CHAPTER 1

RAY OPTICS

1.1 POSTULATES OF RAY OPTICS

EXERCISE 1.1-1

Proof of Snell's Law

The pathlength is given by $n_1 d_1 \sec\theta_1 + n_2 d_2 \sec\theta_2$. (1)

The pathlength is a function of θ_1 and θ_2, which are related by

$d_1 \tan\theta_1 + d_2 \tan\theta_2 = d$. (2)

The pathlength is minimized when $\frac{\partial}{\partial\theta_1}[n_1 d_1 \sec\theta_1 + n_2 d_2 \sec\theta_2] = 0$,
i.e., when $n_1 d_1 \sec\theta_1 \tan\theta_1 + n_2 d_2 \sec\theta_2 \tan\theta_2 (\partial\theta_2/\partial\theta_1) = 0$. (3)

From (2), we have $\frac{\partial}{\partial\theta_1}[d_1 \tan\theta_1 + d_2 \tan\theta_2] = 0$,

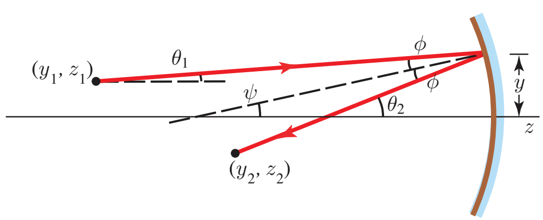
so that $d_1 \sec^2\theta_1 + d_2 \sec^2\theta_2 (\partial\theta_2/\partial\theta_1) = 0$ and $\frac{\partial\theta_2}{\partial\theta_1} = -\frac{d_1 \sec^2\theta_1}{d_2 \sec^2\theta_2}$.

Substituting into (3), we have $n_1 d_1 \sec\theta_1 \tan\theta_1 - n_2 \frac{d_1 \sec^2\theta_1 \tan\theta_2}{\sec\theta_2} = 0$,
whereupon $n_1 \tan\theta_1 = n_2 \sec\theta_1 \sin\theta_2$, from which $n_1 \sin\theta_1 = n_2 \sin\theta_2$, which is Snell's law.

1.2 SIMPLE OPTICAL COMPONENTS

EXERCISE 1.2-1

Image Formation by a Spherical Mirror



A ray originating at $P_1 = (y_1, z_1)$ at angle θ_1 meets the mirror at height
$y \approx y_1 + \theta_1 z_1$. (1)

The angle of incidence at the mirror is $\phi = \psi - \theta_1 \approx \frac{y}{-R} - \theta_1$.
The reflected ray makes angle θ_2 with the z axis:

$$\theta_2 = 2\phi + \theta_1 = 2\left[\frac{y}{-R} - \theta_1\right] + \theta_1 = \frac{2y}{-R} - \theta_1 = \frac{2(y_1 + \theta_1 z_1)}{-R} - \theta_1.$$

Substituting $f = \frac{-R}{2}$, we have $\theta_2 = \frac{y_1 + \theta_1 z_1}{f} - \theta_1$. (2)

The height y_2 can be determined from $\frac{y + (-y_2)}{z_2} \approx \theta_2$. (3)

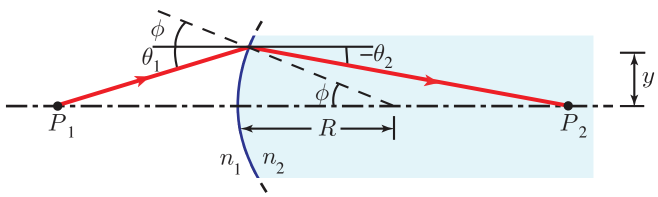
Substituting from (1) and (2) into (3), we have $y_1 + \theta_1 z_1 - y_2 = z_2 \left[\frac{y_1 + \theta_1 z_1}{f} - \theta_1\right]$
and $y_2 = y_1 - \frac{z_2\, y_1}{f} + \theta_1 \left[z_1 - \frac{z_1\, z_2}{f} + z_2\right]$.
If $\left[z_1 - \frac{z_1\, z_2}{f} + z_2\right] = 0,$ or $\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f}$, we have

$$y_2 = y_1 \left(1 - \frac{z_2}{f}\right), \tag{4}$$

which is independent of θ_1.
From (4) it is clear that $\frac{z_2}{f} = 1 - \frac{y_2}{y_1}$, so that $y_2 = -\frac{z_2}{z_1} y_1$.

EXERCISE 1.2-2

Image Formation



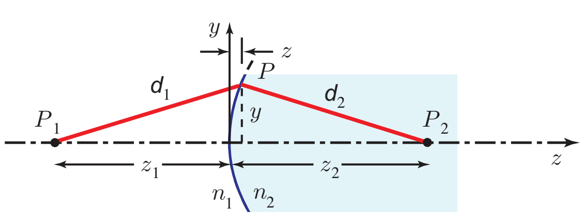
a) From Snell's law, we have $n_1 \sin(\theta_1 + \phi) = n_2 \sin[\phi - (-\theta_2)]$. Since all angles are small, the paraxial version of Snell's Law is $n_1(\theta_1 + \phi) \approx n_2(\phi + \theta_2)$, or $\theta_2 \approx (n_1/n_2)\theta_1 + [(n_1 - n_2)/n_2]\phi$.
Because $\phi \approx y/R$, we obtain $\theta_2 \approx \frac{n_1}{n_2}\,\theta_1 - \frac{n_2 - n_1}{n_2 R}\,y$, which is (1.2-8).

b) Substituting $\theta_1 \approx y/z_1$ and $(-\theta_2) \approx y/z_2$ into (1.2-8),
we have $-y/z_2 \approx \frac{(n_1/n_2)\,y}{z_1} - \frac{n_2 - n_1}{n_2 R}\,y$, from which (1.2-9) follows.

c) With reference to Fig. 1.2-13(b), for the ray passing through the origin 0, we have angles of incidence and refraction given by y_1/z_1 and $-y_2/z_2$, respectively, so that the paraxial Snell's Law leads to (1.2-10). Rays at other angles are also directed from P_1 to P_2, as can be shown using a derivation similar to that followed in Exercise 1.2-1.

EXERCISE 1.2-3

Aberration-Free Imaging Surface In accordance with Fermat's principle, we require



that the optical path length obey $n_1 d_1 + n_2 d_2 =$ constant $= n_1 z_1 + n_2 z_2$. This constitutes an equation defining the surface, which can be written in Cartesian coordinates as

$$n_1\sqrt{(z+z_1)^2+y^2}+n_2\sqrt{(z_2-z)^2+y^2}=n_1z_1+n_2z_2. \qquad (1)$$

Given z_1 and z_2, (1) relates y to z and thus defines the surface.

EXERCISE 1.2-4

Proof of the Thin Lens Formulas

A ray at angle θ_1 and height y refracts at the first surface in accordance with (1.2-8) and its angle is altered to $\theta = \dfrac{\theta_1}{n} - \dfrac{n-1}{nR_1}\,y\,,$ (1)
where R_1 is the radius of the first surface $(R_1 < 0)$.

At the second surface, the angle is altered again to $\theta_2 = n\theta - \dfrac{1-n}{R_2}\,y\,,$ (2)
where R_2 is the radius of the second surface $(R_2 > 0)$. We have assumed that the ray height is not altered since the lens is thin.

Substituting (1) into (2) we obtain:

$$\theta_2 = n\left[\frac{\theta_1}{n} - \frac{n-1}{nR_1}\,y\right] - \frac{1-n}{R_2}\,y = \theta_1 - (n-1)\,y\left[\frac{1}{R_1}-\frac{1}{R_2}\right].$$

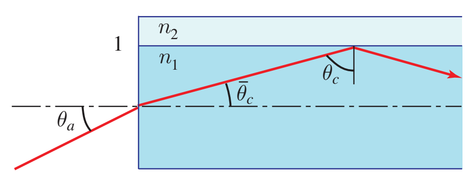
Using (1.2-11), we invoke $\theta_2 = \theta_1 - (y/f)$. (3)

If $\theta_1 = 0$, then $\theta_2 = (-y/f)$, and $z_2 \approx (y/{-\theta_2}) = f$, where f is the focal length. In general $\theta_1 \approx \dfrac{y}{z_1}$ and $-\theta_2 = \dfrac{y}{z_2}$. Therefore from (3), $\dfrac{-y}{z_2} = \dfrac{y}{z_1} - \dfrac{y}{f}$, from which (1.2-13) follows. Equation (1.2-14) can be proved by use of an approach similar to that used in Exercise 1.2-1.

EXERCISE 1.2-5

Numerical Aperture and Angle of Acceptance of an Optical Fiber

Applying Snell's law at the air/core surface:
$\sin\theta_a = n_1 \sin\bar\theta_c = n_1\cos\theta_c$ (1)



Since $\sin\theta_c = n_2/n_1,\ \cos\theta_c = \sqrt{1-(n_2/n_1)^2}\,.$

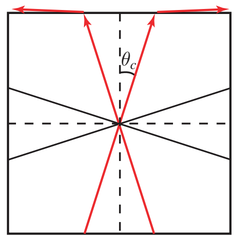
Therefore, from (1), NA $\equiv \sin\theta_a = n_1\sqrt{1-(n_2/n_1)^2} = \sqrt{n_1^2-n_2^2}\,.$

For silica glass with $n_1 = 1.475$ and $n_2 = 1.460$, the numerical aperture NA = 0.21 and the acceptance angle $\theta_a = 12.1°$.

EXERCISE 1.2-6

Light Trapped in a Light-Emitting Diode

a) The rays within the six cones of half angle $\theta_c = \sin^{-1}(1/n)$ $(= 16.1°$ for GaAs) are refracted into air in all directions, as shown in the illustration. The rays outside these six cones are internally reflected. Since $\theta_c < 45°$, the cones do not overlap and the reflected rays remain outside the cones and continue to reflect internally without refraction. These are the trapped rays.



b) The area of the spherical cap atop one of these cones is $A = \int_0^{\theta_c} 2\pi r \sin\theta \, r \, d\theta = 2\pi r^2(1 - \cos\theta_c)$, while the area of the entire sphere is $4\pi r^2$. Thus, the fraction of the emitted light that lies within the solid angle subtended by one of these cones is $A/4\pi r^2 = \frac{1}{2}(1 - \cos\theta_c)$ (see Sec. 18.1B). Thus, the ratio of the extracted light to the total light is $6 \times \frac{1}{2}(1 - \cos\theta_c) = 3(1 - \cos\theta_c)$ $(= 0.118$ for GaAs). Thus, 11.8% of the light is extracted for GaAs.
Note that this derivation is valid only for $\theta_c < 45°$ or $n > \sqrt{2}$.

1.3 GRADED-INDEX OPTICS

EXERCISE 1.3-1

The GRIN Slab as a Lens

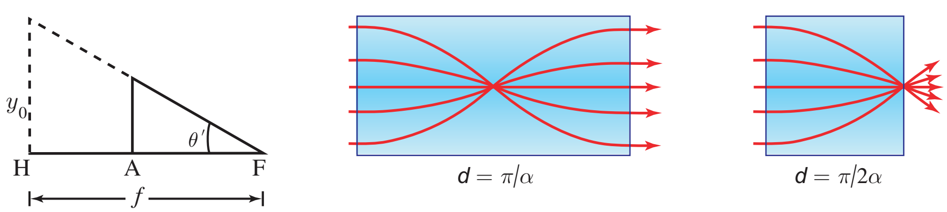
Using (1.3-11) and (1.3-12), with $\theta_0 = 0$ and $z = d$, we have $y(d) = y_0 \cos(\alpha d)$ and $\theta(d) = -y_0\alpha \sin(\alpha d)$. Rays refract into air at an angle $\theta' \approx n_0|\theta(d)| = n_0 y_0 \alpha \sin(\alpha d)$.

Therefore, $\overline{AF} \approx \dfrac{y(d)}{\theta'} = \dfrac{y_0 \cos(\alpha d)}{n_o y_0 \alpha \sin(\alpha d)} = \dfrac{1}{n_0 \alpha \tan(\alpha d)}$ and

$f = \dfrac{y_0}{\theta'} = \dfrac{1}{n_0 \alpha \sin(\alpha d)}$, so that

$$\overline{AH} = f - \overline{AF} = \frac{1}{n_0\alpha}\left[\frac{1}{\sin(\alpha d)} - \frac{1}{\tan(\alpha d)}\right] = \frac{1}{n_0\alpha}\,\frac{1 - \cos(\alpha d)}{\sin(\alpha d)}$$
$$= \frac{1}{n_0\alpha}\,\frac{2\sin^2(\alpha d/2)}{2\sin(\alpha d/2)\cos(\alpha d/2)} = \frac{1}{n_0\alpha}\tan(\alpha d/2).$$

Trajectories:

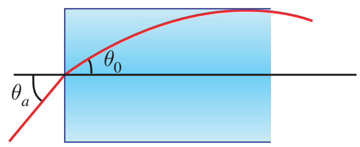


EXERCISE 1.3-2

Numerical Aperture of the Graded-Index Fiber

Using (1.3-11) with $y_0 = 0$, we obtain $y(z) = (\theta_0/\alpha)\sin(\alpha z)$. The ray traces a sinusoidal trajectory with amplitude θ_0/α that must not exceed the radius a. Thus $\theta_0/\alpha = a$.
The acceptance angle is therefore $\theta_a \approx n_0\theta_0 = n_0\alpha a$.

For a step-index fiber (Exercise 1.2-5),
$\theta_a = \sqrt{n_1^2 - n_2^2} = \sqrt{(n_1+n_2)(n_1-n_2)}$.
If $n_1 \approx n_2$, $\theta_a \approx \sqrt{2n_1(n_1-n_2)}$.
If $n_1 = n_0$ and $n_2 = n_0(1-\alpha^2 a^2/2)$,
$\theta_a \approx \sqrt{2n_0(\alpha^2 a^2 n_0/2)} = \alpha\, a\, n_0$, which is the same acceptance angle as for the graded-index fiber.



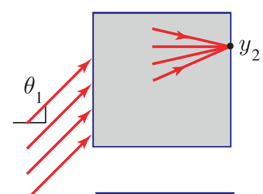
1.4 MATRIX OPTICS

EXERCISE 1.4-1

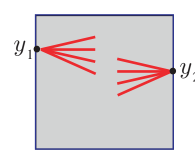
Special Forms of the Ray-Transfer Matrix

Using the basic equations
$y_2 = Ay_1 + B\,\theta_1$ and $\theta_2 = C\,y_1 + D\,\theta_1$, we obtain:

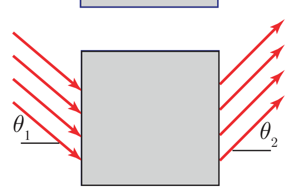
- If $A = 0$, then $y_2 = B\,\theta_1$, i.e., for a given θ_1, we see that y_2 is the same regardless of y_1.
This is a focusing system.



- If $B = 0$, then $y_2 = A\,y_1$, i.e., for a given y_1, we see that y_2 is the same regardless of θ_1.
This is an imaging system.



- If $C = 0$, then $\theta_2 = D\,\theta_1$, i.e., we see that all parallel rays remain parallel.



- If $D = 0$, then $\theta_2 = C\,y_1$, i.e., we see that all rays originating from a point become parallel.