Chapter 2

Problems for Chapter 2

Problem 2-1. For many semiconductor materials used in contemporary electronics, the relationship between the momentum and energy of an electron is given by the implicit formula:

$$\frac{\mathbf{p^2}}{2m} = E\left(1 + \frac{E}{E_g}\right) \tag{2.1}$$

where m is the so-called effective mass of the electron and $1/E_g$ is the so-called non-parabolicity parameter. The formula has two solutions for unknown E: for electrons ($E > 0$) and another for the so-called holes ($E < 0$).

(a) Find both solutions for E.

(b) Derive the expression for $\mathbf{v}$ only for electrons.

(c) Determine the electron velocity in free space and compare it with the expression derived in (b).

As an example, consider GaAs, for which $m = 0.067\ m_0$ and $E_g = 1.42$ eV. Note: the expression for kinetic energy of an electron in a free space is

$$E = \frac{\mathbf{p^2}}{2m_0}\,. \tag{2.2}$$

Solution

(a) Equation (2.1) is the quadratic equation with respect to unknown E. It can be rewritten as

$$\frac{E^2}{E_g} + E - \frac{\mathbf{p}^2}{2m} = 0\,.$$

The quadratic equation of the canonical form $ax^2 + bx + c = 0$ has solutions

$$x_{1,2} == \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

For the case of consideration $a = 1/E_g$, $b = 1$, $c = -\mathbf{p}^2/2m$. Thus, we find *two solutions* of the energy dispersion as

$$E \equiv E_{el} = -\frac{E_g}{2} + \frac{E_g}{2}\sqrt{1 + \frac{2\mathbf{p^2}}{mE_g}}\,, \tag{2.3}$$

$$E \equiv E_h = -\frac{E_g}{2} - \frac{E_g}{2}\sqrt{1 + \frac{2\mathbf{p^2}}{mE_g}}\,. \tag{2.4}$$

In Fig. 2.1 these solutions are presented by solid lines. The two solutions obtained for E can be interpreted as the energy branch for electrons ($E > 0$, the thick solid line in Fig. 2.1) and the energy branch for the so-called holes ($E < 0$, thin solid line). These energy branches are separated by the energy gap equal to E_g. The energy structure with two bands of allowed energies separated by an energy gap is typical for semiconductor materials. They are discussed in Chapter 4 in detail.

It is instructive to compare the obtained $E(\mathbf{p})$-dependences with the kinetic energy of the electrons in a free space (Eq. (2.2)). Consider, for example, the upper electron branch of Eq. (2.3). At small momentum ($|\mathbf{p}| \to 0$) we find *parabolic dependence*, $E_{el} \approx \mathbf{p}^2/2m$, similar to Eq. (2.2). However, the parameter m differs from the mass of the free electron. At large momentum $|\mathbf{p}| \to \infty$, the electron energy becomes the *linear function* of $|\mathbf{p}|$:

$$E_{el} = |\mathbf{p}|\sqrt{\frac{E_g}{2m}}\,. \tag{2.5}$$

Note, all dependences (2.3)-(2.5) are functions of absolute value of the momentum $|\mathbf{p}| \equiv p = \sqrt{p_x^2 + p_y^2 + p_z^2}$. The vector character of $E(\mathbf{p})$ becomes to be important when one calculates vector parameters such as the electron velocity, $\mathbf{v}(\mathbf{p})$.

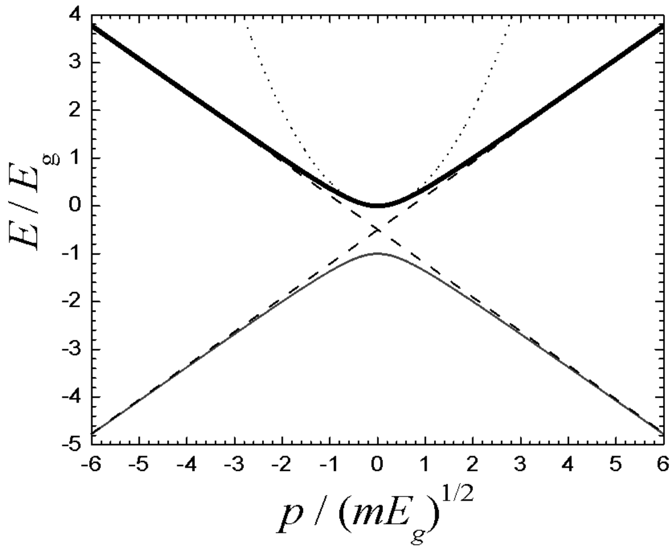

Figure 2.1: Solid lines present dependences (2.3) and (2.4). The dotted line shows parabolic dependence, $E = \mathbf{p}^2/2m$, which is close to a solid line at small momentum. The dashed straight lines show asymptotes of Eqs. (2.3) and (2.4) at $|\mathbf{p}| \to \infty$ where they approach solid lines.

(b) According to the definition, the electron velocity is given by derivative of the energy E with respect to the momentum $\mathbf{p}$:

$$\mathbf{v}(\mathbf{p}) = \frac{dE}{d\mathbf{p}}\,. \tag{2.6}$$

Straightforward calculation of the derivative gives:

$$\mathbf{v}(\mathbf{p}) = \frac{d}{d\mathbf{p}}\left(-\frac{E_g}{2} + \frac{E_g}{2}\sqrt{1+\frac{2\mathbf{p}^2}{mE_g}}\right) = \frac{\mathbf{p}}{m}\frac{1}{\sqrt{1+\frac{2\mathbf{p}^2}{mE_g}}}\,. \tag{2.7}$$

This dependence is illustrated in Fig. 2.2. One of the conclusions, which follows from the latter result, is that the velocity vector $\mathbf{v}(\mathbf{p})$ is always collinear with the momentum $\mathbf{p}$.

(c) Typical for free electrons linear dependence of the velocity on the momentum takes place only at small momentum: $\mathbf{v}(\mathbf{p}) = \frac{\mathbf{p}}{m}$. At large momentum, the velocity saturates: $|\mathbf{v}(\mathbf{p})| \approx v_s = \sqrt{\frac{E_g}{2m}}$. That is, it is not possible to accelerate the electrons beyond a given value. For parameters characteristic for GaAs material, the saturation velocity is $v_s = 1.36 \times 10^8$ cm/s.

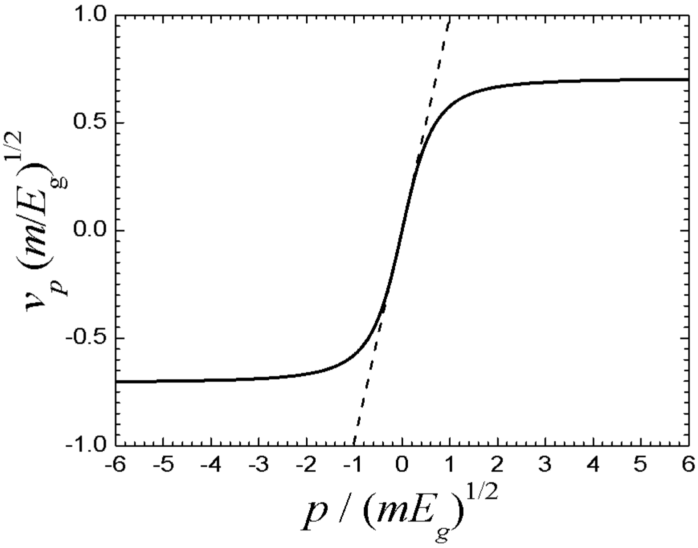

Figure 2.2: Solid line presents dependence (2.7). The dashed straight line shows the velocity of electrons in a free space.

Problem 2-2. Some metals and semiconductor materials instead of having an isotropic parabolic energy dispersion, $E = \mathbf{p^2}/2m$, have anisotropic parabolic energy dispersion,

$$E = \left(\frac{1}{m^*}\right)_{ij} p_i p_j \,, \tag{2.8}$$

i.e., for such cases "the electron mass" is no longer a scalar, but instead is a tensor. Let the reciprocal effective-mass tensor $(1/m)_{ij}$ be

$$\left(\frac{1}{m^*}\right)_{ij} = \begin{pmatrix} m_t^{-1} & 0 \\ 0 & m_l^{-1} \end{pmatrix}, \tag{2.9}$$

where $i, j = x, y$. Here, the parameters m_t and m_l are the so-called transverse and longitudinal effective masses of the conduction electron. For this case, the dispersion relation (2.8) is simplified to

$$E = \frac{{p_x}^2}{2m_t} + \frac{{p_y}^2}{2m_l}. \tag{2.10}$$

For the given momentum vector $\mathbf{p} = \{|\mathbf{p}|\sin\theta, |\mathbf{p}|\cos\theta\}$

(a) calculate velocity $\mathbf{v} = dE/d\mathbf{p}$, and

(b) Find the momentum vectors and the velocity vectors corresponding to momentum vectors with three values of θ, $\theta = 30^0$, 45^0 and 60^0 and take $|\mathbf{p}| = p_0$. Consider the electrons in Ge for which $m_t = 0.082\, m_0$ and $m_l = 1.64\, m_0$.

Notice that the directions of the velocity vectors and momentum vectors do not coincide (i.e., they are not collinear).

Solution

(a) For the energy dispersion of Eq. (2.10) according to definition (2.6) we find

$$v_x = \frac{d}{dp_x} E = \frac{p_x}{m_t} = \frac{|\mathbf{p}| \cos\theta}{m_t}\,, \tag{2.11}$$

$$v_y = \frac{d}{dp_y} E = \frac{p_y}{m_l} = \frac{|\mathbf{p}| \sin\theta}{m_l}\,, \tag{2.12}$$

$$\mathbf{v} = \left\{ \frac{|\mathbf{p}| \cos\theta}{m_t},\ \frac{|\mathbf{p}| \sin\theta}{m_l} \right\}\,. \tag{2.13}$$

Obviously, the direction of the velocity vector and the momentum vector do not coincide (these vectors are not collinear). Equation (2.13) can be rewritten as

$$|\mathbf{v}| = |\mathbf{p}| \left(\frac{\cos^2\theta}{m_t^2} + \frac{\sin^2\theta}{m_l^2} \right)$$

and finally as

$$|\mathbf{v}| = \frac{|\mathbf{p}|}{m^*}\,,$$

where m^* depends on the direction of the momentum $\mathbf{p}$:

$$\frac{1}{m^*} = \sqrt{\frac{\cos^2\theta}{m_t^2} + \frac{\sin^2\theta}{m_l^2}}\,.$$

(b) Using Eq. (2.13) for $\theta = 30^0, 45^0$, and 60^0 we find respectively:

$$\mathbf{p_1} = p_0 \left\{ \frac{1}{2}, \frac{\sqrt{3}}{2} \right\} \quad \mathbf{v_1} = \frac{p_0}{m_0} \left\{ \frac{1}{2}\frac{m_0}{m_t}, \ \frac{m_0}{m_l}\frac{\sqrt{3}}{2} \right\}$$

$$\mathbf{p_2} = p_0 \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\} \quad \mathbf{v_2} = \frac{p_0}{m_0} \left\{ \frac{1}{\sqrt{2}}\frac{m_0}{m_t}, \ \frac{m_0}{m_l}\frac{1}{\sqrt{2}} \right\}$$

$$\mathbf{p_3} = p_0 \left\{ \frac{\sqrt{3}}{2}, \frac{1}{2} \right\} \quad \mathbf{v_3} = \frac{p_0}{m_0} \left\{ \frac{\sqrt{3}}{2}\frac{m_0}{m_t}, \ \frac{m_0}{m_l}\frac{1}{2} \right\}\,.$$

In the case of Ge we obtain

$$\mathbf{v_1} = 26.32 \times \frac{p_0}{m_0} \{0.999, 0.034\}$$