Answers to selected exercises for chapter 1

1.1 Apply $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, then

$$\begin{aligned} & f_1(t) + f_2(t) \\ & \quad = A_1 \cos\omega t \cos\phi_1 - A_1 \sin\omega t \sin\phi_1 + A_2 \cos\omega t \cos\phi_2 - A_2 \sin\omega t \sin\phi_2 \\ & \quad = (A_1\cos\phi_1 + A_2\cos\phi_2)\cos\omega t - (A_1\sin\phi_1 + A_2\sin\phi_2)\sin\omega t \\ & \quad = C_1\cos\omega t - C_2\sin\omega t, \end{aligned}$$

where $C_1 = A_1\cos\phi_1 + A_2\cos\phi_2$ and $C_2 = A_1\sin\phi_1 + A_2\sin\phi_2$. Put $A = \sqrt{C_1^2 + C_2^2}$ and take ϕ such that $\cos\phi = C_1/A$ and $\sin\phi = C_2/A$ (this is possible since $(C_1/A)^2 + (C_2/A)^2 = 1$). Now $f_1(t)+f_2(t) = A(\cos\omega t\cos\phi - \sin\omega t \sin\phi) = A\cos(\omega t + \phi)$.

1.2 Put $c_1 = A_1 e^{i\phi_1}$ and $c_2 = A_2 e^{i\phi_2}$, then $f_1(t) + f_2(t) = (c_1 + c_2)e^{i\omega t}$. Let $c = c_1 + c_2$, then $f_1(t) + f_2(t) = ce^{i\omega t}$. The signal $f_1(t) + f_2(t)$ is again a time-harmonic signal with amplitude $|\,c\,|$ and initial phase $\arg c$.

1.5 The power P is given by

$$\begin{aligned} P &= \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} A^2\cos^2(\omega t + \phi_0)\,dt = \frac{A^2\omega}{4\pi}\int_{-\pi/\omega}^{\pi/\omega}(1+\cos(2\omega t + 2\phi_0))\,dt \\ &= \frac{A^2}{2}. \end{aligned}$$

1.6 The energy-content is $E = \int_0^\infty e^{-2t}\,dt = \frac{1}{2}$.

1.7 The power P is given by

$$P = \frac{1}{4}\sum_{n=0}^{3} |\cos(n\pi/2)|^2 = \tfrac{1}{2}.$$

1.8 The energy-content is $E = \sum_{n=0}^\infty e^{-2n}$, which is a geometric series with sum $1/(1-e^{-2})$.

1.9 **a** If $u(t)$ is real, then the integral, and so $y(t)$, is also real.
b Since

$$\left|\int u(\tau)\,d\tau\right| \le \int |\,u(\tau)\,|\,d\tau,$$

it follows from the boundedness of $u(t)$, so $|\,u(\tau)\,| \le K$ for some constant K, that $y(t)$ is also bounded.
c The linearity follows immediately from the linearity of integration. The time-invariance follows from the substitution $\xi = \tau - t_0$ in the integral $\int_{t-1}^t u(\tau - t_0)\,d\tau$ representing the response to $u(t-t_0)$.
d Calculating $\int_{t-1}^t \cos(\omega\tau)\,d\tau$ gives the following response: $(\sin(\omega t) - \sin(\omega t - \omega))/\omega = 2\sin(\omega/2)\cos(\omega t - \omega/2)/\omega$.
e Calculating $\int_{t-1}^t \sin(\omega\tau)\,d\tau$ gives the following response: $(-\cos(\omega t) + \cos(\omega t - \omega))/\omega = 2\sin(\omega/2)\sin(\omega t - \omega/2)/\omega$.
f From the response to $\cos(\omega t)$ in d it follows that the amplitude response is $|\,2\sin(\omega/2)/\omega\,|$.
g From the response to $\cos(\omega t)$ in d it follows that the phase response is $-\omega/2$ if $2\sin(\omega/2)/\omega \ge 0$ and $-\omega/2 + \pi$ if $2\sin(\omega/2)/\omega < 0$. From

phase and amplitude response the frequency response follows: $H(\omega) = 2\sin(\omega/2)e^{-i\omega/2}/\omega$.

1.11 **a** The frequency response of the cascade system is $H_1(\omega)H_2(\omega)$, since the reponse to $e^{i\omega t}$ is first $H_1(\omega)e^{i\omega t}$ and then $H_1(\omega)H_2(\omega)e^{i\omega t}$.
b The amplitude response is $|H_1(\omega)H_2(\omega)| = A_1(\omega)A_2(\omega)$.
c The phase response is $\arg(H_1(\omega)H_2(\omega)) = \Phi_1(\omega) + \Phi_2(\omega)$.

1.12 **a** The amplitude response is $|1+i||e^{-2i\omega}| = \sqrt{2}$.
b The input $u[n] = 1$ has frequency $\omega = 0$, initial phase 0 and amplitude 1. Since $e^{i\omega n} \mapsto H(e^{i\omega})e^{i\omega n}$, the response is $H(e^0)1 = 1+i$ for all n.
c Since $u[n] = (e^{i\omega n} + e^{-i\omega n})/2$ we can use $e^{i\omega n} \mapsto H(e^{i\omega})e^{i\omega n}$ to obtain that $y[n] = (H(e^{i\omega})e^{i\omega n} + H(e^{-i\omega})e^{-i\omega n})/2$, so $y[n] = (1+i)\cos(\omega(n-2))$.
d Since $u[n] = (1+\cos 4\omega n)/2$, we can use the same method as in b and c to obtain $y[n] = (1+i)(1+\cos(4\omega(n-2)))/2$.

1.13 **a** The power is the integral of $f^2(t)$ over $[-\pi/|\omega|, \pi/|\omega|]$, times $|\omega|/2\pi$. Now $\cos^2(\omega t+\phi_0)$ integrated over $[-\pi/|\omega|, \pi/|\omega|]$ equals $\pi/|\omega|$ and $\cos(\omega t)\cos(\omega t+\phi_0)$ integrated over $[-\pi/|\omega|, \pi/|\omega|]$ is $(\pi/|\omega|)\cos\phi_0$. Hence, the power equals $(A^2 + 2AB\cos(\phi_0) + B^2)/2$.
b The energy-content is $\int_0^1 \sin^2(\pi t)\,dt = 1/2$.

1.14 The power is the integral of $|f(t)|^2$ over $[-\pi/|\omega|, \pi/|\omega|]$, times $|\omega|/2\pi$, which in this case equals $|c|^2$.

1.16 **a** The amplitude response is $|H(\omega)| = 1/(1+\omega^2)$. The phase response is $\arg H(\omega) = \omega$.
b The input has frequency $\omega = 1$, so it follows from $e^{i\omega t} \mapsto H(\omega)e^{i\omega t}$ that the response is $H(1)ie^{it} = ie^{i(t+1)}/2$.

1.17 **a** The signal is not periodic since $\sin(2N) \neq 0$ for all integer N.
b The frequency response $H(e^{i\omega})$ equals $A(e^{i\omega})e^{i\Phi e^{i\omega}}$, hence, we obtain that $H(e^{i\omega}) = e^{i\omega}/(1+\omega^2)$. The response to $u[n] = (e^{2in} - e^{-2in})/2i$ is then $y[n] = (e^{2i(n+1)} - e^{-2i(n+1)})/(10i)$, so $y[n] = (\sin(2n+2))/5$. The amplitude is thus $1/5$ and the initial phase $2 - \pi/2$.

1.18 **a** If $u(t) = 0$ for $t < 0$, then the integral occurring in $y(t)$ is equal to 0 for $t < 0$. For $t_0 \geq 0$ the expression $u(t-t_0)$ is also causal. Hence, the system is causal for $t_0 \geq 0$.
b It follows from the boundedness of $u(t)$, so $|u(\tau)| \leq K$ for some constant K, that $y(t)$ is also bounded (use the triangle inequality and the inequality from exercise 1.9b). Hence, the system is stable.
c If $u(t)$ is real, then the integral is real and so $y(t)$ is real. Hence, the system is real.
d The response is

$$y(t) = \sin(\pi(t-t_0)) + \int_{t-1}^{t} \sin(\pi\tau)\,d\tau = \sin(\pi(t-t_0)) - 2(\cos \pi t)/\pi.$$

1.19 **a** If $u[n] = 0$ for $n < 0$, then $y[n]$ is also equal to 0 for $n < 0$ whenever $n_0 \geq 0$. Hence, the system is causal for $n_0 \geq 0$.
b It follows from the boundedness of $u[n]$, so $|u[n]| \leq K$ for some constant K and all n, that $y[n]$ is also bounded (use the triangle inequality):

$$|y[n]| \leq |u[n-n_0]| + \left|\sum_{l=n-2}^{n} u[l]\right| \leq K + \sum_{l=n-2}^{n} |u[l]| \leq K + \sum_{l=n-2}^{n} K,$$

which equals $4K$. Hence, the system is stable.
c If $u[n]$ is real, then $u[n-n_0]$ is real and also the sum in the expression for $y[n]$ is real, hence, $y[n]$ is real. This means that the system is real.
d The response to $u[n] = \cos \pi n = (-1)^n$ is

$$\begin{aligned} y[n] &= (-1)^{n-n_0} + \sum_{l=n-2}^{n} (-1)^l = (-1)^{n-n_0} + (-1)^n(1-1+1) \\ &= (-1)^n(1+(-1)^{n_0}). \end{aligned}$$