1 An Overview of Robotic Mechanical Systems

1.1 Machine: A historical account

Here is an account of the definitions of machine, taken from (Dudiţă and Diaconescu, 1987):

Different definitions of machine have been given by scholars for more than two millennia, starting with Vitruvius in 28 B.C., namely,

- *A machine is a combination (system, assemblage) of moving material bodies* (Vitruvius, 28 B.C.; Hachette, 1811; Borgnis, 1818; Beck, 1859; Reuleaux, 1875; Koenigs, 1901)
- *A machine is generally composed of three parts: a motor part, a transmission part, and an execution part* (Euler, 1753; Bogolyubov, 1976)
- *A machine produces mechanical work, or performs productive operations, actions, or effects* (Vitruvius, 28 B.C.; Poncelet, 1824; Reuleaux, 1900; Koenigs, 1901; Bogolyubov, 1976)
- *A machine transforms or transmits forces* (Vitruvius, 28 B.C.; Leupold, 1724; Euler, 1753; Bogolyubov, 1976; Reuleaux, 1900; Koenigs, 1901)
- *A machine is characterized by deterministic motions* (Hachette, 1811; Leupold, 1724; Reuleaux, 1875; Borgnis, 1818; Reuleaux, 1900)
- *A machine is an artifact* (Leupold, 1724)

Beck, Th., 1875, *Beiträge zur Geschichte des Maschinenbaues*, J. Springer, Berlin.

Bogolyubov, A. N., 1976, *Teoriya mekhanismov v istoricheskom razvitii* (Theory of Mechanisms and its Historical Development), Nauka, Moscow (in Russian).

Borgnis, G. A., 1818, *Traité Complet de Mécanique Appliquée aux Arts. Traité des Compositions des Machines,* Paris.

Dudiţă, Fl. and Diaconescu, D., 1987, *Optimizarea Structurală a Mecanismelor* (*Optimization of Mechanisms*), in Romanian, Ed. Tehnicá (Publishers), Bucharest.

Hachette, 1811, *Traité Élémentaire des Machines,* Paris.

Koenigs, F., 1901, "Etude critique sur la théorie générale des mécanismes," *Comptes Rendus de l'Académie des Sciences*, Vol. 133.

Leupold, J., 1724, *Theatrum Machinarium Generale,* Leipzig.

Poncelet, J. V., 1824, *Traité de Mécanique Appliquée aux Machines,* Liège.

Reuleaux, F., 1875, *Theoretische Kinematik,* Braunschweig.

Reuleaux, F., 1900, *Lehrbuch der Kinematik,* Braunschweig.

Vitruvius, P. M., 28 B.C., *De Architectura*, Libri X.

1.3 Definitions for "machine," "mechanism," and "linkage":

Machine

- Definitions in Merriam Webster's Collegiate Dictionary (on-line, 2002):
  - (archaic): a constructed thing whether material or immaterial.
  - an assemblage of parts that transmit forces, motion, and energy one to another in a predetermined manner
  - an instrument (as a lever) designed to transmit or modify the application of power, force, or motion

  - a mechanically, electrically, or electronically operated device for performing a task (a calculating machine, a card-sorting machine)

  Comment: comprehensive definitions when considered as a whole

- *An apparatus* for transformation of power, materials, and information to substitute or simplify physical or intellectual work (Frolov, 1987). *Comment: a comprehensive definition, that includes computers*
- *Mechanical system* that performs a specific task, such as the forming of material, and the transference and transformation of *motion* and *force*, Vol. 38, Nos. 7–10 (2003) of *Mechanism and Machine Theory* on Standardization of Terminology. *Comment: leaves computers out*
- An *apparatus* for applying mechanical power, having several parts, each with definite function (The Concise Oxford Dictionary). *Comment: same as above*
- An *apparatus* consisting of interrelated parts with separate functions, used in the performance of some kind of work (The Random House College Dictionary). *Comment: ditto*
- Any *system* in which a specific correspondence exists between an input form of energy or information and the corresponding ones at the output (Loosely translated from Le Petit Robert). *Comment: as comprehensive as Frolov's*

Mechanism

- A piece of machinery (Merriam Webster's Collegiate Dictionary (on-line, 2002)). *Comment: too vague*
- Definitions in Vol. 38, Nos. 7–10 (2003) of *Mechanism and Machine Theory* on Standardization of Terminology.
  - *System* of bodies designed to convert *motions* of, and *forces* on, one or several bodies into constrained motions of, and *forces* on, other bodies. *Comment: English could be terser, but idea is fine.*
  - *Kinematic chain* with one of its components (*link* or *joint*) connected to the *frame*. *Comment: confuses mechanism with its representation as a kinematic chain*
- Structure, adaptation of parts of machine; system of mutually adapted parts working together (as) in machine (The Concise Oxford Dictionary)
- An assembly of moving parts performing a complete functional motion (The Random House College Dictionary)
- A combination layout of pieces or elements, assembled with the goal of (producing) an operation as a unit (Loosely translated from Le Petit Robert)

Comment: In all above definitions, the concept of goal or task is present

Linkage

- Definitions in Merriam Webster's Collegiate Dictionary (on-line, 2002):
  - a system of links. *Comment: concise and comprehensive*
  - a system of links or bars which are jointed together and more or less constrained by having a link or links fixed and by means of which straight or nearly straight lines or other point paths may be traced. *Comment: unnecessarily cumbersome and limited to path-generating linkages*
- *Kinematic chain* whose *joints* are equivalent to *lower pairs* only (Vol. 38, Nos. 7–10 (2003) of *Mechanism and Machine Theory* on Standardization of Terminology). *Comment: confuses linkage with its representation*

The Concise Oxford Dictionary of Current English, 1995, Clarendon Press, Oxford

Frolov, K. V., (editor), 1987, *Teoriya Mechanismov i Mashin* (Theory of Mechanisms and Machines), Vyschaya Shkola, Moscow (in Russian)

Mechanism and Machine Theory on Standardization of Terminology, 2003, Vol. 38, Nos. 7–10

Le Petit Robert, 1994, Dictionnaires Le Robert, Paris

Random House Webster's College Dictionary, 1997, Random House, New York

Merriam Webster's Collegiate Dictionary, on-line 2002

1.9 Here we want to estimate the time required to multiply two floating-point numbers. Since this time is very small, it is more suitable to perform a large number of multiplications, say 10^7, which will require a total time of the order of seconds.

- The Microsoft Visual C++6.0 program below was run on a Pentium IV 2.0:

```
#include <iostream>
  #include <time.h>
  int main()
  {
    clock_t start, end;
    start =clock();
    float a, i, CLOCKS_PER_mSEC=CLOCKS_PER_SEC/1000;
    for(i = 1; i <= 10000000; i++)
    a = 5 * 5;
    std::cout << ``a= `` << a << std::endl;
    end = clock();
    long duration=(long)(end-start)/CLOCKS_PER_mSEC;
    std::cout << ``Time: `` << duration<<''ms''<<std::endl;
    return 0;
  }
```

A Pentium IV 2.0 processor required 125 ms to perform the 10^7 multiplications, which amounts to 1.25×10^{-8} s/mult, or 8×10^7 mult/s.

- The C program below was run for the same purpose on the CLUMEQ (Consortium Laval UQAM McGill and Eastern Quebec for High Performance Computing) supercomputer, AMD Athlon 1900+ cluster:

```
# include "stdio.h"
# include "math.h"
# include <sys/time.h>
# include <sys/resource.h>
# define RUSAGE_SELF  0
\* calling process *\
main()
{
    int getrusage(int who, struct rusage *rusage);
    long diffsec, diffmsec;
    float a;
    int i;
    struct rusage time_begin, time_end;
    struct rusage *pb, *pe;
    pb=&time_begin;
```

```
    getrusage(RUSAGE_SELF, pb);
    for(i=1; i<=100000000; i++)
    {
    a = 5*5;
    }
    printf("a=%f \
t\n",a);
    pe=&time_end;
    getrusage(RUSAGE_SELF,pe);

    diffsec=time_end.ru_utime.tv_sec-time_begin.ru_utime.tv_sec;
    diffmsec=(time_end.ru_utime.tv_usec-time_begin.ru_utime.tv_usec);
    printf("User time used:%d microsec\n",diffmsec);
}
```

The CLUMEQ supercomputer required 40 ms to perform the 10^7 multiplications, which amounts to 4×10^{-9} s/mult, or 2.5×10^8 mult/s.

- On June 20, 2011, Fox News reported that *Kei*—Japanese for 10^{16}—was capable of 8×10^{15} flops[1]—or eight petaflops—per second.

1.10 Here we want to estimate the time required to add two floating-point numbers. Since this time is very small, it is more suitable to perform a bunch of additions, say 10^7, which should amount to a total time of the order of seconds.

- The Microsoft Visual C++6.0 program below was run on a Pentium IV 2.0:

```
#include <iostream>
    #include <time.h>
    int main()
    {
        clock_t start, end;
        start = clock();
        float a, i, CLOCKS_PER_mSEC=CLOCKS_PER_SEC/1000;
        for(i=1; i<=10000000; i++)
        a = 5+5;
        std::cout << "a= "<< a << std::endl;
        end = clock();
        long duration = (long)(end-start)/CLOCKS_PER_mSEC;
        std::cout << "Time: "<< duration<<"ms"<<std::endl;
        return 0;
    }
```

A Pentium IV 2.0 processor required 125 ms to perform the 10^7 additions, which amounts to 1.25×10^-8 s/add, or 8×10^7 add/s.

- The C program below was run for the same purpose on the CLUMEQ supercomputer, AMD Athlon 1900+ cluster:

```
#include "stdio.h"
#include "math.h"
```

[1]See the Index for a definition of *flop*.

```
#include <sys/time.h>
#include <sys/resource.h>
#define RUSAGE_SELF  0    /* calling process */
main()
{
    int getrusage(int who, struct rusage *rusage);
    long diffsec, diffmsec;
    float a;
    int i;
    struct rusage time_begin, time_end;
    struct rusage *pb, *pe;
    pb=&time_begin;
    getrusage(RUSAGE_SELF, pb);
    for(i=1; i<=10000000; i++)
    {
    a = 5+5;
    }

    printf("a=%f \t\n",a);
    pe=&time_end;
    getrusage(RUSAGE_SELF,pe);

    diffsec=time_end.ru_utime.tv_sec-time_begin.ru_utime.tv_sec;
    diffmsec=time_end.ru_utime.tv_usec-time_begin.ru_utime.tv_usec;
    printf("User time used:%d microsec\n",diffmsec);
}
```

The CLUMEQ supercomputer required 40 ms to perform the 10^7 additions, which amounts to 4×10^{-9} add/mult, or 2.5×10^8 add/s.

1.11 What we must find here is the largest floating-point number ϵ that, when added to a given number a, leaves this number unchanged, i.e.,

$$a + \epsilon = a$$

- The Microsoft Visual C++6.0 program below was run on a Pentium IV 2.0 processor at 550 MHz:

```
#include <iostream>
    //#include <time.h>
    int main()
    {
        long i;
            //double a=1;
            float a=1;
            for(i=1; i<=10000000; i++)
            {
                a = a/10;
            if (a==0)
                break;
            }
        std::cout << "smallest floating-point"<< i-1 << std::endl;
        return 0;
    }
```

The value of ϵ reported in single-precision arithmetic was $1.0e-045$; in double-precision...... $1.0e-323$.

- The program below was run on the CLUMEQ supercomputer, AMD Athlon 1900+ cluster:

```
#include "stdio.h"
#include "math.h"
#include <sys/resource.h>
main()
{
    int i;
        float a=1;
        for(i=1; i<=10000000; i++)
        {
            a=a/10;
        if (a==0)
            break;
        }

    printf("smallest floating-point=%d \n",i-1);
}
```

This machine reported exactly the same values as its Pentium counterpart.

2 Mathematical Background

2.6 Let $\mathbf{w}$ be perpendicular to $\mathbf{v}$ ($\mathbf{w}^T\mathbf{v} = \mathbf{v}^T\mathbf{w} = 0$). Since we can find two such vectors, let us call them $\mathbf{w}_1$ & $\mathbf{w}_2$, which lie in the plane normal to $\mathbf{v}$, but are otherwise arbitrary—no need to assume that these two vectors are mutually orthogonal. We then have

$$\mathbf{T} = \mathbf{1} + \mathbf{u}\mathbf{v}^T$$

Multiplying both sides by $\mathbf{w}_i$, we obtain

$$\mathbf{T}\mathbf{w}_i = \mathbf{w}_i + \mathbf{u}\underbrace{\mathbf{v}^T\mathbf{w}_i}_{0} = \mathbf{w}_i, \quad i = 1, 2$$

Therefore, $\mathbf{w}_1$ & $\mathbf{w}_2$ are eigenvectors of $\mathbf{T}$, of eigenvalues $\lambda_1 = \lambda_2 = 1$.

Moreover, let

$$\mathbf{w}_3 \equiv \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

Then,

$$\mathbf{T}\mathbf{w}_3 = \mathbf{w}_3 + \mathbf{u}\mathbf{v}^T\mathbf{w}_3 = \frac{\mathbf{u}}{\|\mathbf{u}\|} + \mathbf{u}\mathbf{v}^T\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\mathbf{u}}{\|\mathbf{u}\|}(1 + \mathbf{v}^T\mathbf{u}) \quad \Rightarrow \quad \mathbf{T}\mathbf{w}_3 = (1 + \mathbf{v}^T\mathbf{u})\mathbf{w}_3$$

and $\mathbf{w}_3$ is the third eigenvector of $\mathbf{T}$, of eigenvalue $\lambda_3 = 1 + \mathbf{v}^T\mathbf{u}$.

2.7 The determinant of a matrix is equal to the product of its eigenvalues. Thus

$$\det(\mathbf{1} + \mathbf{u}\mathbf{v}^T) = \det(\mathbf{T}) = \lambda_1\lambda_2\lambda_3 = 1 + \mathbf{v}^T\mathbf{u} = 1 + \mathbf{u}\cdot\mathbf{v}$$

2.8 From $\mathbf{Q} = \mathbf{R}_1\mathbf{R}_2$, we have

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{R}_1\mathbf{R}_2(\mathbf{R}_1\mathbf{R}_2)^T = \mathbf{R}_1\underbrace{\mathbf{R}_2\mathbf{R}_2^T}_{\mathbf{1}}\mathbf{R}_1^T = \mathbf{R}_1\mathbf{R}_1^T = \mathbf{1}$$

and thus $\mathbf{Q}$ is orthogonal. Moreover, we have

$$\det(\mathbf{Q}) = \det(\mathbf{R}_1)\det(\mathbf{R}_2) = (-1)(-1) = 1$$

Thus, $\mathbf{Q}$ is a rotation.

Let $\mathbf{u}$ and ϕ be the unit vector parallel to the axis of rotation and the angle of rotation, respectively; then,

$$\mathbf{Q} = (\mathbf{1} - 2\mathbf{e}\mathbf{e}^T)(\mathbf{1} - 2\mathbf{f}\mathbf{f}^T) = \mathbf{1} - 2\mathbf{e}\mathbf{e}^T - 2\mathbf{f}\mathbf{f}^T + 4(\mathbf{e}^T\mathbf{f})\mathbf{e}\mathbf{f}^T$$

Hence,

$$\text{vect}(\mathbf{Q}) = -\frac{1}{2}4(\mathbf{e}^T\mathbf{f})\mathbf{e}\times\mathbf{f} = \mathbf{u}\sin\phi$$

and

$$\text{tr}(\mathbf{Q}) = 3 - 2 - 2 + 4(\mathbf{e}^T\mathbf{f})^2 = 4(\mathbf{e}^T\mathbf{f})^2 - 1 = 1 + 2\cos\phi$$

Therefore,

$$\sin\phi = \|\text{vect}(\mathbf{Q})\| = 2|\mathbf{e}^T\mathbf{f}|\ \|\mathbf{e}\times\mathbf{f}\|, \quad 0 \le \phi < \pi$$

whence

$$\mathbf{u} = -\frac{\mathbf{e}\times\mathbf{f}}{\|\mathbf{e}\times\mathbf{f}\|}\ \text{sign}(\mathbf{e}^T\mathbf{f})$$

where $\text{sign}(\mathbf{e}^T\mathbf{f})$ is the *signum function*, which is $+1$ if $\mathbf{e}^T\mathbf{f} > 0$, -1 if $\mathbf{e}^T\mathbf{f} < 0$, and undefined if $\mathbf{e}^T\mathbf{f} = 0$. Hence, $\mathbf{u}$ is undefined if $\mathbf{e}^T\mathbf{f} = 0$, which indicates that $\mathbf{Q} = \mathbf{1} - 2\mathbf{e}\mathbf{e}^T - 2\mathbf{f}\mathbf{f}^T$, in which case $\mathbf{Q}$ is symmetric, as in Exercise 2.13, and

$$\mathbf{u} = \mathbf{e}\times\mathbf{f}, \quad \phi = \pi$$

Hence, if $\mathbf{e}^T\mathbf{f} \neq 0$,

$$\cos\phi = 2(\mathbf{e}^T\mathbf{f})^2 - 1, \quad \Rightarrow \quad \phi = \tan^{-1}\left[\frac{2|\mathbf{e}^T\mathbf{f}|\ \|\mathbf{e}\times\mathbf{f}\|}{2(\mathbf{e}^T\mathbf{f})^2 - 1}\right]$$

2.9 The product $\mathbf{Q} = \mathbf{R}_1\mathbf{R}_2$ is readily computed as

$$\mathbf{Q} = (\mathbf{1} - 2\mathbf{e}\mathbf{e}^T)(\mathbf{1} - 2\mathbf{f}\mathbf{f}^T) = \mathbf{1} - 2\mathbf{e}\mathbf{e}^T - 2\mathbf{f}\mathbf{f}^T + 4(\mathbf{e}^T\mathbf{f})\mathbf{e}\mathbf{f}^T$$

Let ϕ be the angle of rotation of $\mathbf{Q}$, and hence,

$$\text{tr}(\mathbf{Q}) = 3 - 2 - 2 + 4(\mathbf{e}^T\mathbf{f})^2 = 4(\mathbf{e}^T\mathbf{f})^2 - 1 = 1 + 2\cos\phi$$

If $\mathbf{e}^T\mathbf{f} \neq 0$, then

$$\cos\phi = 2(\mathbf{e}^T\mathbf{f})^2 - 1,$$

Because $|\cos\phi| \le 1$, we can write the above expression in the form

$$-1 \le 2(\mathbf{e}^T\mathbf{f})^2 - 1 \le 1 \quad \text{or} \quad 0 \le |\mathbf{e}^T\mathbf{f}| \le 1.$$

Since $\mathbf{e}$ and $\mathbf{f}$ are unit vectors, this condition is always respected for any $\mathbf{e}$ or $\mathbf{f}$. However,

$$\text{vect}(\mathbf{Q}) = 4(\mathbf{e}^T\mathbf{f})\,\text{vect}(\mathbf{e}\mathbf{f}^T) = -2(\mathbf{e}^T\mathbf{f})\,\mathbf{e}\times\mathbf{f}$$

where we have recalled relation (2.60). The axis of rotation of $\mathbf{Q}$ should thus be parallel to $\mathbf{e}\times\mathbf{f}$, which is the sole condition for $\mathbf{Q}$ to be factorable as $\mathbf{R}_1\mathbf{R}_2$.

2.10 (a) With $\mathbf{A} = \mathrm{CPM}(\mathbf{a})$, the equation can be rewritten as

$$(\mathbf{1} + \mathbf{A})\mathbf{v} = \mathbf{b}$$

which was defined as $\mathbf{B}$ in Example 2.3.2. In that example, it was shown that $\mathbf{1}+\mathbf{A}$ is always invertible, and hence, $\mathbf{v}$ can be expressed as

$$\mathbf{v} = (\mathbf{1} + \mathbf{A})^{-1}\mathbf{b} \equiv \frac{1}{1 + \|\mathbf{a}\|^2}[(1 + \|\mathbf{a}\|^2)\mathbf{1} - \mathbf{A} + \mathbf{A}^2]\mathbf{b}$$

which, upon expansion, yields

$$\mathbf{v} = \mathbf{b} + \frac{1}{1 + \|\mathbf{a}\|^2}(-\mathbf{A}\mathbf{b} + \mathbf{A}^2\mathbf{b})$$

and, finally,

$$\mathbf{v} = \frac{-\mathbf{a} \times \mathbf{b} + \mathbf{b} + (\mathbf{a}^T\mathbf{b})\mathbf{a}}{1 + \|\mathbf{a}\|^2}$$

(b) For $\mathbf{v}$ to be orthogonal to $\mathbf{a}$, one must have $\mathbf{v}^T\mathbf{a} = 0$, which means that, when "dotted" with $\mathbf{a}$, the numerator of the above expression must vanish, i.e.,

$$-\underbrace{(\mathbf{a} \times \mathbf{b})^T\mathbf{a}}_{0} + \mathbf{b}^T\mathbf{a} + \mathbf{a}^T\mathbf{b}\|\mathbf{a}\|^2 = 0$$

which thus reduces to

$$(\mathbf{a}^T\mathbf{b})(1 + \|\mathbf{a}\|^2) = 0$$

As the second factor above cannot vanish, the condition sought is that $\mathbf{a}$ and $\mathbf{b}$ be mutually orthogonal.

(c) For $\mathbf{v}$ to be orthogonal to $\mathbf{b}$, a similar relation follows upon "dotting" the numerator of $\mathbf{v}$ with $\mathbf{b}$, which leads to

$$-\underbrace{(\mathbf{a} \times \mathbf{b})^T\mathbf{b}}_{0} + \|\mathbf{b}\|^2 + (\mathbf{a}^T\mathbf{b})^2 = 0$$

which holds if and only if the sum of two positive quantities vanishes. Obviously, this sum cannot vanish, and hence, $\mathbf{v}$ cannot be orthogonal to $\mathbf{b}$.

2.11 Let λ be an eigenvalue of $\mathbf{E}$, and $\mathbf{u}$ the corresponding unit eigenvector, i.e.,

$$\mathbf{E}\mathbf{u} = \lambda\mathbf{u} \tag{1}$$

Multiplying eq.(1) by $\mathbf{E}$, we obtain

$$\mathbf{E}^2\mathbf{u} = \lambda^2\mathbf{u} \tag{2}$$

and multiplying a second time by $\mathbf{E}$,

$$\mathbf{E}^3\mathbf{u} = \lambda^3\mathbf{u} \tag{3}$$

However, from eq.(2.51)

$$\mathbf{E}^3 = -\mathbf{E} \tag{4}$$

Substituting eq.(4) into eq.(3), we obtain

$$-\mathbf{E}\mathbf{u} = \lambda^3\mathbf{u} \tag{5}$$

Moreover, from eq.(1) we have $\mathbf{E}\mathbf{u} = \lambda\mathbf{u}$, and thus eq.(5) reduces to

$$\lambda(1 + \lambda^2)\mathbf{u} = \mathbf{0} \tag{6}$$

However, $\|\mathbf{u}\| = 1$ and hence, $\mathbf{u} \neq \mathbf{0}$. Therefore, we have from eq.(6)

$$P(\lambda) \equiv \lambda(1+\lambda^2) = 0 \tag{7}$$

Equation (7) is the characteristic equation of $\mathbf{E}$, and $P(\lambda)$ its characteristic polynomial. The three roots of eq.(7) are readily obtained as

$$\lambda_1 = 0, \quad \lambda_2 = j, \quad \lambda_3 = -j$$

The eigenvector $\mathbf{u}_1$ corresponding to λ_1 is, obviously, $\mathbf{e}$, for $\mathbf{Ee} = \mathbf{0}$.

Now, to prove that the eigenvectors associated with the two complex eigenvalues are complex and mutually orthogonal, assume that $\mathbf{v}$ and $\mathbf{w}$ are two 3-dimensional complex vectors. Moreover, the former is assumed to be the eigenvector of $\mathbf{E}$ associated with $\lambda_2 = j$, the latter with $\lambda_3 = -j$, i.e.,

$$\mathbf{Ev} = j\mathbf{v}, \quad \mathbf{Ew} = -j\mathbf{w} \tag{8}$$

The scalar product p of $\mathbf{v}$ times $\mathbf{w}$ is a complex number, given by

$$p = \mathbf{v}^*\mathbf{w} \tag{9}$$

where, if, for example, $\mathbf{v} = \mathbf{v}_r + j\mathbf{v}_c$, with $\mathbf{v}_r$ and $\mathbf{v}_c$ denoting two 3-dimensional real vectors, then, the transpose conjugate of $\mathbf{v}$ is

$$\mathbf{v}^* = \mathbf{v}_r^T - j\mathbf{v}_c^T \tag{10}$$

Moreover, the transpose conjugate $\overline{p}$ of p, being a scalar, is simply its complex conjugate, which equals the transpose-conjugate of the product on the right-hand side of eq.(10), namely,

$$\overline{p} \equiv (\mathbf{v}^*\mathbf{w})^* = \mathbf{w}^*\mathbf{v} \tag{11}$$

Now, multiplying both sides of the second of eqs.(8) by j, we obtain

$$\mathbf{w} = j\mathbf{Ew} \tag{12}$$

Substituting $\mathbf{w}$ by the above expression in eq.(10), an alternative expression for p is found:

$$p = \mathbf{v}^*(j\mathbf{Ew}) \equiv j\mathbf{v}^*\mathbf{Ew} \tag{13}$$

Likewise, multiplying both sides of the first of eqs.(8) by j, we obtain

$$\mathbf{v} = -j\mathbf{Ev} \tag{14}$$

which, when substituted into eq.(11), yields an alternative expression for $\overline{p}$:

$$\overline{p} = \mathbf{w}^*(-j\mathbf{Ev}) \equiv -j\mathbf{w}^*\mathbf{Ev} \tag{15}$$

Now, upon taking the complex-conjugate of both sides of eq.(13), one more expression for $\overline{p}$ is obtained, namely,

$$\overline{p} = -j\mathbf{w}^*(\mathbf{E}^T\mathbf{v}) \equiv j\mathbf{w}^*\mathbf{Ev} \tag{16}$$

Notice that the two expressions for $\overline{p}$, eqs.(15) and (16), differ only in the sign, i.e.,

$$\overline{p} = -\overline{p} \quad \Rightarrow \quad p = -p$$

which holds if and only if $p = 0$, thereby showing that, in fact, the two eigenvectors of $\mathbf{E}$ in question, $\mathbf{v}$ and $\mathbf{w}$, are mutually orthogonal. In the next step, these two eigenvectors are found. The real and imaginary parts of $\mathbf{v}$ were defined above as $\mathbf{v}_r$ and $\mathbf{v}_c$, respectively. A similar definition is applied now to the real and imaginary parts of $\mathbf{w}$, namely, $\mathbf{w}_r$ and $\mathbf{w}_c$, respectively. Substitution of $\mathbf{v}$ in terms of its two components, real and imaginary, into the first of eqs.(8) yields

$$\mathbf{E}(\mathbf{v}_r + j\mathbf{v}_c) = j\mathbf{v}_r - \mathbf{v}_c \tag{17}$$

whence two real equations follow, one for the real, one for the complex part:

$$\mathbf{E}\mathbf{v}_r = -\mathbf{v}_c, \quad \mathbf{E}\mathbf{v}_c = \mathbf{v}_r \tag{18}$$

Next, upon multiplying both sides of the first of the above equations by $\mathbf{E}$, one obtains

$$\mathbf{E}^2\mathbf{v}_r = -\mathbf{E}\mathbf{v}_c \equiv -\mathbf{v}_r$$

where the second of eqs.(18) has been recalled, the foregoing equation thus becoming

$$(\mathbf{E}^2 + \mathbf{1})\mathbf{v}_r = \mathbf{0}$$

and, in light of the expression for the square of a 3×3 skew-symmetric matrix, eq.(2.39), the above equations leads to

$$\mathbf{e}\,\underbrace{(\mathbf{e}^T\mathbf{v}_r)}_{0} = \mathbf{0}$$

which implies that $\mathbf{v}_r$ is normal to $\mathbf{e}$. A similar reasoning leads to the same conclusion about $\mathbf{v}_c$, namely, that $\mathbf{v}_c$ is also normal to $\mathbf{e}$. Moreover, the first of eqs.(18) leads to

$$\mathbf{v}_c = -\mathbf{e} \times \mathbf{v}_r$$

whence $\mathbf{v}_c$ is also normal to $\mathbf{v}_r$. Therefore, any two orthonormal vectors lying in the plane normal to $\mathbf{e}$ are candidates for the real and complex components of $\mathbf{v}$. A similar reasoning leads to the conclusion that any two orthonormal vectors lying in the plane normal to $\mathbf{e}$ are candidates for the two components of $\mathbf{w}$. However, $\mathbf{v}$ and $\mathbf{w}$ are known to be mutually orthogonal. Upon imposing the condition $\mathbf{v}^*\mathbf{w} = 0$, the relation below follows:

$$\mathbf{v}_r^T\mathbf{w}_r + \mathbf{v}_c^T\mathbf{w}_c + j(\mathbf{v}_r^T\mathbf{w}_c - \mathbf{v}_c^T\mathbf{w}_r) = 0$$

which readily leads to two real equations, one for the real part, one for its complex counterpart:

$$\mathbf{v}_r^T\mathbf{w}_r + \mathbf{v}_c^T\mathbf{w}_c = 0, \quad \mathbf{v}_r^T\mathbf{w}_c - \mathbf{v}_c^T\mathbf{w}_r = 0$$

The two above equations are verified under the conditions $\mathbf{w}_r = \mathbf{v}_c$ and $\mathbf{w}_c = \mathbf{v}_r$, i.e.,

$$\mathbf{w} = \mathbf{v}_c + j\mathbf{v}_r$$

In conclusion: any pair of orthonormal unit vectors normal to $\mathbf{e}$ constitute the real and imaginary parts of the eigenvector associated with $\lambda_2 = j$. The same vectors, but with their real and imaginary roles exchanged, constitute the real and imaginary parts of the eigenvector associated with $\lambda_3 = -j$.

2.12 From eq.(2.53), we have

$$\mathbf{Q} = e^{\mathbf{E}\phi} = f(\mathbf{E})$$

where $\mathbf{E}$ is the cross-product matrix of the unit vector $\mathbf{e}$ with eigenvalues $\{\,0,\, j,\, -j\,\}$—see Exercise 2.10.

Let λ_k be an eigenvalue of $\mathbf{E}$ and μ_k be an eigenvalue of $\mathbf{Q}$. Then, by the Cayley-Hamilton Theorem, we have

$$\mu_k = e^{\lambda_k\phi} = f(\lambda_k)$$

Therefore,

$$\mu_1 = e^0 = 1, \qquad \mu_2 = e^{j\phi}, \qquad \mu_3 = e^{-j\phi}$$