Problem 1.1 An imaging lens in a digital camera has a focal length of 6 cm. How far should the lens be from the camera's CCD array to focus on an object

(a) 12 cm in front of the lens?
(b) 15 cm in front of the lens?

Solution:

(a) The lens equation (Eq. (1.1)) is $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$.

Here, $f = 6$ cm and $d_o = 12$ cm, so $d_i = \boxed{12 \text{ cm}}$.

(b) The lens equation (Eq. (1.1)) is $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$.

Here, $f = 6$ cm and $d_o = 15$ cm, so $d_i = \boxed{10 \text{ cm}}$.

Problem 1.2 An imaging lens in a digital camera has a focal length of 4 cm.

How far should the lens be from the camera's CCD array to focus on an object

(a) 12 cm in front of the lens;
(b) 8 cm in front of the lens.

Solution:

(a) The lens equation (Eq. (1.1)) is $\frac{1}{d_\mathrm{o}} + \frac{1}{d_\mathrm{i}} = \frac{1}{f}$.

Here, $f = 4$ cm and $d_\mathrm{o} = 12$ cm, so $d_\mathrm{i} = \boxed{6 \text{ cm}}$.

(b) The lens equation (Eq. (1.1)) is $\frac{1}{d_\mathrm{o}} + \frac{1}{d_\mathrm{i}} = \frac{1}{f}$.

Here, $f = 4$ cm and $d_\mathrm{o} = 8$ cm, so $d_\mathrm{i} = \boxed{8 \text{ cm}}$.

Problem 1.3 The following program loads an image stored in `clown.mat` as $I_o(x, y)$, passes it through an imaging system with the PSF given by Eq. (1.6), and displays $I_o(x, y)$ and $I_i(x, y)$. Parameters Δ, D, d_i, and λ (all in mm) are specified in the program's first line.

```
clear;Delta=0.0002;D=0.03;lambda=0.0000005;di=0.003;T=round(0.01/Delta);
for I=1:T;for J=1:T;x2y2(I,J)=(I-T/2).*(I-T/2)+(J-T/2).*(J-T/2);end;end;
gamma=pi*D/lambda*sqrt(x2y2./(x2y2+di*di/Delta/Delta));
h=2*besselj(1,gamma)./gamma;
h(T/2,T/2)=(h(T/2+1,T/2)+h(T/2-1,T/2)+h(T/2,T/2+1)+h(T/2,T/2-1))/4;
h=h.*h;H=h(T/2-5:T/2+5,T/2-5:T/2+5);load clown.mat;Y=conv2(X,H);
figure,imagesc(X),axis off,colormap(gray),figure,imagesc(Y),axis off,colormap(gray)
```

Run the program and display $I_o(x, y)$ (input) and $I_i(x, y)$ (output).

Solution: $I_o(x, y)$ is at left and $I_i(x, y)$ is at right.

The image formed by the optical system is blurred, as expected.

Problem 1.4 Compare the azimuth resolution of a real-aperture radar with that of a synthetic-aperture radar, with both pointed at the ground from an aircraft at a range $R = 5$ km. Both systems operate at $\lambda = 3$ cm and utilize a 2-m-long antenna.

Solution: For the real-aperture radar,

$$\Delta Y'_{\min} = \frac{\lambda R}{l_y} = \frac{3 \times 10^{-2} \times 5 \times 10^3}{2} = 75 \text{ m}.$$

For the SAR,

$$\Delta Y'_{\min} = \frac{l_y}{2} = \frac{2}{2} = 1 \text{ m}.$$

Problem 1.5 A 2-m-long antenna is used to form a synthetic-aperture radar from a range of 100 km. What is the length of the synthetic aperture?

Solution: Scaling the range in Fig. 1-21 from 400 km down to 100 km leads to a synthetic aperture shorter by the same factor. Hence, the synthetic aperture is of length 8 km/4 = 2 km.

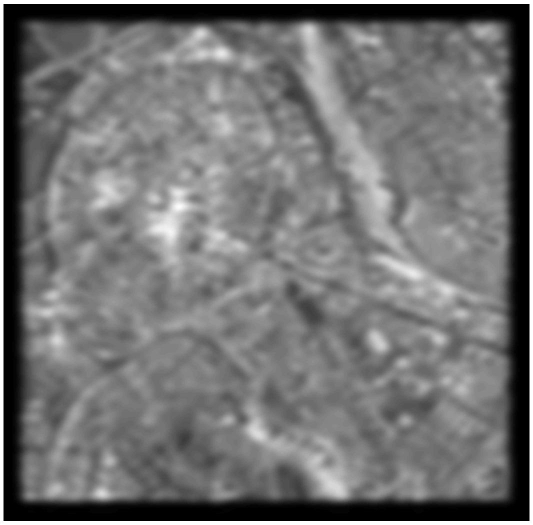
Problem 1.6 The following program loads an image stored in `sar.mat` as $I_\mathrm{o}(x,y)$, passes it through an imaging system with the PSF given by Eq. (1.15), and displays $I_\mathrm{o}(x,y)$ and $I_\mathrm{i}(x,y)$. Parameters Δ, τ, and l are specified in the program's first line.

```
clear;Delta=0.1;l=5;tau=1;I=[-15:15];z=pi*1.8*Delta*I/l;load sar.mat;
hy=sin(pi*z)./(pi*z);hy(16)=1;hy=hy.*hy;hx=exp(-2.77*Delta*Delta*I.*I/tau/tau);
H=hy'*hx;Y=conv2(X,H);
figure,imagesc(X),axis off,colormap(gray),figure,imagesc(Y),axis off,colormap(gray)
```

Run the program and display $I_\mathrm{o}(x,y)$ (input) and $I_\mathrm{i}(x,y)$ (output).

Solution: $I_\mathrm{o}(x,y)$ is at left and $I_\mathrm{i}(x,y)$ is at right.

The image formed by the radar system is blurred, as expected.



Problem 1.7 (This problem assumes prior knowledge of the 1-D Fourier transform (FT)). The basic CT problem is to reconstruct $\alpha(\xi,\eta)$ in Eq. (1.18) from $p(r,\theta)$. One way to do this is as follows:

(a) Take the FT of Eq. (1.18), transforming r to f. Define $p(-r,\ \theta) = p(r,\ \theta+\pi)$.
(b) Define and substitute $\mu = f\cos\theta$ and $\nu = f\sin\theta$ in this FT.
(c) Show that the result defines 2 FTs, transforming ξ to μ and η to ν, and that $\mathbf{A}(\mu,\nu) = \mathbf{P}(f,\theta)$. Hence, $\alpha(\xi,\eta)$ is the inverse FT of $\mathbf{P}(f,\theta)$.

Solution:

(a) The FT of Eq. (1.18) taking r to f is

$$\mathbf{P}(f,\theta) = \boldsymbol{\mathfrak{F}}\{p(r,\theta)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(\xi,\eta)\; e^{-j2\pi f(\xi\cos\theta+\eta\sin\theta)}\; d\xi\; d\eta.$$

(b) Substituting gives

$$\mathbf{P}(f,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(\xi,\eta)\; e^{-j2\pi\mu\xi} e^{-j2\pi\nu\eta}\; d\xi\; d\eta.$$

(c) $\mathbf{P}(f,\theta) = \boldsymbol{\mathfrak{F}}_{\xi\to\mu}\{\boldsymbol{\mathfrak{F}}_{\eta\to\nu}\{\alpha(\xi,\eta)\}\} = \mathbf{A}(\mu,\nu)$,

Problem 1.8 The following program loads an image stored in `mri.mat` as $I_o(x, y)$, passes it through an imaging system with the PSF given by Eq. (1.20), and displays $I_o(x, y)$ and $I_i(x, y)$. Parameters Δ, N, and dk are specified in the program's first line.

```
clear;N=16;Delta=0.01;dk=1;I=[-60:60];load mri.mat;
h=dk*sin(pi*N*dk*I*Delta)./sin(pi*dk*I*Delta);h(61)=N;H=h'*h;Y=conv2(X,H);
figure,imagesc(X),axis off,colormap(gray),figure,imagesc(Y),axis off,colormap(gray)
```

Run the program and display $I_o(x, y)$ (input) and $I_i(x, y)$ (output).

Solution: $I_o(x, y)$ is at left and $I_i(x, y)$ is at right.

The image formed by the MRI system is blurred, as expected.

Problem 1.9 This problem shows how beamforming works on a linear array of transducers, as illustrated in Fig. 1-35, in a medium with a wave speed of 1540 m/s. We are given a linear array of transducers located 1.54 cm apart along the x axis, with the nth transducer located at $x = 1.54n$ cm. Outputs $\{y_n(t)\}$ from the transducers are delayed and summed to produce the signal $y(t) = \sum_n y_n(t - 0.05n)$. In what direction (angle from perpendicular to the array) is the array focused?

Solution: Consider a plane wave (impulse in space and time) $\delta(t - x\sin\theta - y\cos\theta)$ arriving at the array from a direction θ (angle from perpendicular to the array). The plane wave hits the nth transducer at $t = n\sin(\theta)\frac{1.54 \text{ cm}}{1540 \text{ m/s}} = 0.1n\sin\theta$. Setting the delay between tranducers $0.05n = 0.1n\sin\theta$ gives $\boxed{\theta = 30^\circ}$.

Problem 2.1 Compute the following convolutions:

(a) $e^{-t}\ u(t) * e^{-2t}\ u(t)$

(b) $e^{-2t}\ u(t) * e^{-3t}\ u(t)$

(c) $e^{-3t}\ u(t) * e^{-3t}\ u(t)$

Solution:

The convolution of two causal signals is $y(t) = u(t)\ \int_0^t h(\tau)\ x(t-\tau)\ d\tau$.

(a): $e^{-t}\ u(t) * e^{-2t}\ u(t) = u(t)\ \int_0^t e^{-\tau}e^{-2(t-\tau)}\ d\tau = e^{-2t}\ u(t)\ \int_0^t e^{\tau}\ d\tau = e^{-2t}\ u(t)\ [e^t - 1] = \boxed{e^{-t}\ u(t) - e^{-2t}\ u(t)}$

(b): $e^{-2t}\ u(t) * e^{-3t}\ u(t) = u(t)\ \int_0^t e^{-2\tau}e^{-3(t-\tau)}\ d\tau = e^{-3t}\ u(t)\ \int_0^t e^{\tau}\ d\tau = e^{-3t}\ u(t)\ [e^t - 1] = \boxed{e^{-2t}\ u(t) - e^{-3t}\ u(t)}$

(c): $e^{-3t}\ u(t) * e^{-3t}\ u(t) = u(t)\ \int_0^t e^{-3\tau}e^{-3(t-\tau)}\ d\tau = e^{-3t}\ u(t)\ \int_0^t\ d\tau = \boxed{te^{-3t}\ u(t)}$

Problem 2.2 Show that the spectrum of $\frac{\sin(20\pi t)}{\pi t}\frac{\sin(10\pi t)}{\pi t}$ is zero for $|f| > 15$ Hz.

Solution:

Using the Fourier transform property $\mathcal{F}[x(t)y(t)] = \mathbf{X}(f) * \mathbf{Y}(f)$,
and the property that the width of a convolution is the sum of the widths,
the bandwidth of the product of two signals is the sum of their bandwidths.

$$\mathcal{F}\left[\frac{\sin(20\pi t)}{\pi t}\right] = \begin{cases} 1, & |f| < 10 \text{ Hz} \\ 0, & |f| > 10 \text{ Hz} \end{cases}$$

and

$$\mathcal{F}\left[\frac{\sin(10\pi t)}{\pi t}\right] = \begin{cases} 1, & |f| < 5 \text{ Hz} \\ 0, & |f| > 5 \text{ Hz.} \end{cases}$$

Hence,

$$\mathcal{F}\left[\frac{\sin(20\pi t)}{\pi t}\frac{\sin(10\pi t)}{\pi t}\right] = 0 \quad \text{for } |f| > 15 \text{ Hz.}$$

Problem 2.3 Using only Fourier transform properties, show that

$$\frac{\sin(10\pi t)}{\pi t}[1 + 2\cos(20\pi t)] = \frac{\sin(30\pi t)}{\pi t}.$$

Solution:

$\mathcal{F}\left[\frac{\sin(10\pi t)}{\pi t}\right] = \begin{cases} 1, & |f| < 5 \\ 0, & |f| > 5 \end{cases}$

Using the modulation property:

$$\mathcal{F}[x(t)\cos(2\pi f_0 t)] = \frac{1}{2}\mathbf{X}(f - f_0) + \frac{1}{2}\mathbf{X}(f + f_0),$$

we have

$$\mathcal{F}\left[\frac{\sin(10\pi t)}{\pi t}\cos(20\pi t)\right] = \begin{cases} \frac{1}{2}, & 5 \text{ Hz} < |f| < 15 \text{ Hz}, \\ 0, & \text{otherwise.} \end{cases}$$

Adding the first equation to double the second equation gives

$$\begin{aligned} \mathcal{F}\left[\frac{\sin(10\pi t)}{\pi t}(1 + 2\cos(20\pi t))\right] &= \begin{cases} 1, & |f| < 5 \text{ Hz} \\ 0, & |f| > 5 \text{ Hz} \end{cases} + \begin{cases} 1, & 5 \text{ Hz} < |f| < 15 \text{ Hz}, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & |f| < 15 \text{ Hz}, \\ 0, & |f| > 15 \text{ Hz}. \end{cases} \end{aligned}$$

The inverse Fourier transform of this result is $\frac{\sin(30\pi t)}{\pi t}$.

The sum of this lowpass filter and bandpass filter is another lowpass filter: