Chapter 1

Introduction

Section 1.1. Graphs

1. This information can be modeled (or represented) by the graph shown in Figure 1.1, where each small circle indicates a box and a line segment between two boxes indicates that these two boxes contain at least one wire segment of the same color.

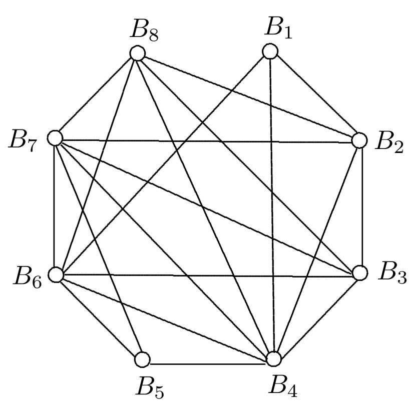

Figure 1.1: The graph in Exercise 1

2. The graph G is shown in Figure 1.2. The degree of $\emptyset$ is 7, the degree of each of $\{1\}$, $\{2\}$ and $\{3\}$ is 4, the degree of each of $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ is 2 and the degree of S is 1. The size of G is 13.

Section 1.2. The Degree of a vertex

3. Denote the degree of the remaining vertices by x. Since there are 8 vertices of degree x, it follows that $5 \cdot 4 + 6 \cdot 5 + 7 \cdot 6 + 8x = 2 \cdot 58$. Thus, $x = 3$ and so the degree of each remaining vertex is 3.

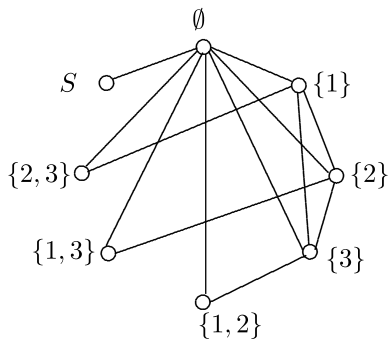

Figure 1.2: The graph G in Exercise 2

4. **Proof.** Assume, to the contrary, that G has at most $k+2$ vertices of degree $k+1$, at most k vertices of degree $k+2$ and at most $k+1$ vertices of degree $k+3$. Since the order of G is $n = 3k+3$, it follows that G has exactly $k+2$ vertices of degree $k+1$, exactly k vertices of degree $k+2$ and exactly $k+1$ vertices of degree $k+3$. In each case, G has an odd number of odd vertices, which is impossible. ■

5. **Proof.** Let G be a graph with r vertices of degree r, $r+1$ vertices of degree $r+1$ and $r+2$ vertices of degree $r+2$. Thus, the order of G is $3r+3$. First, we show that r is odd. Assume, to the contrary, that r is even. Then G contains an odd number $r+1$ of odd vertices, which is impossible by Corollary 1.5. Thus, r is odd and G contains $2r+2$ vertices of odd degree. ■

6. Let G be the graph of order $2k$ with $V(G) = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k\}$. For each i with $1 \le i \le k$, join each vertex u_i to the i vertices $v_1, v_2, \ldots, v_i$. Then $\deg u_i = i$ and $\deg v_i = k+1-i$ for $1 \le i \le k$.

Section 1.3. Isomorphic Graphs

7. (a) $G_1 \cong G_2$.

   (b) $H_1 \not\cong H_2$. For example, H_1 has two vertices of degree 4, while H_2 has three vertices of degree 4.

8. (a) There are 34 such graphs, each of which has size m for some m with $0 \le m \le 10$. By using complementary graphs the number of graphs of order 5 and size m equals the number of graphs of order 5 and size $10-m$ (see Figure 1.3).

   (b) The minimum size of a graph G of order 5 such that every graph of order 5 and size 5 is isomorphic to some subgraph of G is 7. First, observe that the graph $G = P_4 \vee K_1$ has the desired property. If the minimum size were 6, then G must consist of a 5-cycle C and an edge joining two

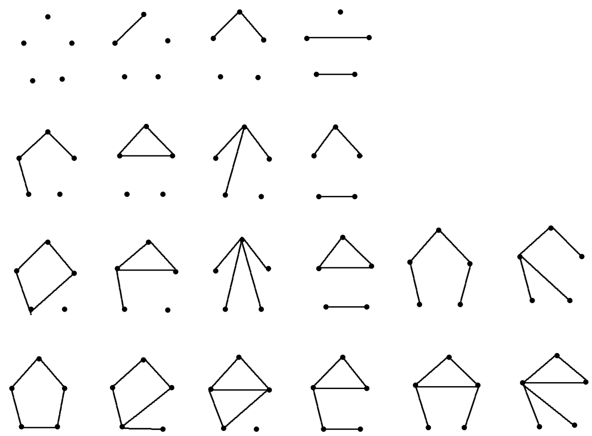

Figure 1.3: Graphs for Exercise 8

nonconsecutive vertices of C. This graph, however, does not have the desired property.

9. (a) **Proof.** Let $\phi : V(G) \to V(H)$ be an isomorphism from G to H. Then $|S| = |T|$ and the mapping $\phi' : S \to T$ defined $\phi'(v) = \phi(v)$ for all $v \in S$ is a bijection from S to T. To show that ϕ' is an isomorphism from $G[S]$ to $H[T]$, it remains to show that ϕ' maps adjacent vertices in $G[S]$ to adjacent vertices in $H[T]$ and maps nonadjacent vertices in $G[S]$ to nonadjacent vertices in $H[T]$. Let $u, v \in V(G[S]) = S$. Since $G[S]$ is an induced subgraph of G, it follows that u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. Since ϕ is an isomorphism from G to H, it follows that u and v are adjacent in G if and only if $\phi(u) = \phi'(u)$ and $\phi(v) = \phi'(v)$ are adjacent in H. Since (1) $\phi'(u), \phi'(v) \in T$ and (2) $H[T]$ is an induced subgraph of H, it follows that $\phi'(u)$ and $\phi'(v)$ are adjacent in H if and only if $\phi'(u)$ and $\phi'(v)$ are adjacent in $H[T]$. Therefore, u and v are adjacent in $G[S]$ if and only if $\phi'(u)$ and $\phi'(v)$ are adjacent in $H[T]$. Therefore, ϕ' is an isomorphism from $G[S]$ to $H[T]$ and so $G[S] \cong H[T]$. ■

   (b) Let $r = 3$. Then $G[S]$ has no edges and $G[T]$ has one edge. Thus $G[S] \ncong G[T]$ and so $G \ncong H$.

10. (a) The three graphs with this property are shown in Figure 1.4(a).

    (b) The graphs G, H, F_1 and F_2 in Figure 1.4(b) have this property.

Section 1.4. Regular Graphs

11. **Proof.** Denote the size of G by m. Thus, $m = rn/2$. The average degree of

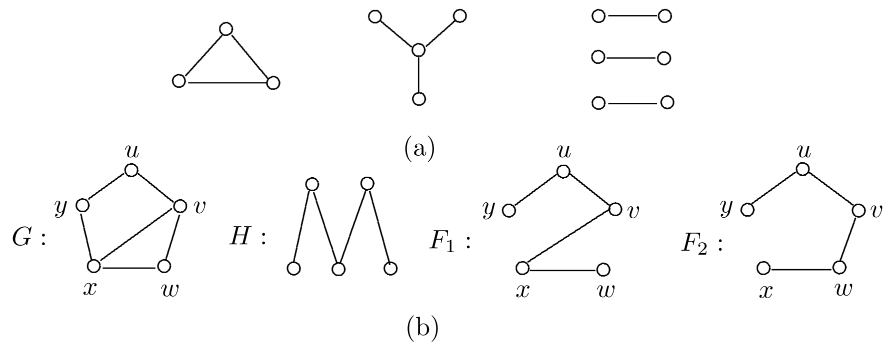

Figure 1.4: The graphs Exercise 10

G is

$$\frac{2m}{n} = 2\left(\frac{rn/2}{n}\right) = r.$$

Since G is not r-regular, G contains a vertex v such that $\deg v \neq r$. We consider two cases.

Case 1. $\deg v > r$. Not all vertices of G have degree r or more, for otherwise,

$$2m = 2\left(\frac{rn}{n}\right) = rn = \sum_{v \in V(G)} \deg v > rn,$$

which is impossible. Hence G contains a vertex u with $\deg u < r$. Therefore, $\Delta(G) \geq \deg v \geq r+1$ and $\delta(G) \leq \deg u \leq r-1$ and so $\Delta(G) - \delta(G) \geq 2$.

Case 2. $\deg v < r$. Not all vertices of G have degree r or less, for otherwise,

$$2m = 2\left(\frac{rn}{n}\right) = rn = \sum_{v \in V(G)} \deg v < rn,$$

which is impossible. Hence, G contains a vertex w with $\deg w > r$. Therefore, $\Delta(G) \geq \deg w \geq r+1$ and $\delta(G) \leq \deg v \leq r-1$ and so $\Delta(G) - \delta(G) \geq 2$. ■

12. Let $k \geq 2$. For $0 \leq i \leq k-1$, let $G_i = C_{3(k-i)} + iC_3$. The graphs $G_0, G_1, \ldots, G_{k-1}$ are pairwise non-isomorphic.

13. Let $V(G) = \{u, v, w, x\}$ and $E(G) = \{uv, vw, wx, xu, vx\}$. Let $e = vx$. Then $G - e = C_4$ and $G - u = C_3$.

14. (a) Let $G_1 = 2C_3$ and $G_2 = C_6$.
    (b) Let $H_1 = 3C_3$ and $H_2 = C_9$.

15. If G itself is r-regular, then there is nothing to prove. So we may assume that G is not r-regular. Let G' be another copy of G and join corresponding

vertices whose degrees are less than r, calling the resulting graph G_1. If G_1 is r-regular, then G_1 has the desired properties. If not, then we continue this procedure until arriving at an r-regular graph G_k, where $k = r - \delta(G)$.

16. For each i with $1 \le i \le j$, let v_i be the vertex of G with $\deg v_i = r - i$ and let G' be another copy of G where the vertex v'_i in G' corresponds to the vertex v_i in G. Let H be the graph of order $2n$ obtained from G and G' by joining v_i to $v'_{j-i+1}, v'_{j-i+2}, \ldots, v'_j$ for $1 \le i \le j$. Then H is an r-regular graph of order $2n$ containing G as an induced subgraph.

17. The Petersen graph.

18. (a) $G_{6,1} = K_6$ and $G_{5,2}$ is the Petersen graph.
    (b) The graph $G_{n,k}$ is an $\binom{n-k}{k}$-regular graph of order $\binom{n}{k}$.

Section 1.5. Bipartite Graphs

19. Let x be the number of vertices of degree 8 in W. Then $n = 10 + 4 + 3 + x$ and $m = 6 \cdot 10 = 60$. Since $m = 4 \cdot 2 + 3 \cdot 4 + 8x = 60$, it follows that $x = 5$ and so $n = 22$.

20. **Proof.** We have seen that the size of the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is $\lfloor n^2/4 \rfloor$. For every bipartite graph with partite sets U and W with $s = |U| \le |W| = t$ and $s + t = n$, clearly $K_{s,t}$ has the maximum size. If $0 \le t - s \le 1$, then $K_{s,t} = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ and the size of $K_{s,t}$ is $\lfloor n^2/4 \rfloor$. Suppose that $t - s \ge 2$. Then $t = \lceil n/2 \rceil + p$ and $s = \lfloor n/2 \rfloor - p$ for some $p \ge 1$. Then the size of $K_{s,t}$ is

$$(\lceil n/2 \rceil + p)\,(\lfloor n/2 \rfloor - p) = \lfloor n^2/4 \rfloor + p\,(\lfloor n/2 \rfloor - \lceil n/2 \rceil) - p^2 < \lfloor n^2/4 \rfloor.$$

Hence, $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the only bipartite graph having size $\lfloor n^2/4 \rfloor$. ■

21. **Proof.** Suppose that the partite sets of a 3-partite graph G of order $n = 3k$ and size m are A, B and C, where $|A| = a$, $|B| = b$ and $|C| = c$ and $a+b+c = 3k$. We may assume that $a \ge b \ge c$. Then $a \ge k$ and $c \le k$. Hence, $a = k + x, c = k - y$ and $b = 3k - a - c = k - x + y$, where $x, y \ge 0$. Then

$$\begin{aligned} m &\le ab + ac + bc = 3k^2 - x^2 + xy - y^2 \\ &= 3k^2 - \left[\left(x - \frac{y}{2}\right)^2 + \frac{3y^2}{4}\right] \le 3k^2, \end{aligned}$$

where $m = 3k^2$ if and only if $x = y = 0$ and so $a = b = c = k$, in which case, $G = K_{k,k,k}$. ■

22. **Proof.** Define a relation R on $V(G)$ by $u \; R \; v$ if $uv \notin E(G)$. The relation R is clearly reflexive and symmetric. For vertices u, v and w of G, if $uv \notin E(G)$ and $vw \notin E(G)$, then $uw \notin E(G)$. Thus, R is transitive and so R is an

equivalence relation on $V(G)$. Hence, $V(G)$ is partitioned into equivalence classes $V_1, V_2, \ldots, V_k$ where $k \geq 2$. Thus, two vertices are related if and only if they belong to the same equivalence class. Also, vertices belonging to different equivalence classes are not related and are therefore adjacent. Thus, G is a complete k-partite graph with partite sets $V_1, V_2, \ldots, V_k$. ■

Section 1.6. Operations on Graphs

23. (Every bipartite graph having no partite set with three or more vertices.) If G has a partite set with three or more vertices, then $\overline{G}$ has a triangle. Thus, we may assume that every partite set has at most two vertices and so the order of G is at most 4. Then $\overline{G}$ is a subgraph of $K_{2,2} = C_4$ and is therefore bipartite.

24. **Proof.** Suppose that G contains a vertices of degree less than k, b vertices of degree greater than k and c vertices of degree k. Thus, $\overline{G}$ contains b vertices of degree less than k, a vertices of degree greater than k and c vertices of degree k. Since the vertices of G and $\overline{G}$ have exactly the same degrees, $a = b$. Hence, the order of G is $2a + c$. Since G has odd order, c is odd. ■

25. (a) The graph $\overline{G}$ is C_7 or $C_3 + C_4$.

    (b) The graph $\overline{G}$ is one of $C_9, C_6 + C_3, C_5 + C_4$ or $3C_3$. Thus, there are four such graphs.

26. **Proof.** Assume, to the contrary, that there exists an r-regular self-complementary graph G of even order $n \geq 2$. Then $\overline{G}$ is $(n-1-r)$-regular and so $r = n-1-r$. Hence, $2r = n - 1$, which is odd. This is impossible. ■

27. **Proof.** We proceed by induction on n. The graph C_5 shows that the result holds for $n = 1$. Assume that there is a regular self-complementary graph H of order 5^k, where $k \geq 1$. Let H_i $(i = 1, 2, \ldots, 5)$ be five copies of H. Let G be the graph consisting of the graphs $H_1, H_2, \ldots, H_5$ together with all edges joining the vertices of H_i and the vertices of H_{i+1} for $i = 1, 2, \ldots, 5$, where $H_6 = H_1$. Then G is a regular self-complementary graph of order 5^{k+1}. ■

28. **Proof.** Suppose that the order n of G_2 is $n = 2k$ and that there are exactly a vertices of degree less than k in G_2. Since G_2 is self-complementary, there are exactly a vertices of degree greater than k in G_2. Furthermore, if $\deg v < k$, then $\deg_{\overline{G}} v = n - \deg v > n - k = k$. Observe that $\overline{G}$ is obtained from $\overline{G}_1$ and $\overline{G}_2$ by joining each vertex in G_2 whose degree is greater than k to every vertex in $\overline{G}_1$ (that is, joining each vertex in $\overline{G}_2$ whose degree is less than k to every vertex in $\overline{G}_1$). Since $G_1 \cong \overline{G}_1$ and $G_2 \cong \overline{G}_2$, the graph $\overline{G}$ is obtained from G_1 and G_2 by joining each vertex in G_2 whose degree is less than k to every vertex in G_1, which is isomorphic to G. ■

29. The following proof essentially makes use of Exercise 28.

    Proof. We use induction on n when (1) $n = 4k$, $k \geq 1$ and (2) $n = 4k + 1$, $k \geq 0$.

(1) Suppose that $n = 4k$. Since P_4 is self-complementary, the statement is true for $k = 1$. Assume that there exists a self-complementary graph G of order $4k$, where $k \geq 1$. We show that there exists a self-complementary graph H of order $4(k+1)$. Let H be the graph obtained from G and $P_4 = (v_1, v_2, v_3, v_4)$ by joining each of v_1 and v_4 to every vertex of G. Then H is self-complementary and the order of H is $4(k+1)$.

(2) Suppose that $n = 4k+1$. Since K_1 is self-complementary, the statement is true for $k = 0$. Now assume that there exists a self-complementary graph G of order $4k+1$, where $k \geq 0$. We show that there exists a self-complementary graph H of order $4(k+1)+1$. Again, let H obtained from G and $P_4 = (v_1, v_2, v_3, v_4)$ by joining each of v_1 and v_4 to every vertex of G. Then H is self-complementary and the order of H is $4(k+1)+1$. ■

30. (a) and (b) see Figures 1.5(a) and (b), respectively.

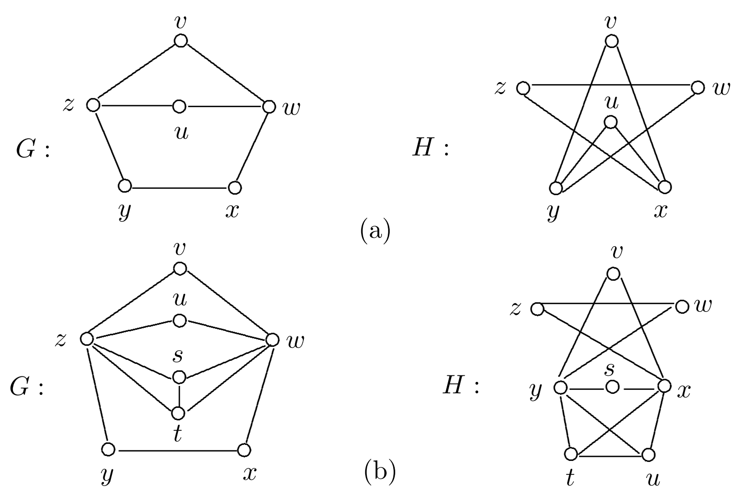

Figure 1.5: The graphs in Exercise 30(a) and (b)

31. **Proof.** Since there is a self-complementary graph of order n for every integer n with $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ by Exercise 29, we may assume that $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$. We verify this for $n \equiv 2 \pmod 4$ by induction. That the result holds for $n = 6$ follows from Exercise 30(a). Assume, for some integer $k \geq 6$ with $k \equiv 2 \pmod 4$, that there exists a graph G_1 of order k and size $\left\lfloor \binom{k}{2}/2 \right\rfloor$ that is isomorphic to a graph $H_1 \subseteq \overline{G}_1$. Add to G_1 the path $P = (p_1, p_2, p_3, p_4)$ of order 4 and join p_1 and p_4 to every vertex of G_1 and denote this graph by G. Then G has order $k+4$, size $\left\lfloor \binom{k+4}{2}/2 \right\rfloor$ and is isomorphic to a graph H of $\overline{G}$, where H is constructed from H_1 by adding (p_2, p_4, p_1, p_3) and joining p_2 and p_3 to every vertex of H_1. Hence, the result holds if $n \equiv 2 \pmod 4$.

The proof for the case when $n \equiv 3 \pmod 4$ is similar. ■

32. The subgraph H in Figure 1.6 is isomorphic to P.

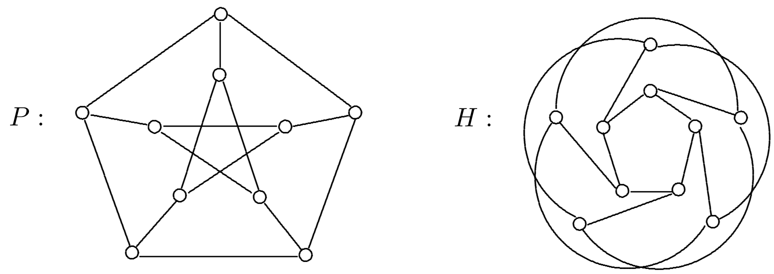

Figure 1.6: The graphs Exercise 32

33. (a) The degree of u_1 in $G_1 + G_2$ is $\deg_{G_1} u_1$.
    (b) The degree of u_1 in $G_1 \vee G_2$ is $\deg_{G_1} u_1 + n_2$
    (c) The degree of (u_1, u_2) in $G_1 \,\square\, G_2$ is $\deg_{G_1} u_1 + \deg_{G_2} u_2$.

34. The graph $P_3 \vee 2P_3$ has order 9 and size 24, the graph $P_3 \,\square\, 2P_3$ has order 18 and size 24, and the graph $Q_1 + Q_2 + Q_3$ has order 14 and size 17.

Section 1.7. Degree Sequences

35. See Figure 1.7. Delete the edges uw and xy of G and add the edges ux and wy. This produces a graph isomorphic to H.

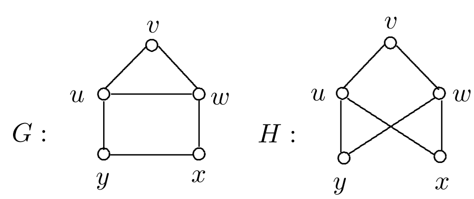

Figure 1.7: The graphs G and H in Exercise 35

36. (a) The minimum number of 2-switches is 1. Label the vertices of G_1 as shown in Figure 1.8(a). Deleting the edges v_1v_2 and v_5v_6 and adding the edges v_1v_6 and v_2v_5 produces a graph G_1' that is isomorphic to H_1.
    (b) The minimum number of 2-switches is 2. Label the vertices of G_2 as shown in Figure 1.8(b). Since G_2 contains three pairwise disjoint triangles and H_2 is bipartite, at least two 2-switches are required to transform G_2 into H_2. Consider the 2-switch in which the edges v_2w_2 and u_3w_3 are deleted from G_2 and the edges u_3w_2 and v_2w_3 are added giving the graph G_2'.

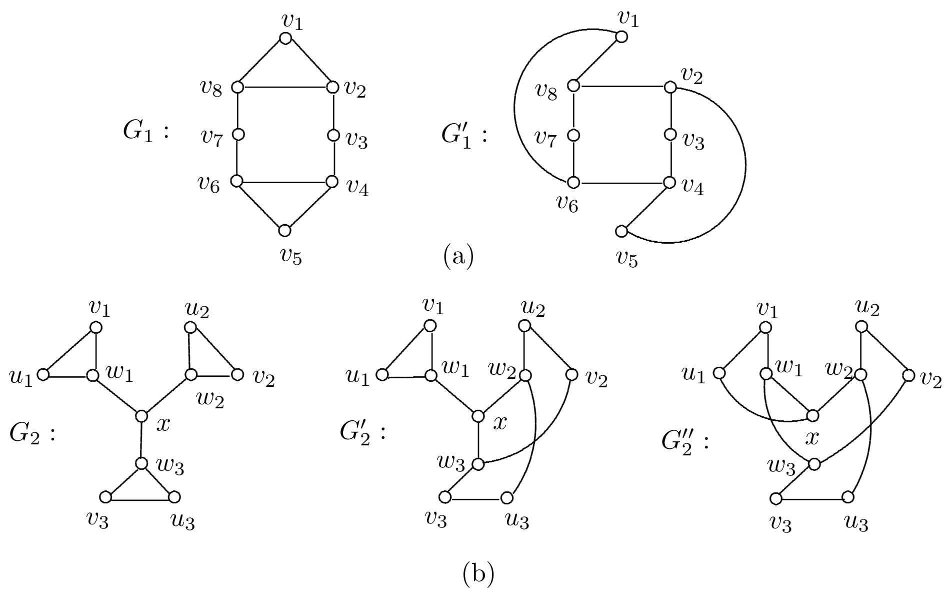

Figure 1.8: The graphs in Exercise 36

Next consider the 2-switch in which the edges u_1w_1 and xw_3 are deleted from G_2' and the edges w_1w_3 and u_1x are added. This produces the graph G_2''. Since $G_2'' \cong H_2$, a minimum number of two 2-switches is required to transform G_2 to H_2.

37. $\mathcal{G}_s = \{C_9, C_4 + C_5, C_3 + C_6, 3C_3\}$ and $G = P_4 = (3C_3, C_3 + C_6, C_9, C_4 + C_5)$.

38. The three graphs G_1, G_2, G_3 in Figure 1.9 all have the degree sequence

$$s : 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3$$

and form a triangle in G.

39. **Proof.** Suppose that this statement is false. Thus, for each graph H with $V(H) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \le i \le n$, there is a vertex v_k $(1 \le k \le n)$ such that both (1) and (2) fail. Among all such graphs H and all such vertices v_k, let G be one with a vertex v_k for which $N_G(v_k)$ has a maximum number of vertices in common with either $W_1 = \{v_1, v_2, \ldots, v_{d_k}\}$ if $k > d_k$ or with $W_2 = \{v_1, v_2, \ldots, v_{d_k+1}\} - \{v_k\}$ if $1 \le k \le d_k$. We consider two cases.

Case 1. $k > d_k$. Thus, $N_G(v_k) \ne W_1$ and the vertex v_k is adjacent to a vertex $v_j \notin W_1$ and is not adjacent to a vertex v_i with $1 \le i \le d_k$ and so $d_i \ge d_j$. Consequently, there is a vertex v_ℓ such that $v_iv_\ell \in E(G)$ and $v_jv_\ell \notin E(G)$. Replacing the edges v_kv_j and v_iv_ℓ by v_kv_i and v_jv_ℓ is a 2-switch in G that produces a graph G_1 with $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ where $\deg_{G_1} v_i = d_i$ for

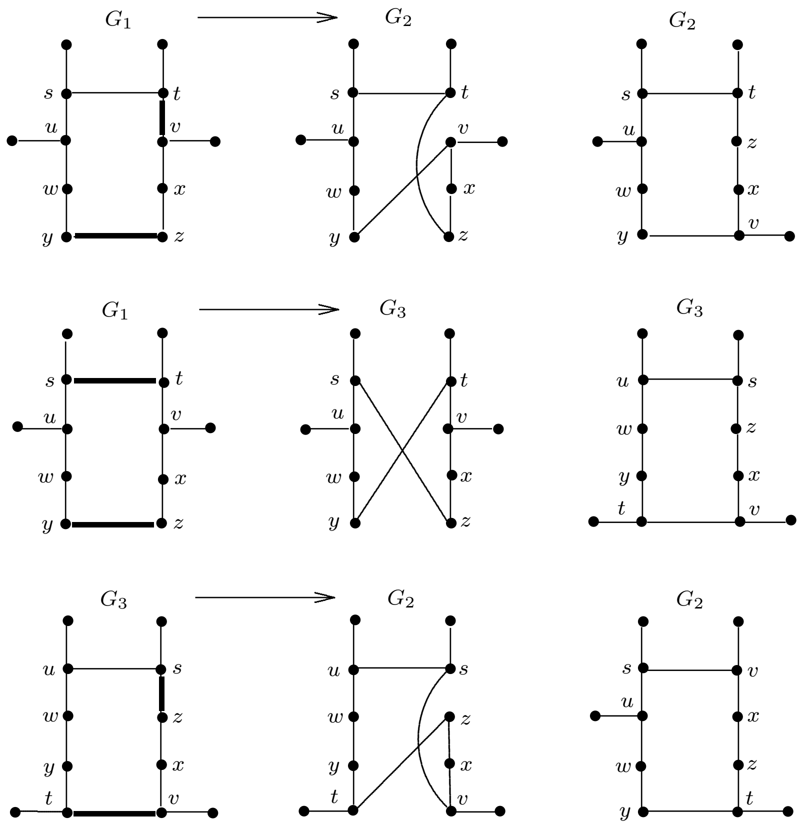

Figure 1.9: The graphs Exercise 38

$1 \le i \le n$ in which $N_{G_1}(v_k)$ has a greater number of vertices in common with W_1. This is a contradiction.

Case 2. $1 \le k \le d_k$. Thus, $N_G(v_k) \ne W_2$ and so the vertex v_k is adjacent to a vertex $v_j \notin W_2$ and is not adjacent to a vertex v_i with $1 \le i \le d_k + 1$ and so $d_i \ge d_j$. Consequently, there is a vertex v_ℓ such that $v_iv_\ell \in E(G)$ and $v_jv_\ell \notin E(G)$. Replacing the edges v_kv_j and v_iv_ℓ by v_kv_i and v_jv_ℓ is a 2-switch in G that produces a graph G_2 with $V(G_2) = \{v_1, v_2, \ldots, v_n\}$ where $\deg_{G_2} v_i = d_i$ for $1 \le i \le n$ in which $N_{G_2}(v_k)$ has a greater number of vertices in common with W_2. This is a contradiction. ■

40. See Figure 1.10.

(a) $G_2 = G_1 - xz + xy$ (see Figure 1.10).

(b) The degree sequence of G_1 is $5, 4, 3, 3, 2, 2, 2, 1$ and the degree sequence of G_3 is $4, 3, 3, 3, 3, 2, 2, 2$. When an edge rotation of G_1 takes place, one

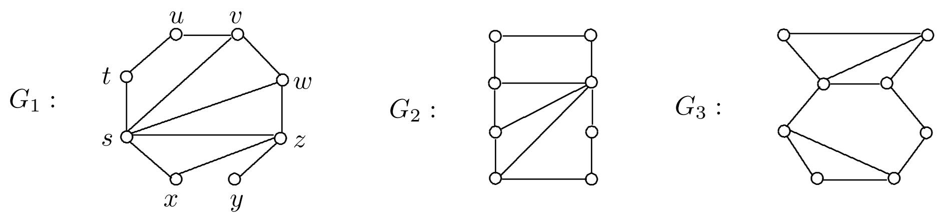

Figure 1.10: The graphs in Exercise 40

vertex of G_1 has its degree increased by 1 and another has its degree decreased by 1. Hence, the degree sequence of G_3 cannot be obtained.

(c) **Proof.** Let G and H be two nonempty, noncomplete graphs of the same order n and same size m. We may assume that $V(G) = V(H) = \{v_1, v_2, \ldots, v_n\}$. Furthermore, $\binom{k}{2} \le m < \binom{k+1}{2}$ for some integer k with $2 \le k \le n-1$. Let $S = \{v_1, v_2, \ldots, v_k\}$ and let F be the graph of order n and size m with $V(F) = \{v_1, v_2, \ldots, v_n\}$ such that $F[S] = K_k$ and v_{k+1} is adjacent to $v_1, v_2, \ldots, v_j$, where $j = m - \binom{k}{2} < k$. We claim that G can be transformed into F by a sequence of edge rotations. Suppose that this is not the case. Among all graphs into which G can be transformed by a sequence of edge rotations, let F' be one for which $F'[S]$ has maximum size. We consider two cases.

Case 1. *The size of* $F'[S]$ *is less than* $\binom{k}{2}$. Then F' contains two nonadjacent vertices v_i and v_j with $1 \le i < j \le k$. If either v_i or v_j is adjacent to a vertex v_ℓ with $\ell > k$, then a graph F'' can be obtained from F' by a single edge rotation so that $v_iv_j \in E(F'')$, producing a contradiction. Otherwise, F' contains an edge v_pv_q where $p, q > k$. Since $v_iv_p \notin E(F')$, a graph F'' containing v_iv_j can be obtained from F' by a single edge rotation so that $v_iv_j \in E(F'')$, again a contradiction.

Case 2. *The size of* $F'[S]$ *is* $\binom{k}{2}$. Among all graphs F' with $F'[S] \cong K_k$ into which G can be transformed by a sequence of edge rotations, let G' be one for which v_{k+1} is adjacent to a maximum number of the vertices $v_1, v_2, \ldots, v_j$, where $j = m - \binom{k}{2}$. Thus $v_{k+1}v_\ell \notin E(G')$ for some ℓ with $1 \le \ell \le j$. If $v_{k+1}v_p \in E(G')$ where $p > j$, then a graph G'' containing $v_{k+1}v_p$ can be obtained from G' by a single edge rotation, a contradiction. Otherwise, v_{k+1} is isolated in G' and G' contains an edge v_pv_q where $p, q > k+1$. Thus, $v_{k+1}v_p \notin E(G')$ and G' can be transformed into a graph containing $v_{k+1}v_p$ by a single edge rotation which in turn can be transformed into a graph containing $v_{k+1}v_\ell$, again producing a contradiction.

Thus, as claimed, G can be transformed into the graph F with the desired properties by a sequence of edge rotations, as can H. Consequently, F can be transformed into the graph H by a sequence of edge rotations, implying that G can be transformed into the graph H by a sequence of edge rotations. ■

41. (b), (c) and (e) are graphical.

42. For a vertex v of a graph G of order n, we have $\deg_G v + \deg_{\overline{G}} v = n - 1$. Thus, $d_1, d_2, \cdots, d_n$ is a degree sequence of a graph G of order n if and only if $(n-1) - d_1, (n-1) - d_2, \cdots, (n-1) - d_n$ is a degree sequence of $\overline{G}$.

43. If there exists a graph G with degree sequence $x, 1, 2, 3, 5, 5$, then the order of G is 6. Since there are at least two vertices of degree 5, it follows that $\delta(G) \geq 2$ and so no vertex of G has degree 1.

44. Since every graph has an even number of odd vertices, x must be even. So $x = 0, 2, 4, 6$ are the only possibilities. We cannot have $x = 0$ since not all the degrees can be different. We now consider $x = 2, 4, 6$, and apply the Havel-Hakimi theorem (Theorem 1.10).

$\underline{x = 2}$
$s : 7, 6, 5, 4, 3, 2, 2, 1$
$s_1 : 5, 4, 3, 2, 1, 1, 0$
$s_2 : 3, 2, 1, 0, 0, 0$
$s_3 : 1, 0, -1, 0, 0$. Since s_3 is not graphical, s is not graphical and so $x \neq 2$.
$\underline{x = 4}$
$s : 7, 6, 5, 4, 4, 3, 2, 1$
$s_1 : 5, 4, 3, 3, 2, 1, 0$
$s_2 : 3, 2, 2, 1, 0, 0$
$s_3 : 1, 1, 0, 0, 0$. Note that s_3 is a degree sequence of $G = K_2 + 3K_1$ and so s_3 is graphical. Therefore, s is graphical and $x = 4$ is possible.
$\underline{x = 6}$
$s : 7, 6, 6, 5, 4, 3, 2, 1$
$s_1 : 5, 5, 4, 3, 2, 1, 0$
$s_2 : 4, 3, 2, 1, 0, 0$.
Ignoring isolated vertices, not all degrees can be different and so s_2 is not graphical. Therefore, s is not graphical and $x \neq 6$.

Therefore, the sequence is graphical for $x = 4$ only.

45. Since $\sum_{i=1}^{3} d_i = 17$ and $3(3-1) + \sum_{i=1}^{7} \min\{3, d_i\} = 16$, it follows from Theorem 1.13 that the sequence $s : 6, 6, 5, 4, 3, 2, 2$ is not graphical.

46. First, there is no such graphical sequence where one term occurs 3 times since no graph contains an odd number of odd vertices. Thus, the only possible graphical sequences are those in which two terms occur twice each. Again, since no graph contains an odd number of odd vertices, the sequence $4, 4, 3, 2, 2$ is not graphical. On the other hand, both sequences $4, 4, 3, 3, 2$ and $4, 3, 3, 2, 2$ are graphical.

47. Let $S = \{a_1, a_2, \cdots, a_n\}$ be a set of positive integers and let $k = \text{lcm}\ \{a_1 + 1, a_2 + 1, \cdots, a_n + 1\}$. (Note that we could also let $k = (a_1+1)(a_2+1)\cdots(a_n+1)$.) Let G be the union of the graphs $\frac{k}{a_i+1}K_{a_i+1}$ for $1 \le i \le n$. Then G has the desired property.

Let $S = \{2, 6, 7\}$. If $k = 1, 3$, then the graph with degree set S has an odd number of odd vertices, which is impossible. If $k = 2$, then a degree sequence of the graph G is $7, 7, 6, 6, 2, 2$. This implies that G has order 6 but maximum degree 7, which is impossible. The sequence

$$s : 7, 7, 7, 7, 6, 6, 6, 6, 2, 2, 2, 2$$

is graphical by Theorem 1.12.

48. The graph F_n constructed in Theorem 1.15 contains exactly two vertices of degree $\lfloor \frac{n}{2} \rfloor$. Since the degrees of the vertices of F_n are $1, 2, \ldots, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \ldots, n-1$, the degrees of the vertices of $\overline{F}_n$ are $n-2, n-3, \ldots, \lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil - 1, \ldots, 1, 0$.

49. **Proof.** Assume first that s_1' and s_2' are bigraphical sequences. Then there exists a bipartite graph G' with partite sets V_1' and V_2' such that $V_1' = \{u_2, u_3, \ldots, u_r\}$ and $V_2' = \{w_1, w_2, \ldots, w_t\}$, where $\deg_{G'} u_i = a_i$ for $2 \le i \le r$ and

$$\deg_{G'} w_j = \begin{cases} b_j - 1 & \text{if } 1 \le j \le a_1 \\ b_j & \text{if } a_1 + 1 \le j \le t. \end{cases}$$

A new bipartite graph G can be constructed from G' by adding a new vertex u_1 and joining u_1 to w_i for $1 \le i \le a_1$. Thus the sequences s_1 and s_2 are bigraphical sequences.

For the converse, suppose that s_1 and s_2 are bigraphical sequences. Hence, there exist bipartite graphs with partite sets $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{w_1, w_2, \ldots, w_t\}$ such that $\deg u_i = a_i$ and $\deg w_j = b_j$ for $1 \le i \le r$ and $1 \le j \le t$. If there exists such a bipartite graph H and a vertex u of H of degree a_1 in V_1 adjacent to vertices of degrees $b_1, b_2, \ldots, b_{a_1}$, then s_1' and s_2' are bigraphical sequences for $H - u$. Hence, we may assume that no bipartite graph with bigraphical sequences s_1 and s_2 has this property. Among all such graphs, let G be one having a vertex u_1 of degree a_1 in V_1 such that the sum of the degrees of the neighbors of u_1 is maximum. Hence there exist integers i and j with $1 \le i < j \le t$ such that $b_i > b_j$ and u_1 is not adjacent to w_i but is adjacent to w_j. Because $\deg w_i > \deg w_j$, there is a vertex $u_k \in V_1$ with $\deg u_k = a_k$ such that w_i is adjacent to u_k but w_j is not adjacent to u_k. Let

$$F = (G - \{u_1 w_j, u_k w_i\}) + \{u_1 w_i, u_k w_j\}.$$

Thus, the bipartite graph F also has bigraphical sequences s_1 and s_2 but the sum of the degrees of the neighbors of u_1 in F is greater than the sum of degrees of the neighbors of u_1 in G. This is a contradiction. ∎

50. These are the two nearly irregular graphs of order 10 and each is the complement of the other.

51. (a) Let $n = r + s$, where $s \geq 0$ is even. Then $G = rK_1 + (s/2)K_2$ has the desired properties.

    (b) Since every even vertex has degree at least 0 and every odd vertex has degree at least 1, the minimum size of such a graph is at least $s/2$. Since the graph $rK_1 + (s/2)K_2$ has this size, the minimum size is $s/2$.

    (c) Suppose first that n is even. Then $n - 1$ is odd. So, every odd vertex has degree at most $n-1$ and every even vertex has degree at most $n-2$. Hence the maximum size of G is at most $\binom{n}{2} - r/2$. Since r and s are both even, the graph $\overline{(r/2)K_2 + sK_1}$ has size $\binom{n}{2} - r/2$, which is the maximum size. Suppose next that n is odd. Then $n - 1$ is even. Every even vertex has degree at most $n-1$ and every odd vertex has degree at most $n - 2$. Hence, the maximum size of G is at most $\binom{n}{2} - s/2$. Since $\overline{(s/2)K_2 + rK_1}$ has size $\binom{n}{2} - s/2$, this is the maximum size.

52. (a) **Proof.** Let $uv \in E(G)$. Let W_{00} be the set of vertices in $V(G) - \{u, v\}$ adjacent to neither u nor v, let W_{10} be the set of vertices in $V(G) - \{u, v\}$ adjacent to u but not v, let W_{01} be the set of vertices in $V(G) - \{u, v\}$ adjacent to v but not u and let W_{11} be the set of vertices in $V(G) - \{u, v\}$ adjacent to both u and v. For $i, j \in \{0, 1\}$, let $|W_{ij}| = a_{ij}$. Then $a_{00} + a_{10} + a_{01} + a_{11} = n - 2$, $a_{10} + a_{11} \geq (2n + 1)/3$ and $a_{01} + a_{11} \geq (2n + 1)/3$. Therefore,

$$a_{01} + a_{10} + 2a_{11} \geq \frac{4n + 2}{3}$$

and so

$$(a_{01} + a_{10} + a_{11}) + a_{11} = (n - 2 - a_{00}) + a_{11} \geq \frac{4n + 2}{3}.$$

Thus,

$$a_{11} \geq \frac{4n + 2}{3} - (n - 2) + a_{00} = \frac{n + 8}{3} + a_{00} \geq \frac{n + 8}{3}.$$

Since $n \geq 4$, W_{11} contains at least four vertices. If any two vertices in W_{11} are adjacent, then u and v belong to a complete subgraph of order 4. Suppose, on the other hand, that no two vertices of W_{11} are adjacent. Then for each $x \in W_{11}$,

$$\deg x \leq n - \frac{n + 8}{3} = \frac{2n - 8}{3},$$

contradicting the assumption that $\deg x \geq (2n + 1)/3$ for every vertex v of G. ∎

(b) Since every vertex in $K_{r,r,r}$ has degree $2r = 2n/3$ and $K_{r,r,r}$ contains no complete subgraph of order 4, the result in (a) is best possible.

Section 1.8. Multigraphs

53. Only the sequence in (c) is a degree sequence of an irregular multigraph (see Figure 1.11). There is no multigraph containing an odd number of odd vertices. So, (a) and (b) are not the degree sequences of any multigraph.

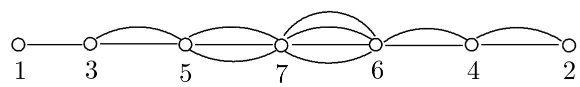

Figure 1.11: The graphs Exercise 53

54. The graphs of order 3 and 4 are shown in Figure 1.7 in the text. See Figure 1.12.

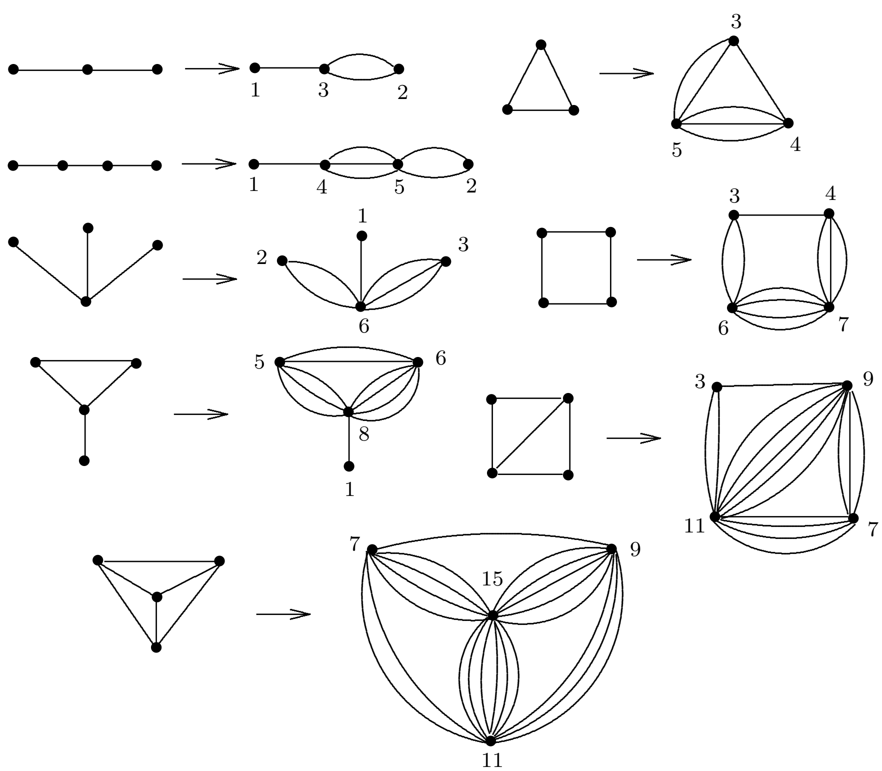

Figure 1.12: Graphs and multigraphs in Exercise 54

55. Each of the sequences in (a), (c), (e), (g) and (h) is the degree sequence of a multigraph. See Figure 1.13.

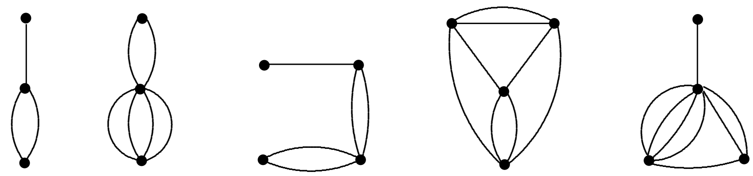

Figure 1.13: Multigraphs in Exercise 55

Since the sum of the degrees of the vertices of a multigraph G is twice the number of edges of G, neither s_4 nor s_6 is the degree sequence of a multigraph. The sequence s_2 is also not the degree sequence of any multigraph G; otherwise, G has $(1+2+5)/2 = 4$ edges and 5 edges incident with a vertex of G, which is impossible.

56. **Proof.** First, let G be a multigraph of order n and size m having degree sequence $s : d_1, d_2, \ldots, d_n$, where $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $2m = \sum_{i=1}^{n} d_i$ is even. Suppose, to the contrary, that $d_1 > \frac{1}{2}\sum_{i=1}^{n} d_i = m$. Then there is a vertex of G incident with more edges than those in G. This is impossible.

We verify the converse by induction on n. Suppose that $s : d_1$ is a sequence of one nonnegative integer such that d_1 is even and $d_1 \leq \frac{1}{2}d_1$. Then $d_1 = 0$, so s is the degree sequence of the only multigraph K_1 of order $n = 1$. For $n = 2$, suppose that $s : d_1, d_2$ is a sequence of a nonnegative integers such that $d_1 \geq d_2$, $\sum_{i=1}^{2} d_i$ is even and $d_1 \leq \frac{1}{2}\sum_{i=1}^{2} d_i$. This implies that $d_1 = d_2$. The multigraph of order 2 whose two vertices are joined by d_1 parallel edges has degree sequence s. Hence, the statement is true for $n = 2$.

Assume for an integer $k \geq 2$ that if $s_1 : e_1, e_2, \ldots, e_k$ is an sequence of k nonnegative integers such that $e_1 \geq e_2 \geq \cdots \geq e_k$, $\sum_{i=1}^{k} e_i$ is even and $e_1 \leq \frac{1}{2}\sum_{i=1}^{k} e_i$, then s_1 is a degree sequence of a multigraph of order k. Let $s : d_1, d_2, \ldots, d_{k+1}$ be a sequence of $k+1$ nonnegative integers such that $d_1 \geq d_2 \geq \cdots \geq d_{k+1}$, $\sum_{i=1}^{k+1} d_i$ is even and $d_1 \leq \frac{1}{2}\sum_{i=1}^{k+1} d_i$. Since $d_1 \geq d_2$ and $d_1 \leq \sum_{i=2}^{k+1} d_i$, the terms $d_2, d_3, \ldots, d_{k+1}$ can be decreased by a total of d_1 resulting in a sequence $d'_2, d'_3, \ldots, d'_{k+1}$ of nonnegative integers such that $\sum_{i=2}^{k+1} d'_i = \sum_{i=2}^{k+1} d_i - d_1 \geq 0$, $d'_2 \geq d'_3 \geq \cdots \geq d'_{k+1}$ where $d'_2 - d'_3 \leq 1$. Because $\sum_{i=1}^{k+1} d_i$ is even, $\sum_{i=2}^{k+1} d_i - d_1 = \sum_{i=2}^{k+1} d'_i$ is even. Consequently, $d'_2 \leq \sum_{i=3}^{k+1} d'_i$ and so $d'_2 \leq \frac{1}{2}\sum_{i=2}^{k+1} d'_i$. By the induction hypothesis, $s' : d'_2, d'_3, \ldots, d'_{k+1}$ is the degree sequence of a multigraph G' of order n. We may assume that $V(G') = \{v_2, v_3, \ldots, v_{k+1}\}$ and $\deg_{G'} v_i = d'_i$ for $2 \leq i \leq k+1$. Let G be the multigraph obtained from G' by adding a new vertex v_1 to G' and joining v_1 to the vertices of G' so that the degrees of $V(G')$ become $d_2, d_3, \ldots, d_{k+1}$. Then $\deg_G v_1 = d_1$ and so $s : d_1, d_2, \ldots, d_{k+1}$ is the degree sequence of G. ■

57. (a) $m = 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 4 = 20$.

(b) At most two edges can be replaced by one edge each and at most two edges can be replaced by two parallel edges. Thus, the size of H is at least $2 \cdot 1 + 2 \cdot 2 + 3 = 9$. This is possible (see the graph H in Figure 1.14).

At least two edges can be replaced by one edge each and at most one edge can be replaced by four parallel edges and, if so, one edge can be replaced by three parallel edges. Thus, the size is at most $2 \cdot 1 + 4 + 3 + 2 = 11$. The graph F in Figure 1.14 shows that this can occur.

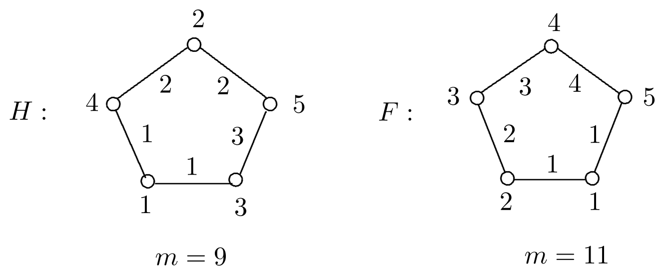

Figure 1.14: Graphs H and F in Exercise 57(b)

(c) Let U and W be the partite sets of G, where $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$ such that $\deg u_1 \geq \deg u_2 \geq \cdots \geq \deg u_s$ and $\deg w_1 \geq \deg w_2 \geq \cdots \geq \deg w_t$. Then the minimum value of m is

$$\min \left\{ \sum_{i=1}^{s} i \deg u_i, \quad \sum_{i=1}^{t} i \deg w_i \right\}.$$