $E$  occurs and either  $F$  or  $G$  occur; therefore, either  $EF$  or  $EG$  occurs and so

$$E(F \cup G) \subset EF \cup EG$$

Similarly, if  $EF \cup EG$  occurs, then either  $EF$  or  $EG$  occurs. Thus,  $E$  occurs and either  $F$  or  $G$  occurs; and so  $E(F \cup G)$  occurs. Hence,

$$EF \cup EG \subset E(F \cup G)$$

which together with the reverse inequality proves the result.

7. If  $(E \cup F)^c$  occurs, then  $E \cup F$  does not occur, and so  $E$  does not occur (and so  $E^c$  does);  $F$  does not occur (and so  $F^c$  does) and thus  $E^c$  and  $F^c$  both occur. Hence,

$$(E \cup F)^c \subset E^c F^c$$

If  $E^c F^c$  occurs, then  $E^c$  occurs (and so  $E$  does not), and  $F^c$  occurs (and so  $F$  does not). Hence, neither  $E$  or  $F$  occurs and thus  $(E \cup F)^c$  does. Thus,

$$E^c F^c \subset (E \cup F)^c$$

and the result follows.

8.  $1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$   
 9.  $F = E \cup FE^c$ , implying since  $E$  and  $FE^c$  are disjoint that  $P(F) = P(E) + P(FE^c)$ .  
 10. Either by induction or use

$$\bigcup_1^n E_i = E_1 \cup E_1^c E_2 \cup E_1^c E_2^c E_3 \cup \dots \cup E_1^c \dots E_{n-1}^c E_n$$

and as each of the terms on the right side are mutually exclusive:

$$\begin{aligned} P\left(\bigcup_1^n E_i\right) &= P(E_1) + P(E_1^c E_2) + P(E_1^c E_2^c E_3) + \dots \\ &\quad + P(E_1^c \dots E_{n-1}^c E_n) \\ &\leq P(E_1) + P(E_2) + \dots + P(E_n) \quad (\text{why?}) \end{aligned}$$

11.  $P\{\text{sum is } i\} = \begin{cases} \frac{i-1}{36}, & i = 2, \dots, 7 \\ \frac{13-i}{36}, & i = 8, \dots, 12 \end{cases}$

12. Either use hint or condition on initial outcome as:

$$\begin{aligned} &P\{E \text{ before } F\} \\ &= P\{E \text{ before } F | \text{initial outcome is } E\}P(E) \\ &\quad + P\{E \text{ before } F | \text{initial outcome is } F\}P(F) \\ &\quad + P\{E \text{ before } F | \text{initial outcome neither } E \text{ or } F\}[1 - P(E) - P(F)] \\ &= 1 \cdot P(E) + 0 \cdot P(F) + P\{E \text{ before } F\} \\ &= [1 - P(E) - P(F)] \end{aligned}$$

Therefore,  $P\{E \text{ before } F\} = \frac{P(E)}{P(E) + P(F)}$

13. Condition an initial toss

$$P\{\text{win}\} = \sum_{i=2}^{12} P\{\text{win} | \text{throw } i\} P\{\text{throw } i\}$$

Now,

$$P\{\text{win}|\text{throw } i\} = P\{i \text{ before } 7\}$$

$$= \begin{cases} 0 & i = 2, 12 \\ \frac{i-1}{5+1} & i = 3, \dots, 6 \\ 1 & i = 7, 11 \\ \frac{13-i}{19-1} & i = 8, \dots, 10 \end{cases}$$

where above is obtained by using Problems 11 and 12.

$$P\{\text{win}\} \approx .49.$$

$$\begin{aligned} 14. \quad P\{A \text{ wins}\} &= \sum_{n=0}^{\infty} P\{A \text{ wins on } (2n+1)\text{st toss}\} \\ &= \sum_{n=0}^{\infty} (1-P)^{2n} P \\ &= P \sum_{n=0}^{\infty} [(1-P)^2]^n \\ &= P \frac{1}{1-(1-P)^2} \\ &= \frac{P}{2P-P^2} \\ &= \frac{1}{2-P} \\ P\{B \text{ wins}\} &= 1 - P\{A \text{ wins}\} \\ &= \frac{1-P}{2-P} \end{aligned}$$

$$\begin{aligned} 16. \quad P(E \cup F) &= P(E \cup FE^c) \\ &= P(E) + P(FE^c) \end{aligned}$$

since  $E$  and  $FE^c$  are disjoint. Also,

$$\begin{aligned} P(E) &= P(FE \cup FE^c) \\ &= P(FE) + P(FE^c) \text{ by disjointness} \end{aligned}$$

Hence,

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

$$\begin{aligned} 17. \quad \text{Prob}\{\text{end}\} &= 1 - \text{Prob}\{\text{continue}\} \\ &= 1 - P(\{H, H, H\} \cup \{T, T, T\}) \\ &= 1 - [\text{Prob}(H, H, H) + \text{Prob}(T, T, T)]. \end{aligned}$$

$$\begin{aligned}\text{Fair coin: Prob}\{\text{end}\} &= 1 - \left[ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \right] \\ &= \frac{3}{4}\end{aligned}$$

$$\begin{aligned}\text{Biased coin: } P\{\text{end}\} &= 1 - \left[ \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \right] \\ &= \frac{9}{16}\end{aligned}$$

18. Let  $B$  = event both are girls;  $E$  = event oldest is girl;  $L$  = event at least one is a girl.

$$(a) P(B|E) = \frac{P(BE)}{P(E)} = \frac{P(B)}{P(E)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$(b) P(L) = 1 - P(\text{no girls}) = 1 - \frac{1}{4} = \frac{3}{4},$$

$$P(B|L) = \frac{P(BL)}{P(L)} = \frac{P(B)}{P(L)} = \frac{1/4}{3/4} = \frac{1}{3}$$

19.  $E$  = event at least 1 six  $P(E)$

$$= \frac{\text{number of ways to get } E}{\text{number of samples pts}} = \frac{11}{36}$$

$D$  = event two faces are different  $P(D)$

$$= 1 - \text{Prob}(\text{two faces the same})$$

$$= 1 - \frac{6}{36} = \frac{5}{6} P(E|D) = \frac{P(ED)}{P(D)} = \frac{10/36}{5/6} = \frac{1}{3}$$

20. Let  $E$  = event same number on exactly two of the dice;  $S$  = event all three numbers are the same;  $D$  = event all three numbers are different. These three events are mutually exclusive and define the whole sample space. Thus,  $1 = P(D) + P(S) + P(E)$ ,  $P(S) = 6/216 = 1/36$ ; for  $D$  have six possible values for first die, five for second, and four for third.

$$\therefore \text{Number of ways to get } D = 6 \cdot 5 \cdot 4 = 120.$$

$$P(D) = 120/216 = 20/36$$

$$\therefore P(E) = 1 - P(D) - P(S)$$

$$= 1 - \frac{20}{36} - \frac{1}{36} = \frac{5}{12}$$

21. Let  $C$  = event person is color blind.

$$\begin{aligned}P(\text{Male}|C) &= \frac{P(C|\text{Male})P(\text{Male})}{P(C|\text{Male})P(\text{Male}) + P(C|\text{Female})P(\text{Female})} \\ &= \frac{.05 \times .5}{.05 \times .5 + .0025 \times .5} \\ &= \frac{2500}{2625} = \frac{20}{21}\end{aligned}$$

22. Let trial 1 consist of the first two points; trial 2 the next two points, and so on. The probability that each player wins one point in a trial is  $2p(1 - p)$ . Now a total of  $2n$  points are played if the first  $(a - 1)$  trials all result in each player winning one of the points in that trial and the  $n$ th trial results in one of the players winning both points. By independence, we obtain

$$\begin{aligned} &P\{2n \text{ points are needed}\} \\ &= (2p(1 - p))^{n-1}(p^2 + (1 - p)^2), \quad n \geq 1 \end{aligned}$$

The probability that  $A$  wins on trial  $n$  is  $(2p(1 - p))^{n-1}p^2$  and so

$$\begin{aligned} P\{A \text{ wins}\} &= p^2 \sum_{n=1}^{\infty} (2p(1 - p))^{n-1} \\ &= \frac{p^2}{1 - 2p(1 - p)} \end{aligned}$$

$$\begin{aligned} 23. &P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\dots P(E_n|E_1\dots E_{n-1}) \\ &= P(E_1) \frac{P(E_1E_2)}{P(E_1)} \frac{P(E_1E_2E_3)}{P(E_1E_2)} \dots \frac{P(E_1\dots E_n)}{P(E_1\dots E_{n-1})} \\ &= P(E_1\dots E_n) \end{aligned}$$

24. Let  $a$  signify a vote for  $A$  and  $b$  one for  $B$ .

$$\begin{aligned} \text{(a)} \quad &P_{2,1} = P\{a, a, b\} = 1/3 \\ \text{(b)} \quad &P_{3,1} = P\{a, a\} = (3/4)(2/3) = 1/2 \\ \text{(c)} \quad &P_{3,2} = P\{a, a, a\} + P\{a, a, b, a\} \\ &= (3/5)(2/4)[1/3 + (2/3)(1/2)] = 1/5 \\ \text{(d)} \quad &P_{4,1} = P\{a, a\} = (4/5)(3/4) = 3/5 \\ \text{(e)} \quad &P_{4,2} = P\{a, a, a\} + P\{a, a, b, a\} \\ &= (4/6)(3/5)[2/4 + (2/4)(2/3)] = 1/3 \\ \text{(f)} \quad &P_{4,3} = P\{\text{always ahead}|a, a\}(4/7)(3/6) \\ &= (2/7)[1 - P\{a, a, a, b, b, b|a, a\} \\ &\quad - P\{a, a, b, b|a, a\} - P\{a, a, b, a, b, b|a, a\}] \\ &= (2/7)[1 - (2/5)(3/4)(2/3)(1/2) \\ &\quad - (3/5)(2/4) - (3/5)(2/4)(2/3)(1/2)] \\ &= 1/7 \\ \text{(g)} \quad &P_{5,1} = P\{a, a\} = (5/6)(4/5) = 2/3 \\ \text{(h)} \quad &P_{5,2} = P\{a, a, a\} + P\{a, a, b, a\} \\ &= (5/7)(4/6)[(3/5) + (2/5)(3/4)] = 3/7 \end{aligned}$$

By the same reasoning we have

(i)  $P_{5,3} = 1/4$

(j)  $P_{5,4} = 1/9$

(k) In all the cases above,  $P_{n,m} = \frac{n-n}{n+n}$

25. (a)  $P\{\text{pair}\} = P\{\text{second card is same denomination as first}\}$   
 $= 3/51$

(b)  $P\{\text{pair}|\text{different suits}\}$   
 $= \frac{P\{\text{pair, different suits}\}}{P\{\text{different suits}\}}$   
 $= P\{\text{pair}\}/P\{\text{different suits}\}$   
 $= \frac{3/51}{39/51} = 1/13$

26.  $P(E_1) = \binom{4}{1} \binom{48}{12} / \binom{52}{13} = \frac{39.38.37}{51.50.49}$

$$P(E_2|E_1) = \binom{3}{1} \binom{36}{12} / \binom{39}{13} = \frac{26.25}{38.37}$$

$$P(E_3|E_1E_2) = \binom{2}{1} \binom{24}{12} / \binom{26}{13} = 13/25$$

$$P(E_4|E_1E_2E_3) = 1$$

$$P(E_1E_2E_3E_4) = \frac{39.26.13}{51.50.49}$$

27.  $P(E_1) = 1$

$P(E_2|E_1) = 39/51$ , since 12 cards are in the ace of spades pile and 39 are not.

$P(E_3|E_1E_2) = 26/50$ , since 24 cards are in the piles of the two aces and 26 are in the other two piles.

$$P(E_4|E_1E_2E_3) = 13/49$$

So

$$P\{\text{each pile has an ace}\} = (39/51)(26/50)(13/49)$$

28. Yes.  $P(A|B) > P(A)$  is equivalent to  $P(AB) > P(A)P(B)$ , which is equivalent to  $P(B|A) > P(B)$ .

29. (a)  $P(E|F) = 0$

(b)  $P(E|F) = P(EF)/P(F) = P(E)/P(F) \geq P(E) = .6$

(c)  $P(E|F) = P(EF)/P(F) = P(F)/P(F) = 1$

30. (a)  $P\{\text{George}|\text{exactly 1 hit}\} = \frac{P\{\text{George, not Bill}\}}{P\{\text{exactly 1}\}}$   
 $= \frac{P\{G, \text{ not } B\}}{P\{G, \text{ not } B\} + P\{B, \text{ not } G\}}$   
 $= \frac{(.4)(.3)}{(.4)(.3) + (.7)(.6)}$   
 $= 2/9$

$$\begin{aligned}
 \text{(b) } P\{G|\text{hit}\} &= P\{G, \text{hit}\}/P\{\text{hit}\} \\
 &= P\{G\}/P\{\text{hit}\} = .4/[1 - (.3)(.6)] \\
 &= 20/41
 \end{aligned}$$

31. Let  $S$  = event sum of dice is 7;  $F$  = event first die is 6.

$$\begin{aligned}
 P(S) &= \frac{1}{6} P(FS) = \frac{1}{36} P(F|S) = \frac{P(F|S)}{P(S)} \\
 &= \frac{1/36}{1/6} = \frac{1}{6}
 \end{aligned}$$

32. Let  $E_i$  = event person  $i$  selects own hat.  $P$  (no one selects own hat)

$$\begin{aligned}
 &= 1 - P(E_1 \cup E_2 \cup \dots \cup E_n) \\
 &= 1 - \left[ \sum_{i_1} P(E_{i_1}) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \dots \right. \\
 &\quad \left. + (-1)^{n+1} P(E_1 E_2 E_n) \right] \\
 &= 1 - \sum_{i_1} P(E_{i_1}) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) \\
 &\quad - \sum_{i_1 < i_2 < i_3} P(E_{i_1} E_{i_2} E_{i_3}) + \dots \\
 &\quad + (-1)^n P(E_1 E_2 E_n)
 \end{aligned}$$

Let  $k \in \{1, 2, \dots, n\}$ .  $P(E_{i_1} E_{i_2} E_{i_k})$  = number of ways  $k$  specific men can select own hats  $\div$  total number of ways hats can be arranged =  $(n - k)!/n!$ . Number of terms in summation  $\sum_{i_1 < i_2 < \dots < i_k}$  = number of ways to choose  $k$  variables out of  $n$  variables =  $\binom{n}{k} = n!/k!(n - k)!$ .

Thus,

$$\begin{aligned}
 &\sum_{i_1 < \dots < i_k} P(E_{i_1} E_{i_2} \dots E_{i_k}) \\
 &= \sum_{i_1 < \dots < i_k} \frac{(n - k)!}{n!} \\
 &= \binom{n}{k} \frac{(n - k)!}{n!} = \frac{1}{k!}
 \end{aligned}$$

$\therefore P(\text{no one selects own hat})$

$$\begin{aligned}
 &= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \\
 &= \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}
 \end{aligned}$$

33. Let  $S$  = event student is sophomore;  $F$  = event student is freshman;  $B$  = event student is boy;  $G$  = event student is girl. Let  $x$  = number of sophomore girls; total number of students =  $16 + x$ .

$$\begin{aligned} P(F) &= \frac{10}{16+x} P(B) = \frac{10}{16+x} P(FB) = \frac{4}{16+x} \\ \frac{4}{16+x} &= P(FB) = P(F)P(B) = \frac{10}{16+x} \\ \frac{10}{16+x} &\Rightarrow x = 9 \end{aligned}$$

34. Not a good system. The successive spins are independent and so

$$\begin{aligned} P\{11\text{th is red} | 1\text{st } 10 \text{ black}\} &= P\{11\text{th is red}\} \\ &= P\left[\frac{18}{38}\right] \end{aligned}$$

35. (a)  $1/16$   
 (b)  $1/16$   
 (c)  $15/16$ , since the only way in which the pattern  $H, H, H, H$  can appear before the pattern  $T, H, H, H$  is if the first four flips all land heads.
36. Let  $B$  = event marble is black;  $B_i$  = event that box  $i$  is chosen. Now

$$\begin{aligned} B &= BB_1 \cup BB_2 P(B) = P(BB_1) + P(BB_2) \\ &= P(B|B_1)P(B_1) + P(B|B_2)P(B_2) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{7}{12} \end{aligned}$$

37. Let  $W$  = event marble is white.

$$\begin{aligned} P(B_1|W) &= \frac{P(W|B_1)P(B_1)}{P(W|B_1)P(B_1) + P(W|B_2)P(B_2)} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5} \end{aligned}$$

38. Let  $T_W$  = event transfer is white;  $T_B$  = event transfer is black;  $W$  = event white ball is drawn from urn 2.

$$\begin{aligned} P(T_W|W) &= \frac{P(W|T_W)P(T_W)}{P(W|T_W)P(T_W) + P(W|T_B)P(T_B)} \\ &= \frac{\frac{2}{7} \cdot \frac{2}{3}}{\frac{2}{7} \cdot \frac{2}{3} + \frac{1}{7} \cdot \frac{1}{3}} = \frac{\frac{4}{21}}{\frac{5}{21}} = \frac{4}{5} \end{aligned}$$



39. Let  $W$  = event woman resigns;  $A, B, C$  are events the person resigning works in store  $A, B, C$ , respectively.

$$\begin{aligned} P(C|W) &= \frac{P(W|C)P(C)}{P(W|C)P(C) + P(W|B)P(B) + P(W|A)P(A)} \\ &= \frac{.70 \times \frac{100}{225}}{.70 \times \frac{100}{225} + .60 \times \frac{75}{225} + .50 \times \frac{50}{225}} \\ &= \frac{70}{225} \bigg/ \frac{140}{225} = \frac{1}{2} \end{aligned}$$

40. (a)  $F$  = event fair coin flipped;  $U$  = event two-headed coin flipped.

$$\begin{aligned} P(F|H) &= \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(F|HH) &= \frac{P(HH|F)P(F)}{P(HH|F)P(F) + P(HH|U)P(U)} \\ &= \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} = \frac{\frac{1}{8}}{\frac{5}{8}} = \frac{1}{5} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(F|HHT) &= \frac{P(HHT|F)P(F)}{P(HHT|F)P(F) + P(HHT|U)P(U)} \\ &= \frac{P(HHT|F)P(F)}{P(HHT|F)P(F) + 0} = 1 \end{aligned}$$

since the fair coin is the only one that can show tails.

41. Note first that since the rat has black parents and a brown sibling, we know that both its parents are hybrids with one black and one brown gene (for if either were a pure black then all their offspring would be black). Hence, both of their offspring's genes are equally likely to be either black or brown.

$$\begin{aligned} \text{(a)} \quad P(2 \text{ black genes} | \text{at least one black gene}) &= \frac{P(2 \text{ black genes})}{P(\text{at least one black gene})} \\ &= \frac{1/4}{3/4} = 1/3 \end{aligned}$$

- (b) Using the result from part (a) yields the following:

$$\begin{aligned} P(2 \text{ black genes} | 5 \text{ black offspring}) &= \frac{P(2 \text{ black genes})}{P(5 \text{ black offspring})} \\ &= \frac{1/3}{1(1/3) + (1/2)^5(2/3)} \\ &= 16/17 \end{aligned}$$

where  $P(5 \text{ black offspring})$  was computed by conditioning on whether the rat had 2 black genes.

42. Let  $B$  = event biased coin was flipped;  $F$  and  $U$  (same as above).

$$\begin{aligned} P(U|H) &= \frac{P(H|U)P(U)}{P(H|U)P(U) + P(H|B)P(B) + P(H|F)P(F)} \\ &= \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{9}{12}} = \frac{4}{9} \end{aligned}$$

43. Let  $B$  be the event that Flo has a blue eyed gene. Using that Jo and Joe both have one blue-eyed gene yields, upon letting  $X$  be the number of blue-eyed genes possessed by a daughter of theirs, that

$$P(B) = P(X = 1 | X < 2) = \frac{1/2}{3/4} = 2/3$$

Hence, with  $C$  being the event that Flo's daughter is blue eyed, we obtain

$$P(C) = P(CB) = P(B)P(C|B) = 1/3$$

44. Let  $W$  = event white ball selected.

$$\begin{aligned} P(T|W) &= \frac{P(W|T)P(T)}{P(W|T)P(T) + P(W|H)P(H)} \\ &= \frac{\frac{1}{5} \cdot \frac{1}{2}}{\frac{1}{5} \cdot \frac{1}{2} + \frac{5}{12} \cdot \frac{1}{2}} = \frac{12}{37} \end{aligned}$$

45. Let  $B_i$  = event  $i$ th ball is black;  $R_i$  = event  $i$ th ball is red.

$$\begin{aligned} P(B_1|R_2) &= \frac{P(R_2|B_1)P(B_1)}{P(R_2|B_1)P(B_1) + P(R_2|R_1)P(R_1)} \\ &= \frac{\frac{r}{b+r+c} \cdot \frac{b}{b+r}}{\frac{r}{b+r+c} \cdot \frac{b}{b+r} + \frac{r+c}{b+r+c} \cdot \frac{r}{b+r}} \\ &= \frac{rb}{rb + (r+c)r} \\ &= \frac{b}{b+r+c} \end{aligned}$$

46. Let  $X(=B \text{ or } =C)$  denote the jailer's answer to prisoner A. Now for instance,

$$\begin{aligned} &P\{A \text{ to be executed} | X = B\} \\ &= \frac{P\{A \text{ to be executed}, X = B\}}{P\{X = B\}} \\ &= \frac{P\{A \text{ to be executed}\}P\{X = B | A \text{ to be executed}\}}{P\{X = B\}} \\ &= \frac{(1/3)P\{X = B | A \text{ to be executed}\}}{1/2}. \end{aligned}$$

Now it is reasonable to suppose that if  $A$  is to be executed, then the jailer is equally likely to answer either  $B$  or  $C$ . That is,

$$P\{X = B|A \text{ to be executed}\} = \frac{1}{2}$$

and so,

$$P\{A \text{ to be executed}|X = B\} = \frac{1}{3}$$

Similarly,

$$P\{A \text{ to be executed}|X = C\} = \frac{1}{3}$$

and thus the jailer's reasoning is invalid. (It is true that if the jailer were to answer  $B$ , then  $A$  knows that the condemned is either himself or  $C$ , but it is twice as likely to be  $C$ .)

47. 1.  $0 \leq P(A|B) \leq 1$
2.  $P(S|B) = \frac{P(SB)}{P(B)} = \frac{P(B)}{P(B)} = 1$
3. For disjoint events  $A$  and  $D$

$$\begin{aligned} P(A \cup D|B) &= \frac{P((A \cup D)B)}{P(B)} \\ &= \frac{P(AB \cup DB)}{P(B)} \\ &= \frac{P(AB) + P(DB)}{P(B)} \\ &= P(A|B) + P(D|B) \end{aligned}$$

Direct verification is as follows:

$$\begin{aligned} &P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B) \\ &= \frac{P(ABC)}{P(BC)} \frac{P(BC)}{P(B)} + \frac{P(ABC^c)}{P(BC^c)} \frac{P(BC^c)}{P(B)} \\ &= \frac{P(ABC)}{P(B)} + \frac{P(ABC^c)}{P(B)} \\ &= \frac{P(AB)}{P(B)} \\ &= P(A|B) \end{aligned}$$

## Chapter 2

$$1. P\{X = 0\} = \frac{\begin{bmatrix} 7 \\ 2 \end{bmatrix}}{\begin{bmatrix} 10 \\ 2 \end{bmatrix}} = \frac{14}{30}$$