"Game Theory" by Maschler, Solan and Zamir

Solution Manual

Chapter 1: The game of chess

1.1 No. The statement in the exercise holds for games without chance moves, but also for games with chance moves, like backgammon. The statement in Theorem 1.4, on the other hand, does not hold in backgammon; in this game, if a player is extremely lucky, he can win regardless of the strategy of the other player. In particular, the two statements cannot be equivalent.

1.2 Tic-Tac-Toe, Four in a row, Nim.

1.3 (a) Denote by N the number of game positions in chess. If after N^2 turns neither White wins nor Black wins, then there is at least one game position that is repeated twice so the play ends with draw. Hence, the conditions of Theorem 1.5 are satisfied and (a) holds.

(b) Assume that one of the players has a winning strategy in finite chess. Then he can guarantee by following this strategy that finite chess ends with his winning- before it reaches a repeated game position. Therefore, a strategy in which he follows his winning strategy as long as no game position is repeated, otherwise he chooses an arbitrary legal action is a winning strategy in chess. Indeed, by following this strategy in chess the game ends before it reaches a repeated game position as if it is finite chess, thus it leads to his winning.

(c) Assume that each player has a strategy guaranteeing at least a draw in finite chess. We show that each player has a strategy guaranteeing at least a draw in chess. We show it for White, the proof for Black is similar and thus it is omitted.
Denote by σ_W a strategy of White in finite chess that guarantees at least a draw. Consider the following strategy $\widehat{\sigma}_W$ for White in chess: implement the strategy σ_W until either the play of chess terminates or a game position repeats itself (at which point, the play of finite chess terminates). If the play of chess arrives to a game position x that has previously appeared, implement the strategy σ_W restricted to the subgame beginning at x until the play arrives at a game position y that has previously appeared, and so on.
We prove that the strategy $\widehat{\sigma}_W$ guarantees at least a draw in chess. Assume by contradiction that $\widehat{\sigma}_W$ does not guarantee at least a

draw. Let $\widehat{\sigma}_B$ be a strategy of Black that wins the game against White playing $\widehat{\sigma}_W$ with minimal number of moves. Since σ_W guarantees at least a draw in finite chess, when the player follow $(\widehat{\sigma}_W, \widehat{\sigma}_B)$, White faces at least one game position, say x, that appears at least twice before the winning of Black. According to $\widehat{\sigma}_W$, White selects the same move in x. Therefore, from the first repetition of x onward, Black could follow the moves he takes after the second repetition of x and win the game using less moves, which contradicts $\widehat{\sigma}_B$ wins with minimal number of moves.

Chapter 2: Utility theory

2.1 (a) Let $\succ$ be a strict preference.
$\succ$ is anti symmetric: if $x \succ y$ then $y \not\succsim x$, therefore $y \not\succ x$.
In addition, $\succ$ is transitive: if $x \succ y$ and $y \succ z$ then $x \succsim y$ and $y \succsim z$. Hence, by the transitivity of $\succsim$, one has $x \succsim z$. It is left to prove that $z \not\succsim x$. Assume to the contrary that $z \succsim x$, so by the transitivity of $\succsim$, one has $z \succsim y$, a contradiction. In conclusion, $z \not\succsim x$ and therefore $x \succ z$.

(b) Let $\approx$ be an indifference relation.
Then $\approx$ is symmetric: if $x \approx y$ then $x \succsim y$ and $y \succsim x$, therefore $y \approx x$.
In addition, $\approx$ is transitive: if $x \approx y$ and $y \approx z$ then $x \succsim y$ and $y \succsim x$, likewise $y \succsim z$ and $z \succsim y$. Hence, by the transitivity of $\succsim$, one has $x \succsim z$ and $z \succsim x$, so $x \approx z$.

2.2 Let O be a set of outcomes. Let $\succsim$ be a complete, reflexive, and transitive relation over O. Assume that u is a utility function representing $\succsim$. We prove that for every monotonically increasing function $v : \mathbf{R} \to \mathbf{R}$, the composition $v \circ u$ is also a utility function representing $\succsim$. Indeed, let $x, y \in O$. Then, since u represents $\succsim$, $x \succsim y$ if and only if $u(x) \geq u(y)$. But the last inequality holds if and only if $v(u(x)) \geq v\left(u(y)\right)$, since v is a monotonically increasing function. In particular, $v \circ u$ is also a utility function representing $\succsim$, as needed.

2.3 Let $O = \mathbf{Q}$, the set of the rational numbers. Define a relation over O as follows. For every $p, q \in O$, $p \succsim q$ if and only if $p \geq q$. We show that there is no utility function representing $\succsim$ that includes only integer values. Assume to the contrary that there is such a utility function. Then $0, 1 \in O$ and $u(0) < u(1)$ two integer values. But, there is an increasing countable series of rational numbers in $(0, 1)$, $p^1 < p^2 < ....$

Hence, $u(p^1) < u^(p^2) < \ldots$ an increasing countable series of integers in $(u(0), u(1))$, a contradiction.

2.4 Assume that a transitive preference relation $\succsim_i$ satisfies the axioms of continuity and monotonicity. Let $A \succsim_i B \succsim_i C$ and $A \succ_i C$. We prove that there is a unique number $\theta_i \in [0,1]$ satisfying $B \approx [\theta_i(A), (1-\theta_i)(C)]$. By the continuity axiom, there is at least one such θ_i. We show that θ_i is unique. Let $\theta_i' \in [0,1]$ such that $B \approx \left[\theta_i'(A), (1-\theta_i')(C)\right]$. Then, by transitivity, $[\theta_i(A), (1-\theta_i)(C)] \approx \left[\theta_i'(A), (1-\theta_i')(C)\right]$. Hence, by the monotonicity axiom, one has $\theta_i' = \theta_i$.

2.5 We show that the axioms of VNM are independent. For each axiom we present a preference relation that does not satisfy that axiom, but satisfies all the other axioms.
Monotonicity: Let $O = \{x, y\}$ be a set of outcomes. Consider a preference relation that is determined by the total probabilities over the outcomes x, y as follows: $x \succ \left[p(x), (1-p)(y)\right] \succ y$ for every $p \in (0,1)$. Furthermore, for every $p, q \in (0,1)$ one has $\left[p(x), (1-p)(y)\right] \succsim \left[q(x), (1-q)(y)\right]$ if and only if $\max\{p, 1-p\} \geq \max\{q, 1-q\}$. The preference relation satisfies all the VNM axioms, but the monotonicity. Indeed, $x \succ y$ but $\left[0.25(x), 0.75(y)\right] \succ \left[0.5(x), 0.5(y)\right]$.
Continuity: Consider a set of outcomes $O = \{x, y, z\}$. Define a lexicographic preference over the set of lotteries over O as follows.

$$\left[p_1(x), p_2(y), p_3(z)\right] \succsim \left[q_1(x), q_2(y), q_3(z)\right] \Longleftrightarrow$$

$$\{p_1 > q_1\} \ \ or \ \ \{p_1 = q_1 \ \ and \ \ p_2 \geq q_2\}.$$

Finally, assume the preference relation is determined by the total probabilities over the outcomes x, y, z. This preference relation satisfies all the VNM axioms, but the continuity. Indeed, $x \succ y \succ z$, however, for every $0 < \theta \leq 1$, one has $[\theta(x), (1-\theta)(z)] \succ y$. In addition, for $\theta = 0$, $y \succ [\theta(x), (1-\theta)(z)]$.
The axiom of simplification of compound lotteries: Consider a set of outcomes $O = \{x, y\}$. Define a preference relation over the compound lotteries over O as follows. A player is indifferent between all simple lotteries (including outcomes). He also indifferent between all compound lotteries. However, he prefers every simple lottery over a compound lottery. Clearly, this preference relation satisfies all the VNM axioms, but the axiom of simplification of compound lotteries.

Independence: Let $O = \{x, y\}$ be a set of outcomes. Consider a preference relation that satisfies the axiom of simplification of compound lotteries such that, for every $p, q \in [0,1]$ one has

$$\left[p(x), (1-p)(y)\right] \succsim \left[q(x), (1-q)(y)\right] \Leftrightarrow \max\{p, 1-p\} \geq \max\{q, 1-q\}.$$

This preference relation satisfies all the VNM axioms, but the independence. Indeed, $[1(x)] \approx \left[1(y)\right]$ but $\left[0.5(x), 0.5\left(\left[1(x)\right]\right)\right] \succ \left[0.5(x), 0.5\left(\left[1(y)\right]\right)\right]$.

2.6 Let u be a linear utility function representing a preference relation $\succsim_i$ of player i. We prove that $\succsim_i$ satisfies the VNM axioms. First, since the relation $\geq$ over $\mathbf{R}$ is complete and transitive then $\succsim_i$ is complete and transitive as well.
Continuity: Let $A \succsim_i B \succsim_i C$ then $u(A) \geq u(B) \geq u(C)$. If $u(A) = u(C)$ then for each $\theta \in [0,1]$ one has $u(B) = \theta u(A) + (1-\theta) u(C)$. Otherwise, $u(A) > u(C)$, so $u(B) = \theta u(A) + (1-\theta) u(C)$ for $\theta = \frac{u(B)-u(C)}{u(A)-u(C)}$. By the linearity of u,

$$u\left(\left[\theta(A), (1-\theta)(C)\right]\right) = \theta u(A) + (1-\theta) u(C) = u(B).$$

So, $B \approx \left[\theta(A), (1-\theta)(C)\right]$.
Monotonicity: Let $\alpha, \beta \in [0,1]$, and $A \succ B$. By the linearity of u,

$$u\left(\left[\alpha(A), (1-\alpha)(B)\right]\right) = \alpha u(A) + (1-\alpha) u(B),$$

as well as,

$$u\left(\left[\beta(A), (1-\beta)(B)\right]\right) = \beta u(A) + (1-\beta) u(B).$$

By $A \succ B$ it follows that $u(A) > u(B)$. Hence,

$$\begin{gathered}\alpha \geq \beta \Longleftrightarrow \\ \alpha u(A) + (1-\alpha) u(B) \geq \beta u(A) + (1-\beta) \Longleftrightarrow \\ u\left(\left[\alpha(A), (1-\alpha)(B)\right]\right) \geq u\left(\left[\beta(A), (1-\beta)(B)\right]\right) \Longleftrightarrow \\ \left[\alpha(A), (1-\alpha)(B)\right] \succsim \left[\beta(A), (1-\beta)(B)\right].\end{gathered}$$

The axiom of simplification of compound lotteries: For $j = 1, ..., J$, let $L_j = \left[p_1^j(A_1), ..., p_k^j(A_k)\right]$ be a simple lottery. Let $L = \left[q_1(L_1), ..., q_J(A_J)\right]$

be a compound lottery. Denote, $\widehat{L} = (r_1(A_1), ..., r_K(A_K))$, where $r_k = \sum_{j=1}^{J} q_j p_k^j$, for every $k = 1, ..., K$. Then'

$$\begin{aligned} u(L) &= q_1 u(L_1) + ... + q_J u(L_J) \\ &= q_1 \left(p_1^1 u(A_1) + ... + p_K^1 (A_K)\right) + ... + q_J \left(p_1^J u(A_1) + ... + p_K^J (A_K)\right) \\ &= r_1 u(A_1) + ... + r_J u(A_J) \\ &= u\left(\widehat{L}\right) \Rightarrow \\ L &\approx \widehat{L} \end{aligned}$$

Independence: Let $\widehat{L} = [q_1(L_1), ..., q_K(L_K)]$ be a compound lottery, and let M be a simple lottery. Assume $L_j \approx_i M$ then $u\left(L_j\right) = u(M)$. Thus,

$$\begin{aligned} u\left(\widehat{L}\right) &= q_1 u(L_1) + ... + q_{j-1} u\left(L_{j-1}\right) + q_j u\left(L_j\right) + q_{j+1} u\left(L_{j+1}\right) + ... + q_J u(L_J) \\ &= q_1 u(L_1) + ... + q_{j-1} u\left(L_{j-1}\right) + q_j u(M) + q_{j+1} u\left(L_{j+1}\right) + ... + q_J u(L_J) \Rightarrow \\ \widehat{L} &\approx \left[q_1(L_1), ..., q_{j-1}\left(L_{j-1}\right), q_j(M), q_{j+1}\left(L_{j+1}\right), ..., q_J(L_J)\right]. \end{aligned}$$

2.7 (a) Let $u(1000) = 1$ and $u(0) = 0$.

$$\begin{aligned} [1(500)] &\approx \left[\tfrac{2}{3}(1000), \tfrac{1}{3}(0)\right] \Rightarrow u(500) = \tfrac{2}{3}u(1000) + \tfrac{1}{3}u(0) = \tfrac{2}{3}, \\ [1(100)] &\approx \left[\tfrac{3}{8}(500), \tfrac{5}{8}(0)\right] \Rightarrow u(100) = \tfrac{3}{8}u(500) + \tfrac{5}{8}u(0) = \tfrac{1}{4}. \end{aligned}$$

(b) L_1 will be preferred by this person, indeed

$$\begin{aligned} u(L_1) &= \tfrac{3}{10}u(1000) + \tfrac{1}{10}u(500) + \tfrac{1}{2}u(100) + \tfrac{1}{10}u(0) = 0.492 > \\ u(L_2) &= \tfrac{2}{10}u(1000) + \tfrac{3}{10}u(500) + \tfrac{2}{10}u(100) + \tfrac{3}{10}u(0) = 0.45 \end{aligned}$$

(c) One can observe that

$$u(100) < u(L_1) < u(500) \quad and \quad u(100) < u(400) < u(500),$$

but it is not possible to ascertain whether $u(L_1) < u(400)$, $u(L_1) = u(400)$ or $u(L_1) > u(400)$ and therefore it is not possible to ascertain which will be preferred.

(d) One can observe that $u(100) < u(L_1) < u(500) < u(600)$, so he will prefer receiving \$600 with probability 1.

2.8 The preferences between the lotteries do not change by the suggested change, since it only define a positive affine transformation over the original utility function.

2.9 A person satisfies the VNM axioms and he prefers $A \succ_i D$. Set $u(A) = 1$ and $u(D) = 0$.

$$C \approx \left[\tfrac{3}{5}(A), \tfrac{2}{5}(D)\right] \Rightarrow u(C) = \tfrac{3}{5}u(A) + \tfrac{2}{5}u(D) = \tfrac{3}{5}$$
$$B \approx \left[\tfrac{3}{4}(A), \tfrac{1}{4}(C)\right] \Rightarrow u(B) = \tfrac{3}{4}u(A) + \tfrac{1}{4}u(C) = \tfrac{9}{10}$$
$$u(L_1) = \tfrac{2}{5}u(A) + \tfrac{1}{5}u(B) + \tfrac{1}{5}u(C) + \tfrac{1}{5}u(D) = \tfrac{35}{50} <$$
$$u(L_2) = \tfrac{2}{5}u(B) + \tfrac{3}{5}u(C) = \tfrac{36}{50}.$$

L_2 will be preferred by this person.

2.10 Assume the preference in 2.9 is $D \succ_i A$ rather than $A \succ_i D$, then L_1 will be preferred. Indeed, set $u(A) = -1$ and $u(D) = 0$. Therfore,

$$C \approx \left[\tfrac{3}{5}(A), \tfrac{2}{5}(D)\right] \Rightarrow u(C) = \tfrac{3}{5}u(A) + \tfrac{2}{5}u(D) = -\tfrac{3}{5}$$
$$B \approx \left[\tfrac{3}{4}(A), \tfrac{1}{4}(C)\right] \Rightarrow u(B) = \tfrac{3}{4}u(A) + \tfrac{1}{4}u(C) = -\tfrac{9}{10}$$
$$u(L_1) = \tfrac{2}{5}u(A) + \tfrac{1}{5}u(B) + \tfrac{1}{5}u(C) + \tfrac{1}{5}u(D) = -\tfrac{35}{50} >$$
$$u(L_2) = \tfrac{2}{5}u(B) + \tfrac{3}{5}u(C) = -\tfrac{36}{50}.$$

2.11 Assume that u is a linear utility function. Let $L = \left[q_1(L_1), q_2(L_2), ..., q_J(L_J)\right]$ be a compound lottery where $L_j = \left[p_1^j(A_1), p_2^j(A_2), ..., p_K^j(A_K)\right]$ for every $j = 1, 2, ..., J$. We prove that $u(L) = \sum_{j=1}^J q_j u\left(L_j\right)$.
By the axiom of simplification of compound lotteries,

$$L \approx \left[\sum_{j=1}^J p_1^j q_j(A_1), \sum_{j=1}^J p_2^j q_j(A_2), ..., \sum_{j=1}^J p_K^j q_j(A_K)\right] \Rightarrow$$
$$u(L) = u\left(\left[\sum_{j=1}^J p_1^j q_j(A_1), \sum_{j=1}^J p_2^j q_j(A_2), ..., \sum_{j=1}^J p_K^j q_j(A_K)\right]\right)$$
$$= \sum_{k=1}^K \left(\sum_{j=1}^J p_K^j q_j\right) u(A_K) = \sum_{j=1}^J q_j \sum_{k=1}^K p_K^j u(A_K)$$
$$= \sum_{j=1}^J q_j u\left(L_j\right).$$

2.12 Assume, $\succsim$ satisfies the VNM axioms. Let u be a linear utility function representing $\succsim$. We prove that

$$\left[\alpha\left(L_1\right),\left(1-\alpha\right)\left(L_3\right)\right] \succ \left[\alpha\left(L_2\right),\left(1-\alpha\right)\left(L_3\right)\right]$$

if and only if

$$\left[\alpha\left(L_1\right),\left(1-\alpha\right)\left(L_4\right)\right] \succ \left[\alpha\left(L_2\right),\left(1-\alpha\right)\left(L_4\right)\right],$$

for every four lotteries L_1, L_2, L_3, L_4 and $\alpha \in [0,1]$.

$$\begin{gathered}\left[\alpha\left(L_1\right),\left(1-\alpha\right)\left(L_3\right)\right] \succ \left[\alpha\left(L_2\right),\left(1-\alpha\right)\left(L_3\right)\right] \overset{\text{by 2.11}}{\Longleftrightarrow} \\ \alpha u\left(L_1\right)+\left(1-\alpha\right)u\left(L_3\right) > \alpha u\left(L_2\right)+\left(1-\alpha\right)u\left(L_3\right) \Leftrightarrow \\ \alpha u\left(L_1\right)+\left(1-\alpha\right)u\left(L_4\right) > \alpha u\left(L_2\right)+\left(1-\alpha\right)u\left(L_4\right) \Leftrightarrow \\ \left[\alpha\left(L_1\right),\left(1-\alpha\right)\left(L_4\right)\right] \succ \left[\alpha\left(L_2\right),\left(1-\alpha\right)\left(L_4\right)\right].\end{gathered}$$

2.13 Suppose a person satisfies VNM axioms. Let u be a linear utility function representing $\succsim$. Assume also that his preferences with respect to lotteries L_1, L_2, L_3, L_4 are $L_1 \succ L_2$ and $L_3 \succ L_4$. We prove that for every $\alpha \in [0,1]$ one has $\left[\alpha(L_1),(1-\alpha)(L_3)\right] \succ \left[\alpha(L_2),(1-\alpha)(L_4)\right]$. For $\alpha = 0$ or $\alpha = 1$, it obviously holds. Let $\alpha \in (0,1)$, then

$$\begin{aligned} L_1 \succ L_2 \Rightarrow u(L_1) > u(L_2) \quad &\Rightarrow \ \alpha u(L_1) > \alpha u(L_2), \text{ in addition,} \\ L_3 \succ L_4 \Rightarrow u(L_3) > u(L_4) \quad &\Rightarrow \ (1-\alpha)u(L_3) > (1-\alpha)u(L_4). \end{aligned}$$

Hence,

$$\alpha u\left(L_1\right)+\left(1-\alpha\right)u\left(L_3\right) > \alpha u\left(L_2\right)+\left(1-\alpha\right)u\left(L_4\right).$$

So, by 2.11,

$$\left[\alpha(L_1),(1-\alpha)(L_3)\right] \succ \left[\alpha(L_2),(1-\alpha)(L_4)\right].$$

2.14 Suppose a person satisfies VNM axioms. Let u be a linear utility function representing $\succsim$. Assume also that his preference with respect to lotteries L_1, L_2 is $L_1 \succ L_2$. We prove that for every $0 < \alpha \leq 1$ one has

$\left[\alpha(L_1),(1-\alpha)(L_2)\right] \succ L_2$.

$$\begin{aligned}
&L_1 \succ L_2 && \Rightarrow\\
&u(L_1) > u(L_2) && \Rightarrow\\
&\alpha u(L_1) > \alpha u(L_2) && \Rightarrow\\
&\alpha u(L_1) + (1-\alpha) u(L_2) > \alpha u(L_2) + (1-\alpha) u(L_2) && \Rightarrow\\
&\alpha u(L_1) + (1-\alpha) u(L_2) > u(L_2) && \Rightarrow\\
&\left[\alpha(L_1),(1-\alpha)(L_2)\right] \succ L_2.
\end{aligned}$$

2.15 Suppose the tennis player satisfies VNM axioms.

(a) Let u be a linear utility function representing $\succsim$. Set $u(win) = 1$ and $u(lose) = 0$.

(b) Denote by $L_{s.great}$ ($L_{s.medium}$) the lottery that takes place when the player strikes the ball with great force (medium force, respectively) at the second attempt. Then

$$\begin{aligned}
L_{s.great} &= \left[0.65\left(\left[0.75(win), 0.25(lose)\right]\right), 0.35(lose)\right]\\
L_{s.medium} &= \left[0.9\left(\left[0.5(win), 0.5(lose)\right]\right), 0.1(lose)\right].
\end{aligned}$$

(c)

$$\begin{aligned}
L_{f.great,s.great} &= \left[0.65\left(\left[0.75(win), 0.25(lose)\right]\right), 0.35\left(L_{s.great}\right)\right]\\
L_{f.great,s.medium} &= \left[0.65\left(\left[0.75(win), 0.25(lose)\right]\right), 0.35\left(L_{s.medium}\right)\right]\\
L_{f.medium,s.great} &= \left[0.9\left(\left[0.5(win), 0.5(lose)\right]\right), 0.1\left(L_{s.great}\right)\right]\\
L_{f.medium,s.medium} &= \left[0.9\left(\left[0.5(win), 0.5(lose)\right]\right), 0.1\left(L_{s.medium}\right)\right]
\end{aligned}$$

(d)

$$
\begin{aligned}
u\left(L_{s.great}\right) &= 0.65 \cdot 0.75u\,(win) + 0.65 \cdot 0.25u\,(lose) + 0.35u\,(lose) \\
&= 0.4875 \\
u\left(L_{s.medium}\right) &= 0.9 \cdot 0.5u\,(win) + 0.9 \cdot 0.5u\,(lose) + 0.1u\,(lose) \\
&= 0.45. \\
u\left(L_{f.great,s.great}\right) &= 0.65 \cdot 0.75u\,(win) + 0.65 \cdot 0.25u\,(lose) + 0.35u\left(L_{s.great}\right) \\
&= 0.6581 \\
u\left(L_{f.medium,s.great}\right) &= 0.9 \cdot 0.5u\,(win) + 0.9 \cdot 0.5u\,(lose) + 0.1u\left(L_{s.grat}\right) \\
&= 0.4988. \\
u\left(L_{f.great,s.medium}\right) &= 0.65 \cdot 0.75u\,(win) + 0.65 \cdot 0.25u\,(lose) + 0.35u\left(L_{s.medium}\right) \\
&= 0.645 \\
u\left(L_{f.medium,s.medium}\right) &= 0.9 \cdot 0.5u\,(win) + 0.9 \cdot 0.5u\,(lose) + 0.1u\left(L_{s.medium}\right) \\
&= 0.495.
\end{aligned}
$$

The preferred lottery is strike the ball with great force in both attempts.

2.16 (a) Let p be the probability distribution over the black and the white chocolate balls in the container. Let n the number of chocolate balls in it. Then the set of outcomes is

$$O = \{p\} \bigcup \{w | w \in \{0, 1, ..., n\}\}$$

where w is the number of white chocolate balls in the container. The preference relation of Ron satisfies: (i) For every $w > w'$ one has $w \succ w'$ since Ron's excitement climbing higher the greater the number of white chocolate balls; and (ii) $p \succ w$ for every $w \in \{0, 1, .., n\}$.

(b) Ron's preferences relation does not satisfy the VNM axioms. He violates the monotonicity axiom according to which receiving a container with only white chocolate balls should be preferred to any lottery over the number of white chocolate balls as opposed to Ron's preferences ($p \succ w$).

2.17 There exists a preference relation over compound lotteries that satisfies VNM axioms and extends the lexicographic preference relation. There are only a finite number of pairs (x, y) such that $x, y = 0, 1, \ldots, 1000$. Indeed, one can define a utility function over the set of outcomes that maintain the lexicographic preference relation of the farmer (e.g. $u(0,0) = 0, u(0,1) = 1, ..., u(0,1000) = 1000, u(1,0) = 1001, ...$). Now, one may define a preference relation over compound lotteries as follows $L \succsim L'$ if and only if $Eu(L) \geq Eu(L')$.

2.18 Let $O = \{(x,y) \,|0 \leq x,y \leq 1000\}$. We next prove that there is no preference relation over compound lotteries over O that satisfies VNM axioms and extends the lexicographic preference relation over O. Assume to the contrary that there is such a preference relation.

(a) Let $(x,y) \in [0,1000]^2$. In particular, $(1000,1000) \succsim (x,y) \succsim (0,0)$. So, by the continuity, there is at least one $\theta_{(x,y)}$, such that $(x,y) \approx \left[\theta_{(x,y)}(1000,1000), \left(1-\theta_{(x,y)}\right)(0,0)\right]$. But, by the monotonicity, $\theta_{(x,y)}$ is unique.

(b) We next prove that the function $(x,y) \mapsto \theta_{(x,y)}$ is injective. Let $(x,y) \neq (x',y')$ then necessarily either $(x,y) \succ (x',y')$ or $(x,y) \prec (x',y')$. Assume w.l.o.g. that $(x,y) \succ (x',y')$ then by transitivity

$$\left[\theta_{(x,y)}(1000,1000), \left(1-\theta_{(x,y)}\right)(0,0)\right] \succ$$
$$\left[\theta_{(x',y')}(1000,1000), \left(1-\theta_{(x',y')}\right)(0,0)\right].$$

Thus, by monotonicity, $\theta_{(x,y)} > \theta_{(x',y')}$.

(c) Set $A_x = \left\{\theta_{(x,y)} | y \in [0,1000]\right\}$. Then, by (b), $\theta_{(x,0)}, \theta_{(x,1000)} \in A_x$ and $\theta_{(x,0)} < \theta_{(x,1000)}$, so A_x contains at least two elements. In addition, if $\theta \in A_x$ and $x \neq x'$, then, by (b), $\theta \notin A_{x'}$, and the sets $\{A_x : x \in [0,1000]\}$ are pairwise disjoint.

(d) Let $x_1 < x_2$, $\theta_1 \in A_{x_1}$ and $\theta_2 \in A_{x_2}$. Then, there are y_1 and y_2 such that $\theta_{(x_1,y_1)} = \theta_1$ and $\theta_{(x_2,y_2)} = \theta_2$. However, since $x_1 < x_2$, one has $(x_1,y_1) \prec (x_2,y_2)$ and thus, by (b), $\theta_1 = \theta_{(x_1,y_1)} < \theta_{(x_2,y_2)} = \theta_2$.

(e) From (c) it follows that, for every $x \in [0,1000]$, there are at least two elements in A_x, say $\theta_x < \theta_{x'}$. By (d), $\{[\theta_x, \theta_{x'}] : x \in [0,1000]\}$ is an infinite set of closed disjoint segments contained in $[0,1000]$. But there can be only countable disjoint segments, a contradiction.

(f) Therefore, there is no utility function over $[0,1000]^2$ that represents the lexicographic preference relation.

(g) The monotonicity and continuity axioms are not satisfied by the preference relation.

2.19 Assume that v is a positive affine transformation of u. Then there are $a > 0$ and $b \in \mathbf{R}$ such that

$$v(\cdot) = a \cdot u(\cdot) + b.$$

In particular,

$$u(\cdot) = \frac{1}{a} v(\cdot) - \frac{b}{a}.$$

Thus, u is a positive affine transformation of v.

2.20 Assume that v is a positive affine transformation of u, and that w is a positive affine transformation of v. Then there are $a, b > 0$ and $c, d \in \mathbf{R}$ such that

$$v(\cdot) = a \cdot u(\cdot) + c$$

and

$$w(\cdot) = b \cdot v(\cdot) + d.$$

In particular,

$$w(\cdot) = ba \cdot u(\cdot) + (cb + d).$$

Thus, w is a positive affine transformation of u.

2.21 Suppose a person's preferences, which satisfies VNM axioms, are representable by two linear utility functions u and v. We prove that v is a positive affine transformation of u. If this person is indifference between each two outcomes, so both utility functions are necessarily constant, and we are done. Otherwise, let A and B be two outcomes such that $A \prec B$. Hence, $u(A) < u(B)$ and $v(A) < v(B)$. Set $a = \frac{v(B)-v(A)}{u(B)-u(A)} > 0$ and $b = v(A) - a \cdot u(A)$. We next prove that $v(\cdot) = a \cdot u(\cdot) + b$.

Case 1:

$$a \cdot u(A) + b = \frac{v(B) - v(A)}{u(B) - u(A)} u(A) + v(A) - \frac{v(B) - v(A)}{u(B) - u(A)} u(A) = v(A).$$

Case 2:

$$a \cdot u(B) + b = \frac{v(B) - v(A)}{u(B) - u(A)} u(B) + v(A) - \frac{v(B) - v(A)}{u(B) - u(A)} u(A) = v(B)$$

Case 3: $A \preceq C \preceq B$. By the continuity axiom, there is $\theta \in [0,1]$ such that $C \approx \left[\theta(B), (1-\theta)(A)\right]$ Therefore

$$\begin{aligned} v(C) &= \theta \cdot v(B) + (1-\theta) v(A) \\ &= \theta(a \cdot u(B) + b) + (1-\theta)(a \cdot u(A) + b) \\ &= a(\theta \cdot u(B) + (1-\theta) u(A)) + b \\ &= a \cdot u(C) + b \end{aligned}$$

(the proof in case that $C \preceq A \prec B$ and $A \prec B \preceq C$ is similar to Case 3).

2.22 Let O be an infinite set of outcomes. Suppose that a player has a complete reflexive and transitive preference relation $\succsim$ over the set of all compound lotteries over a finite number of simple lotteries over a finite set of outcomes in O. Assume that the preference relation $\succsim$ satisfies VNM axioms, and also satisfies the property that O contains a most-preferred outcome A_K, and least preferred outcome A_1.

(a) We prove that there exists a linear utility function that represents the player preference's relation. First, set $u(A_K) = 1$ and $u(A_1) = 0$. For every $B \in O$ an outcome, one has $A_1 \preceq B \preceq A_K$. By the continuity axiom, there is $\theta_B \in [0,1]$ such that $B \approx [\theta_B(A_K), (1-\theta_B)(A_1)]$. By 2.4, θ_B is unique. Set $u(B) = \theta_B$.
For every simple lottery over a finite set of outcomes in O. Define,

$$u([p_1(B_1), p_2(B_2), ..., p_n(B_n)]) = \sum_{i=1}^{n} p_i u(B_i) = \sum_{i=1}^{n} p_i \theta_{B_i}.$$

u is a linearly utility function, we show that u represents the preference relation. Let

$$L = [p_1(B_1), p_2(B_2), ..., p_n(B_n)]$$

and

$$L' = [q_1(B_1), q_2(B_2), ..., q_n(B_n)]$$

be two lotteries (note that one can assume without loss of generality that both lotteries are over the same set of outcome). Then, $L \succsim L'$ if and only if, by the independence axiom,

$$\Big[p_1([\theta_{B_1}(A_K), (1-\theta_{B_1})(A_1)]), ..., p_n([\theta_{B_n}(A_K), (1-\theta_{B_n})(A_1)])\Big] \succsim \Big[q_1([\theta_{B_1}(A_K), (1-\theta_{B_1})(A_1)]), ..., q_n([\theta_{B_n}(A_K), (1-\theta_{B_n})(A_1)])\Big]$$

if and only if, by The axiom of simplification of compound lotteries,

$$\left[\left(\sum_{i=1}^{n} p_i\theta_{B_i}\right)(A_K), \left(1 - \sum_{i=1}^{n} p_i\theta_{B_i}\right)(A_1)\right] \succsim \left[\left(\sum_{i=1}^{n} q_i\theta_{B_i}\right)(A_K), \left(1 - \sum_{i=1}^{n} q_i\theta_{B_i}\right)(A_1)\right]$$

if and only if, by the monotonicty axiom,

$$\sum_{i=1}^{n} p_i \theta_{B_i} \geq \sum_{i=1}^{n} q_i \theta_{B_i}$$

, if and only if $u(L) \geq u(L')$, as needed.

(b) To prove that if u and v are two linear utility functions that represents the player preference's relation, then v is a positive affine transformation of u, see Exercise 2.21.

(c) We prove that there exists a unique linear utility function (up to a positive affine transformation) representing the player preference's relation. By (a) it follows that there is a linear utility function u representing the player preference's relation. By (b), any other linear utility function representing the player preference's relation must be a positive affine transformation of u. Thus u is unique up to a positive affine transformation .

2.23 We prove that a player is a risk averse if and only if for each $p \in [0,1]$ and every pair of outcomes $x, y \in \mathbf{R}$, $u_i([p(x),(1-p)(y)]) \leq u_i([1(px+(1-p)y)])$. If a player is a risk averse then the inequality for each $p \in [0,1]$ and every pair of outcomes $x, y \in \mathbf{R}$ follows directly by the definition of risk averse. We prove the opposite direction. Assume that for each $p \in [0,1]$ and every pair of outcomes $x, y \in \mathbf{R}$, $u_i([p(x),(1-p)(y)]) \leq u_i([1(px+(1-p)y)])$. We prove that the player is a risk averse, that is for every lottery $L = [p_1(x_1), p_2(x_2), ..., p_n(x_n)]$, one has

$$u_i\left([p_1(x_1), p_2(x_2), ..., p_n(x_n)]\right) \leq u_i\left(\left[1(\sum_{k=1}^{n} p_k x_k)\right]\right).$$

We prove that the last inequality holds for every lottery by induction over n. For $n = 2$, it holds by the above assumption. Assume it holds for every $n < N$. We prove for $n = N$. Let

$$L = [p_1(x_1), p_2(x_2), ..., p_N(x_N)]$$

be a lottery. Then,

$$u_i\left(\left[p_1\left(x_1\right), p_2\left(x_2\right), \ldots, p_N\left(x_N\right)\right]\right)$$

$$= u_i\left(\left(\sum_{i=1}^{N-1} p_i\right)\left(\left[\frac{p_1}{\sum_{i=1}^{N-1} p_i}\left(x_1\right), \ldots, \frac{p_{N-1}}{\sum_{i=1}^{N-1} p_i}\left(x_{N-1}\right)\right]\right), p_N\left(x_N\right)\right) \quad (1)$$

$$= \left(\sum_{i=1}^{N-1} p_i\right) u_i\left(\left[\frac{p_1}{\sum_{i=1}^{N-1} p_i}\left(x_1\right), \ldots, \frac{p_{N-1}}{\sum_{i=1}^{N-1} p_i}\left(x_{N-1}\right)\right]\right) + p_N u_i\left(x_N\right) \quad (2)$$

$$\leq \left(\sum_{i=1}^{N-1} p_i\right) u_i\left(\left[1\left(\frac{p_1}{\sum_{i=1}^{N-1} p_i} x_1 + \cdots + \frac{p_(N-1)}{\sum_{i=1}^{N-1} p_i} x_{N-1}\right)\right]\right) + p_N u_i\left(x_N\right) \quad (3)$$

$$\leq u_i\left(\left[1\left(\left(\sum_{i=1}^{N-1} p_i\right)\left(\frac{p_1}{\sum_{i=1}^{N-1} p_i} x_1 + \cdots + \frac{p_{N-1}}{\sum_{i=1}^{N-1} p_i} x_{N-1}\right) + p_N x_N\right)\right]\right) \quad (4)$$

$$= u_i\left(\left[1\left(\sum_{k=1}^{n} p_k x_k\right)\right]\right),$$

where (1) follows by the axiom of simplification of compound lotteries, (2) follows by the linearity of the utility function, and (3) and (4) by the induction hypothesis. The proof for risk neutral/risk seeking is similar; however the inequalities in (3) and (4) should be replaced by equalities/reverse inequalities.

2.24 (a) The function $u(x) = 2x + 5$ is an increasing linear function therefore it represents a risk neutral player.

(b) The function $u(x) = -7x + 5$ is decreasing in x which contradicts the assumption that a player prefers receiving more. In particular the function cannot represent a preference relation.

(c) The function $u(x) = 7x - 5$ is an increasing linear function therefore it represents a risk neutral player.

(d) The function $u(x) = x^2$ is decreasing in $x < 0$ which contradicts the assumption that a player prefers receiving more. In particular the function cannot represent a preference relation.

(e) The function $u(x) = x^3$ is concave for $x < 0$ and convex for $x > 0$ (since $u''(x) = 6x$) hence a player is not risk neutral/risk averse/risk seeking.

(f) The function $u(x) = e^x$ is convex since $u''(x) = e^x > 0$, so it represents a risk seeking player.

(g) The function $u(x) = \ln x$ is defined only for $x > 0$. In the definition range, the function is concave since $u''(x) = -\frac{1}{x^2} < 0$, so it represents a risk averse player.

(h) The function $u(x) = x$ for $x \geq 0$ and $u(x) = 6x$ for $x < 0$ represents a risk averse player. Indeed, consider a lottery over two outcome x, y with probabilities $p, 1-p$, respectively. If either $x, y \geq 0$ or $x, y \leq 0$ then the player is indifference between receiving the lottery or its expected value (since the utility function over these ranges are linear). On the other hand, if $x > 0$ and $y < 0$ then (since $x < 6x$ and $6y < y$)

$$\begin{aligned} u\left(\left[p(x), (1-p)(y)\right]\right) &= px + (1-p)6y \\ &< \min\left\{p6x + (1-p)6y, px + (1-p)y\right\} \\ &= \min\left\{6\left[px + (1-p)y\right], \left[px + (1-p)y\right]\right\} \\ &\leq u\left(px + (1-p)y\right). \end{aligned}$$

(i) The function $u(x) = 6x$ for $x \geq 0$ and $u(x) = x$ for $x < 0$ represents a risk seeking player. Indeed, consider a lottery over two outcome x, y with probabilities $p, 1-p$, respectively. If either $x, y \geq 0$ or $x, y \leq 0$ then the player is indifference between receiving the lottery or its expected value (since the utility function over these ranges are linear). On the other hand, if $x > 0$ and $y < 0$ then

$$\begin{aligned} u\left(\left[p(x), (1-p)(y)\right]\right) &= px + (1-p)6y \\ &> \max\left\{p6x + (1-p)6y, px + (1-p)y\right\} \\ &= \max\left\{6\left[px + (1-p)y\right], \left[px + (1-p)y\right]\right\} \\ &\geq u\left(px + (1-p)y\right). \end{aligned}$$

(j) The function $u(x) = x^{3/2}$ for $x \geq 0$ and x for $x < 0$ represents a risk seeking player. Indeed, for $x > 0$ one has $u''(x) = 3/4x^{-1/2} > 0$, and for every $x < 0$ one has $u''(x) = 0$.

(k) The function $u(x) = x\ln(2+x)$ for $x \geq 0$ and $u(x) = x$ for $x < 0$ represents a player that is represents a risk averse player.

Indeed, for $x > 0$ one has $u''(x) < 0$, and for every $x < 0$ one has $u''(x) = 0$.

2.25 Let $U : \mathbf{R} \to \mathbf{R}$ be a concave function, let X be a random variable with a finite expected value, and let Y be a random variable that is independent of X and has an expected value 0. Define $Z = X + Y$. We prove that $E\left[U(X)\right] \geq E\left[U(Z)\right]$. Let $x \in \mathbf{R}$. Then

$$\begin{aligned}
&E\left[U(Z)|X = x\right] \\
&= E\left[U(X+Y)|X = x\right] \\
&= E\left[U(x+Y)|X = x\right] \\
&= E\left[U(x+Y)\right] && (5)\\
&\leq U\left(E[x+Y]\right) && (6)\\
&= U\left(x + E[Y]\right) \\
&= U\left(x\right) && (7)
\end{aligned}$$

where (5) holds since X and Y are independent, (6) holds since U is concave, and (7) holds by the assumption that $E\left[Y\right] = 0$. Hence, by the law of total expectation, one has

$$\begin{aligned}
&E\left[U\left(Z\right)\right] \\
&= E\left[E\left[U\left(Z\right)|X\right]\right] \\
&\leq E\left[U\left(X\right)\right].
\end{aligned}$$

2.26 Let $U : \mathbf{R} \to \mathbf{R}$ be a concave function, let X be a random variable with a normal distribution, expected value μ and a standard deviation σ. Let $\lambda > 1$ and let Y be a random variable with a normal distribution, expected value μ and a standard deviation $\lambda\sigma$.

(a) For every $c > 0$ one has $\mu + c < \mu + \lambda c$ and $\mu - c > \mu - \lambda c$. Therefore, since U is concave, one has

$$U\left(\mu + c\right) + U\left(\mu - c\right) \geq U\left(\mu + \lambda c\right) + U\left(\mu - \lambda c\right).$$

(b)

$$\begin{aligned}
&\int_{-\infty}^{\infty} u(x)\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}dx \\
&= \int_{-\infty}^{\mu} u(x)\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}dx + \int_{\mu}^{\infty} u(x)\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}dx \\
&= \int_{0}^{\infty} u(\mu+t)\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-t^2}{2\sigma^2}}dt + \int_{0}^{\infty} u(\mu-t)\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-t^2}{2\sigma^2}}dt \\
&= \int_{0}^{\infty} \left\{u(\mu+t)+u(\mu-t)\right\}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-t^2}{2\sigma^2}}dt \\
&= \int_{0}^{\infty} \left\{u\left(\mu+\frac{y}{\lambda}\right)+u\left(\mu-\frac{y}{\lambda}\right)\right\}\frac{1}{\sqrt{2\pi}\lambda\sigma}e^{\frac{-y^2}{2(\lambda\sigma)^2}}dy \\
&\geq \int_{0}^{\infty} \left\{u\left(\mu+\lambda\frac{y}{\lambda}\right)+u\left(\mu-\lambda\frac{y}{\lambda}\right)\right\}\frac{1}{\sqrt{2\pi}\lambda\sigma}e^{\frac{-y^2}{2(\lambda\sigma)^2}}dy \\
&\geq \int_{0}^{\infty} \left\{u(\mu+y)+u(\mu-y)\right\}\frac{1}{\sqrt{2\pi}\lambda\sigma}e^{\frac{-y^2}{2(\lambda\sigma)^2}}dy \\
&= \int_{-\infty}^{\infty} u(y)\frac{1}{\sqrt{2\pi}\lambda\sigma}e^{\frac{-(y-\mu)^2}{2(\lambda\sigma)^2}}dy
\end{aligned}$$

(c)

$$\begin{aligned}
E\left[U(X)\right] &= \int_{-\infty}^{\infty} u(x)\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}dx \\
&\geq \int_{-\infty}^{\infty} u(y)\frac{1}{\sqrt{2\pi}\lambda\sigma}e^{\frac{-(y-\mu)^2}{2(\lambda\sigma)^2}}dy = E\left[U(Y)\right]
\end{aligned}$$

2.27 Let $U(x) = 1 - e^{-x}$.

(a) The player is risk averse. Indeed, $U''(x) = -e^{-x} < 0$.

(b) For every $a \in (0,1)$ and $p \in (0,1)$, denote by $X_{a,p}$ a random variable such that $P(X_{a,p} = 1-a) = \frac{1-p}{2}$, $P(X_{a,p} = 1) = p$, and $P(X_{a,p} = 1+a) = \frac{1-p}{2}$. Then $E\left[X_{a,p}\right] = 1$, and $Var(X_{a,p}) = (1-p)a^2$.

(c) Let $c^2 = (1-p)a^2$. Then,

$$\begin{aligned}
E\left[U(X_{a,p})\right] &= \frac{1-p}{2}U(1-a) + pU(1) + \frac{1-p}{2}U(1+a) \\
&= \frac{1-p}{2}(1-e^{-1+a}) + p(1-e^{-1}) + \frac{1-p}{2}(1-e^{-1-a}) \\
&= 1 - \frac{e^{-1}}{2}((1-p)(e^{-a}+e^{a}+2)-2) \\
&= 1 - \frac{e^{-1}}{2}((1-p)\frac{a^2}{a^2}(e^{-a}+e^{a}+2)-2) \\
&= 1 - \frac{e^{-1}}{2}(\frac{c^2}{a^2}(e^{-a}+e^{a}+2)-2).
\end{aligned}$$

(d) Let $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{2}$, $p_1 = \frac{1}{2}$, $p_2 = \frac{7}{8}$ then $E\left[X_{a_1,p_1}\right] = E\left[X_{a_2,p_2}\right] = 1$ and $Var\left[X_{a_1,p_1}\right] = Var\left[X_{a_2,p_2}\right] = 1/32$. However,

$$E\left[U(X_{a_1,p_1})\right] < E\left[U(X_{a_2,p_2})\right].$$

(e) Let $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{2}$, $p_1 = \frac{5}{8}$, $p_2 = \frac{7}{8}$ then $E\left[X_{a_1,p_1}\right] = E\left[X_{a_2,p_2}\right] = 1$ and $Var\left[X_{a_1,p_1}\right] < Var\left[X_{a_2,p_2}\right]$. However,

$$E\left[U(X_{a_1,p_1})\right] < E\left[U(X_{a_2,p_2})\right].$$

2.28 Let U_i be a monotonically increasing, strictly concave and twice continuously differentiable function over **R**.

(a) Assume the player has $\$x$, and is required to participate in a lottery in which he stands to gain or lose a small amount $\$h$, with equal probabilities. Denote by Y the amount of money the player will have after the lottery is conducted. Then, the expected value of Y is $E[Y] = \frac{1}{2}(x+h) + \frac{1}{2}(x-h) = x$, and the variance of Y is $Var(Y) = h^2$.

(b) The utility of the lottery is

$$u_i(Y) = \frac{1}{2}U_i(x+h) + \frac{1}{2}U_i(x-h).$$

Thus, the utility loss due to the fact that he is required to participate in the lottery is

$$\begin{aligned}
\Delta u_h &= U_i(x) - u_i(Y) = U_i(x) - \left(\frac{1}{2}U_i(x+h) + \frac{1}{2}U_i(x-h)\right) \\
&= -\frac{(U_i(x+h)-U_i(x)) - (U_i(x)-U_i(x-h))}{2}.
\end{aligned}$$

(c)

$$\lim_{h\to 0}\frac{\Delta u_h}{h^2} = \lim_{h\to 0} -\frac{(U_i(x+h)-U_i(x))-(U_i(x)-U_i(x-h))}{2h^2}$$
$$= \lim_{h\to 0} -\frac{\frac{U_i(x+h)-U_i(x)}{h}-\frac{U_i(x)-U_i(x-h)}{h}}{2h}$$
$$= -\frac{U_{i''}(x)}{2}.$$

(d) Denote by $y_{x,h}$ the amount of money that satisfies $U_i(y_{x,h}) = u_i(Y)$, and by Δ_{x_h} the difference $\Delta_{x_h} = x - y_{x,h}$. Then $\Delta_{x_h} \geq 0$. Indeed, since U_i is a monotonically increasing and strictly concave function then it necessarily represents a risk averse player. Therefore, $u_i(Y) \leq U_i(E[Y])$. Hence,

$$U_i(y_{x,h}) = u_i(Y) \leq U_i(E[Y]) = U_i(x).$$

Finally, from the fact that U_i is monotonically increasing, it follows that $y_{x,h} \leq x$, so $\Delta_{x_h} = x - y_{x,h} \geq 0$.

(e)

$$\lim_{h\to 0}\frac{\Delta_{x_h}}{Var(Y)} = \lim_{h\to 0}\frac{\Delta_{x_h}}{h^2}\cdot\frac{\Delta_{u_h}}{\Delta_{u_h}}$$
$$= \lim_{h\to 0}\left(\frac{\Delta_{u_h}}{h^2}\right) / \left(\frac{\Delta_{u_h}}{\Delta_{x_h}}\right)$$
$$= \lim_{h\to 0} -\frac{U_i''(x)}{2U_i'(x)}$$
$$= \frac{1}{2}r_{U_i}(x).$$

(f) (a) Let $U_i(x) = x^\alpha$ for $0 < \alpha < 1$, then

$$r_{U_i}(x) = -\frac{\alpha(\alpha-1)x^{\alpha-2}}{\alpha x^{\alpha-1}} = \frac{1-\alpha}{x}.$$

(b) Let $U_i(x) = 1 - e^{-\alpha x}$ for $\alpha > 0$, then $r_{U_i}(x) = -\frac{-\alpha^2 e^{-\alpha x}}{\alpha e^{-\alpha x}} = \alpha$.

(g) The function $U_i(x) = x^\alpha$ for $0 < \alpha < 1$ exhibits decreasing absolute risk aversion.
The function $U_i(x) = 1 - e^{-\alpha x}$ for $\alpha > 0$ exhibits constant absolute risk aversion.

2.29 Apparently, the preferences expressed by the Second World War pilots violated the monotonicity axiom, since $Life \succ Death$ but they preferred $[1/2(Life), 1/2(Death)] \succ [3/4(Life), 1/4(Death)]$. However, it seems more likely that they violated the axiom of simplification of compound lotteries, so they were not indifference between the lottery

$$[3/4(Life), 1/4(Death)]$$

and the lottery

$$[3/4([1(Life)]), 1/4([1(Death)])].$$

Meaning, it seems that if the pilots were faced the lottery

$$[3/4(Life), 1/4(Death)]$$

they would have preferred it over the lottery

$$[1/2(Life), 1/2(Death)],$$

but they preferred the lottery

$$[1/2(Life), 1/2(Death)]$$

over the lottery

$$[3/4([1(Life)]), 1/4([1(Death)])].$$

Chapter 3: Extensive-form games

3.1 Let Player I be the player that takes the first move in the game. In the game tree depicted below we represent the action taken by the current player as the piles which remain after the player removes matches. For instance, if at the first move Player I removed 2 matches from the pile contains 3 piles then the action is $1, 1, 2$. The outcome in the leaves represent the winning strategy. The bold lines represent the best reply of each player. To simplify the game tree, we first consider two subgames. Denote by t_3^i a similar game that starts with only one pile which contains 3 matches. Denote by i the opening player and by $-i$ his opponent. As one can see in the appropriate game tree, the opening player has a winning strategy.

t_3^i :