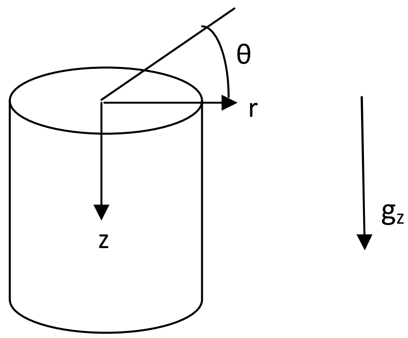
Problem 1.1: Solution

Write the mass, momentum and energy conservation equations for an incompressible, constant property and Newtonian fluid, for the following systems:

a) Downward flow in a vertical pipe mass (continuity)



let $\vec{V} = (u, v, w)$

u: r-component

v: θ-component

w: z-component

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u) + \frac{1}{r}\cancel{\frac{\partial}{\partial \theta}}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r u) + \frac{\partial w}{\partial z} = 0$$

Momentum (Navier-Stokes):

r: $$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \theta}} - \frac{v^2}{r} + w\frac{\partial u}{\partial z}\right) = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}} + \frac{\partial^2 u}{\partial z^2} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \theta}}\right] - \frac{\partial p}{dr} + \cancel{\rho g_r}$$

$$\rightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \theta}} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}} + \frac{\partial^2 u}{\partial z^2} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \theta}}\right] - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

θ: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \theta}} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}} + \frac{\partial^2 v}{\partial z^2} + \cancel{\frac{2}{r^2}\frac{\partial u}{\partial \theta}}\right] - \cancel{\frac{1}{r\rho}\frac{\partial p}{\partial \theta}}$$

One may use physical argument and conclude that for axi-symmetric flow $v=0$ everywhere.

z: $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \theta}} + w\frac{\partial w}{\partial z} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}} + \frac{\partial^2 w}{\partial z^2}\right] - \frac{1}{\rho}\frac{\partial p}{\partial z} + g_z$$

energy:

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \alpha\nabla^2 T + \frac{1}{\rho c_p}\frac{\mathrm{D}P}{\mathrm{D}t} + \frac{\nu}{c_p}\Phi$$

Where $\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{dT}{dt} + \vec{V}\cdot\nabla T$

$$= \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\cancel{\frac{\partial T}{\partial \theta}} + w\frac{\partial T}{\partial z}$$

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}} + \frac{\partial^2 T}{\partial z^2}$$

$$\frac{\mathrm{D}P}{\mathrm{D}t} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} + \frac{v}{r}\cancel{\frac{\partial p}{\partial \theta}} + w\frac{\partial p}{\partial z}$$

$$\Phi = 2\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r}\cancel{\frac{\partial v}{\partial \theta}} + \frac{u}{r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$+\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\cancel{\frac{\partial u}{\partial \theta}}\right]^2 + \left[\frac{1}{r}\cancel{\frac{\partial w}{\partial \theta}} + \frac{\partial v}{\partial z}\right]^2$$

$$+\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right]^2 - \frac{2}{3}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\cancel{\frac{\partial v}{\partial \theta}} + \frac{\partial w}{\partial z}\right]^2$$

b) Downward flow in the above vertical pipe, where the hydrodynamic entrance effects have all disappeared (assume fully developed flow → $\frac{\mathrm{d}\vec{V}}{\mathrm{dz}} = 0$)

Mass:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} = 0$$

Note that one may argue that for axi-symmetric flow *v=0.*

Momentum:

r: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\cancel{\frac{\partial u}{\partial \theta}} - \frac{v^2}{r} + \cancel{w\frac{\partial u}{\partial z}} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru)\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 u}{\partial \theta^2}} + \cancel{\frac{\partial^2 u}{\partial z^2}} - \frac{2}{r^2}\cancel{\frac{\partial v}{\partial \theta}}\right] - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\rightarrow \cancel{\frac{\partial u}{\partial t}} + u\cancel{\frac{\partial u}{\partial r}} + \frac{v}{r}\cancel{\frac{\partial u}{\partial \theta}} - \cancel{\frac{v^2}{r}} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru)\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 u}{\partial \theta^2}} - \frac{2}{r^2}\cancel{\frac{\partial v}{\partial \theta}}\right] - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

Θ: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right] - \frac{1}{r\rho}\frac{\partial p}{\partial \theta}$$

(All the terms in this equation will vanish, leading to $\frac{\partial p}{\partial \theta} = 0$, which implies that P does not depend on θ.)

z: $\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] - \frac{1}{\rho}\frac{\partial p}{\partial z} + g_z$

The only terms that are finite in this equation are:

$$\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] - \frac{1}{\rho}\frac{\partial p}{\partial z} + g_z = 0$$

energy:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta} + w\frac{\partial T}{\partial z} = \alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial t^2}\right] + \frac{1}{\rho c_p}\left[\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} + \frac{v}{r}\frac{\partial p}{\partial \theta} + w\frac{\partial p}{\partial z}\right]$$

$$+2\mu\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)^2\right] + \mu\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}\right]^2 + \left[\frac{1}{r}\frac{\partial w}{\partial \theta}\right]^2$$

$$+\mu\left[\frac{\partial w}{\partial r}\right]^2 - \frac{2}{3}\mu\left[\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta}\right]^2$$

After neglecting terms that are zero, we get:

$$\frac{\partial T}{\partial t} + w\frac{\partial T}{\partial z} = \alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial t^2}\right] + \frac{1}{\rho c_p}\left[w\frac{\partial p}{\partial z}\right] + \mu\left[\frac{\partial w}{\partial r}\right]^2$$

c) Repeat part b assuming axi-symmetric flow and assuming the hydrodynamic and thermal entrance effects have all disappeared.

Mass:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) = 0 \rightarrow u = 0$$

Momentum:

r: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru)\right)\right] - \frac{1}{\rho}\frac{\partial p}{\partial r}$

Θ: $\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right)\right]$

z: $\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] - \frac{1}{\rho}\frac{\partial p}{\partial z} + g_z$

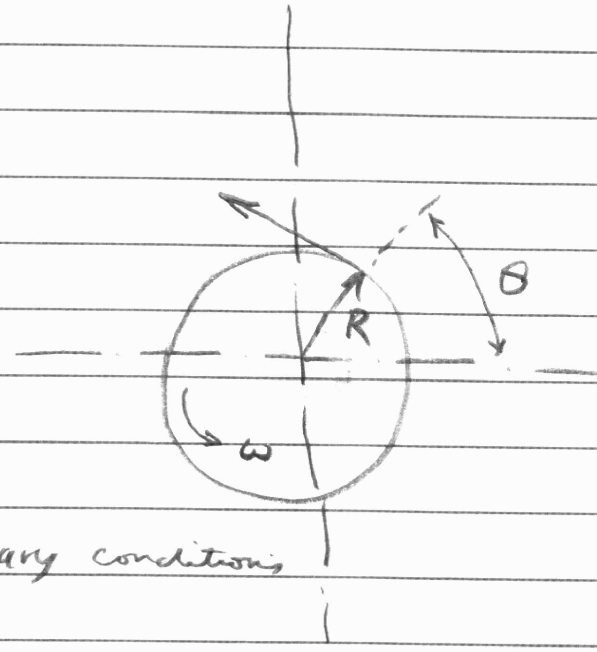
Note that in these equations $u = 0, v = 0$ everywhere. Terms involving u and v all disappear.

Energy:

$$w\frac{\partial T}{\partial z} = \alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] + \frac{1}{\rho c_p}\left[w\frac{\partial p}{\partial z}\right] + \mu\left[\frac{\partial w}{\partial r}\right]^2$$

Notes:

1. In axi-symmetric flow all θ-dependent terms must be dropped.
2. In the last equation the term $\frac{\partial T}{\partial z}$ must follow the forthcoming Eq. (4.2.22) for thermally developed assumption to hold. Also, fully-developed and thermally-developed flows imply steady-state, therefore time-dependent terms should also disappear. However, in practice slow transients (quasi-steady state) flows can behave approximately like fully-developed and thermally-developed flows.



The boundary conditions are

at $r = R$

$$\begin{cases} u_r = 0 \\ u_\theta = R\omega \\ T = T_s \end{cases}$$

at $r \to \infty$

$u_r = 0$

$P = P_\infty$

$T = T_\infty$

The mass and momentum equations are (B.1), and (B.2) ~ (B.4). The momentum equations, neglecting gravity, will give

$$P = P(r) \qquad (1)$$

$$\rho\left(u_r \frac{du_r}{dr} - \frac{u_\theta^2}{r}\right) = -\frac{\partial P}{\partial r} + \mu\left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r u_r)\right]\right\} \qquad (2)$$

$$\rho\left(u_r \frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r}\right) = \mu\left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r u_\theta)\right]\right\} \qquad (3)$$

The energy equation will be based on Eq. (C.1) of Appendix C, which reduces to

$$\rho C_p \left(u_r \frac{dT}{dr} \right) = k \frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \mu \phi$$

$$\phi = 2 \left[\left(\frac{\partial u_r}{\partial r} \right)^2 + \left(\frac{u_r}{r} \right)^2 \right] + \left[r \frac{d}{dr} \left(\frac{u_\theta}{r} \right) \right]^2 - \frac{2}{3} \left[\frac{1}{r} \frac{d}{dr} (r u_r) \right]^2$$

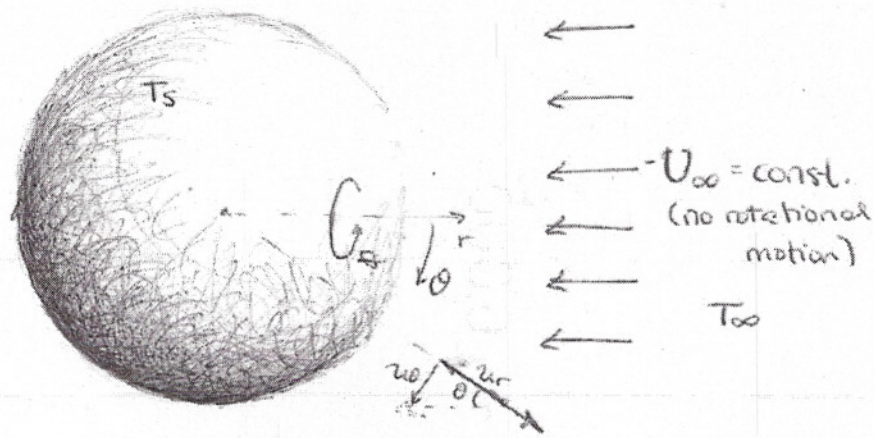
Note: Based on the physics of the problem, one may argue that we must have $u_r = 0$ everywhere. All the terms that include u_r or its derivatives as a multiplier will then be dropped from the above equations.

Problem 1.3

3 Problem

$T_s \neq T_\infty$

quiescent, ∞-large laminar fluid



axi-symmetric about $r \to u_\phi, \frac{\partial \cdot}{\partial \phi} = 0$

const. properties $\to \rho, \mu, k =$ const.

Newtonian / incomp $\to$ appendix eqn's

1) conservation & BC's (spherical coord.)

BC's

@ $r = R \to u_r = u_\theta = 0$ (no slip)

$T = T_s$

@ $r = \infty \to u_r = -U_\infty \cos\theta$

$u_\theta = U_\infty \sin\theta$

$T = T_\infty$

$$\sigma_\theta = 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \cancel{\frac{u_r}{r}}^{\,0}\right)\Big|_{r=R} + \lambda \nabla\cdot\vec{U}$$

$$\sigma_\phi = 2\mu\left(\frac{\cancel{u_r}^{\,0} + u_\theta\cot\theta}{r}\right)\Big|_{R=R} + \lambda\nabla\cdot\vec{U}$$

$$q_r'' = -k\frac{\partial T}{\partial r}\Big|_{r=R} \qquad q_\phi'' = 0$$

$$q_\theta'' = -k\frac{1}{r}\frac{\partial T}{\partial \theta}\Big|_{r=R}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\Big|_{r=R} + \frac{1}{r}\cancel{\frac{\partial u_r}{\partial \theta}}^{\,0}\Big|_{r=R}\right]$$

$$\sigma_r = \mu 2\frac{\partial u_r}{\partial r}\Big|_{r=R} + \left[\frac{\lambda}{r^2}\frac{\partial}{\partial r}(r^2 u_r)\Big|_{r=R} + \frac{\lambda}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)\Big|_{r=R}\right]$$

$= \lambda\nabla\cdot\vec{U}$

$\tau_{\phi\theta} = \tau_{\theta\phi} = 0 \quad \tau_{r\phi} = \tau_{\phi r} = 0$ ~~[illegible]~~

mass

$$\cancel{\frac{\partial\rho}{\partial t}} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho u_\theta\sin\theta) + \frac{1}{r\sin\theta}\cancel{\frac{\partial}{\partial\phi}}(\rho \cancel{u_\phi}) = 0$$

$$\longrightarrow \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{\rho}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) = 0$$

$$\longrightarrow \frac{1}{r}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) = 0$$

r:
$$\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\cancel{\frac{u_\phi}{r\sin\theta}}\cancel{\frac{\partial u_r}{\partial \phi}}-\frac{u_\theta^2-\cancel{u_\phi^2}}{r}\right]=-\frac{\partial P}{\partial r}$$
$$+\mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2u_r)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\cancel{\frac{\partial^2 u_r}{\partial\phi^2}}\right]+\rho g_r$$

$$\rightarrow \rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}\right]=-\frac{\partial P}{\partial r}+\mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2u_r)\right.$$
$$\left.+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)\right]+\rho g_r$$

θ:
$$\rho\left[\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_ru_\theta}{r}\right]=-\frac{1}{r}\frac{\partial P}{\partial\theta}+\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)\right.$$
$$\left.+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)\right]+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right]+\rho g_\theta$$

φ: $0=\rho g_\phi$

energy

$$\rho C_p\left[\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial\theta}\right]=k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)\right]+\mu\Phi$$

where

$$\Phi=2\left[\left(\frac{\partial u_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)^2+\left(\frac{u_r+u_\theta\cot\theta}{r}\right)^2\right]+\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right.$$
$$\left.-\frac{2}{3}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)\right]\right]$$

b). steady state ($\frac{\partial}{\partial t}=0$)

mass same as (a)

momentum

$$\rho\left[u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r}\right]=-\frac{\partial P}{\partial r}+\mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2u_r)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)\right]$$
$$+g_r\rho$$

θ:

$$\rho\left[u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r}\right] = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta)\right) + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right] + \rho g_\theta$$

ϕ: $\quad 0 = \rho g_\phi$

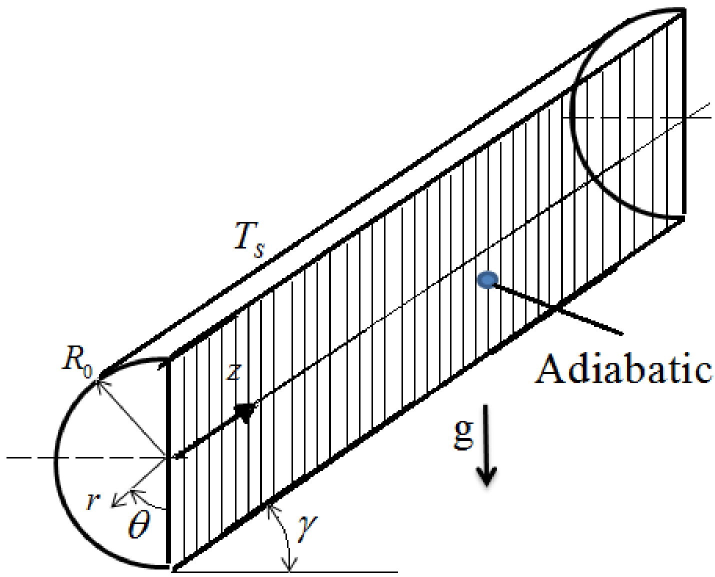
energy

$$\rho C_p\left[u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta}\right] = k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right)\right] + \mu\Phi$$

$\Phi \rightarrow$ same as (a)

Problem 1.3 2nd Ed.. Consider the flow passage shown in the figure. The flow passage has a semi-circular cross-section, and its axis makes an angle of inclination equal to γ with respect to the horizontal plane. The curved surface of the flow passage is at a constant temperature T_s, and the flat part of the flow passage surface is adiabatic. An incompressible, constant-property fluid flows through the flow passage. Using polar cylindrical coordinates, write the complete momentum and energy conservation equations, along with boundary conditions representing the surface of the flow channel.

Hint: Start from the equations in Appendices B and C.



Solution:

Equations of Motion for $\mu = constant$

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}\right)=-\frac{\partial P}{\partial r}+\mu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right]+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right\}+\rho g\cos\gamma\cos\theta$$

(B.2)

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right]+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right\}-\rho g\cos\gamma\sin\theta$$

(B.3)

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right\}-\rho g\sin\gamma$$

(B.4)

Boundary Conditions:

$u_r=0; u_\theta=0; u_z=0 \ \ at \ r=R_0$
$u_r=0; u_\theta=0; u_z=0 \ \ at \ \theta=0$
$u_r=0; u_\theta=0; u_z=0 \ \ at \ \theta=\pi$

Equations of Motion for *constant properties*

$$\rho C_P\left(\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial \theta}+u_z\frac{\partial T}{\partial z}\right)=k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right]+\mu\Phi \qquad \text{(C-1)}$$

$$\Phi=2\left[\left(\frac{\partial u_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)^2+\left(\frac{\partial u_z}{\partial z}\right)^2\right]+\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]^2+\left[\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right]^2+$$
$$\left[\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right]^2-\frac{2}{3}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right]^2$$

(C-2)

Boundary Conditions:

Boundary Conditions:

$T=T_s \ at \ r=R_0$

$\frac{\partial T}{\partial r}=0 \ \ at \ \theta=0$

$\frac{\partial T}{\partial r}=0 \ \ at \ \ \theta=\pi$

Problem 1.4

$$\nabla \cdot \underline{\underline{\tau}} = \frac{\partial \tau_{jk}}{\partial x_j} \vec{e}_k$$

For $j=1$, which corresponds to the x coordinate, we will have:
(Note that in general $\tau_{ij} = \tau_{ji}$)

$$(\nabla \cdot \underline{\underline{\tau}})_x = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \vec{e}_y + \frac{\partial \tau_{xz}}{\partial z} \right) \vec{e}_x$$

$$= \left[\frac{\partial}{\partial x} \left(2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \vec{U} \right) + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] \right] \vec{e}_x \qquad (1)$$

Likewise

$$(\nabla \cdot \underline{\underline{\tau}})_y = \frac{\partial \tau_{yx}}{\partial x} \vec{e}_y + \frac{\partial \tau_{yy}}{\partial y} \vec{e}_y + \frac{\partial \tau_{yz}}{\partial z} \vec{e}_y$$

$$= \left\{ \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot \vec{U} \right] + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \right\} \vec{e}_y \qquad (2)$$

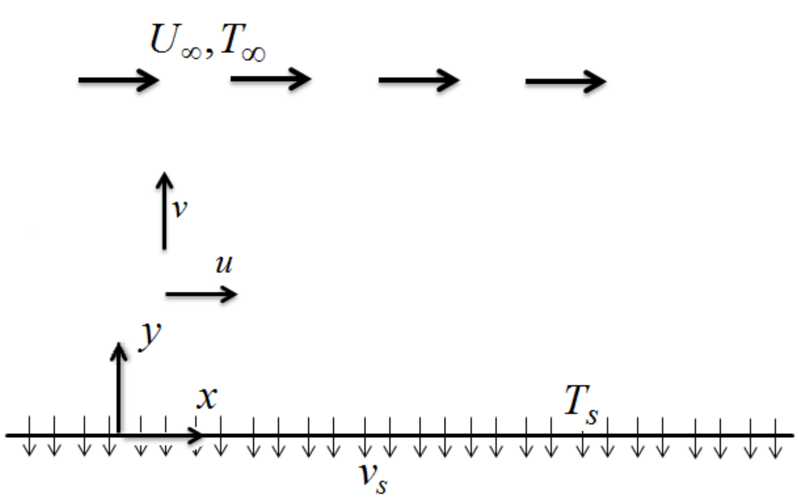
$$(\nabla \cdot \underline{\underline{\tau}})_z = \frac{\partial \tau_{zx}}{\partial x} \vec{e}_z + \frac{\partial \tau_{zy}}{\partial y} \vec{e}_z + \frac{\partial \tau_{zz}}{\partial z} \vec{e}_z$$

$$= \left\{ \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot \vec{U} \right] \right\} \vec{e}_z \qquad (3)$$

Problem 1.4 2nd Ed.. Consider the flow of an incompressible and constant-property fluid parallel to an infinitely large flat surface. The flow field is two-dimensional. At wall suction of fluid takes place at a uniform rate such that the fluid has a velocity equal to v_s at $y=0$. The surface temperature is uniform at T_s. Far from the surface the fluid has a velocity of U_∞ and a temperature of T_∞.

1. Prove that for all parameters to be independent of x we must have $v=constant=v_s$ everywhere.
2. Simplify and solve the momentum equation in x direction, and derive an expression for the velocity profile $u(y)$.
3. Assuming that viscous dissipation is negligible, simplify and solve the energy equation and derive an expression for the temperature profile.
4. Assume that the fluid is atmospheric air, and $v_s = 0.005\ m/s; T_\infty = 300\ K; T_s = 320\ K$

   Find the distance from the wall where $u/U_\infty = 0.99$ and $(T - T_s)/(T_\infty - T_s) = 0.99$



Problem 1.yyy

First consider the continuity equation

$$\cancelto{0}{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$\Rightarrow \quad \frac{\partial v}{\partial y} = \text{const}$$

$$v = \text{const.} = v_s \qquad (2)$$

Therefore v is everywhere equal to v_s.

Momentum equation in x-direction:

$$u \cancelto{0}{\frac{\partial u}{\partial x}} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \quad v_s \frac{du}{dy} = \nu \frac{d^2 u}{dy^2} \qquad (3)$$

This is a second-order ode. The general solution is

$$u = C_1 e^{\frac{v_s}{\nu} y} + C_2 \qquad (4)$$

at $\cancel{x}y = 0$ we have $u = 0 \Rightarrow C_1 + C_2 = 0$

$$\Rightarrow \quad u = C_2 \left[1 - e^{v_s y/\nu} \right] \qquad (5)$$

at $y \to \infty$ we have $u \to U_\infty$. Because $v_s < 0$, then $\lim\limits_{y \to \infty} u = C_2 = U_\infty$

$$\Rightarrow \quad u = U_\infty \left[1 - e^{v_s y/\nu} \right] \qquad (6)$$

Now consider the thermal energy equation

$$\cancelto{0}{u \frac{\partial T}{\partial x}} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (7)$$

$$v_s \frac{dT}{dy} = \alpha \frac{d^2 T}{dy^2} \quad (8)$$

Define

$$\theta = \frac{T - T_s}{T_\infty - T_s} \quad (9)$$

$$\Rightarrow \quad \frac{d^2\theta}{dy^2} = \frac{v_s}{\alpha} \frac{d\theta}{dy} \quad (10)$$

$$\theta = C_3 e^{\frac{v_s}{\alpha} y} + C_4$$

at $y = 0 \quad \theta = 0 \quad (11)$

at $y \to \infty, \quad \theta = 1 \quad (12)$

$$\Rightarrow \quad \theta = 1 - e^{v_s y/\alpha} \quad (13)$$

Note that the above analysis is correct when v_s is negative (suction). If $v_s > 0$ is defined for suction, then Eqs. (6) & (13) become

$$u/U_\infty = 1 - e^{-v_s y/\nu} \quad (6')$$

$$\theta = 1 - e^{-v_s y/\alpha} \quad (13')$$

For air at $T_f = \frac{1}{2}(T_s + T_\infty) = 310\,K$

$\nu = 1.67 \times 10^{-5}\ m^2/s$

$\alpha = 2.305 \times 10^{-5}\ m^2/s$

For $v_s = -0.005\ m/s$:

From Eq. (6'), $U/U_\infty = 0.99$ at $y = 1.52 \times 10^{-2}\ m$

From Eq. (13'), $\theta = 0.99$ at $y = 0.0212\ m$

Problem 1.5

$$I = \underline{\underline{\tau}} : (\nabla \vec{U}) = \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

This can be cast as:

$$I = \tau_{xx} \frac{\partial u}{\partial x} + \tau_{xy} \frac{\partial u}{\partial y} + \tau_{xz} \frac{\partial u}{\partial z} + \tau_{yx} \frac{\partial v}{\partial x} + \tau_{yy} \frac{\partial v}{\partial y} + \tau_{yz} \frac{\partial v}{\partial z} + \tau_{zx} \frac{\partial w}{\partial x} + \tau_{zy} \frac{\partial w}{\partial y} + \tau_{zz} \frac{\partial w}{\partial z}$$

Now:

$$\tau_{xx} \frac{\partial u}{\partial x} = \left[-\left(P + \frac{2}{3} \mu \nabla \cdot \vec{U}\right) + 2\mu \frac{\partial u}{\partial x} \right] \frac{\partial u}{\partial x}$$

$$\tau_{yy} \frac{\partial v}{\partial y} = \left[-\left(P + \frac{2}{3} \mu \nabla \cdot \vec{U}\right) + 2\mu \frac{\partial v}{\partial y} \right] \frac{\partial v}{\partial y}$$

$$\tau_{zz} \frac{\partial w}{\partial z} = \left[-\left(P + \frac{2}{3} \mu \nabla \cdot \vec{U}\right) + 2\mu \frac{\partial w}{\partial z} \right] \frac{\partial w}{\partial z}$$

$$\Rightarrow \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}$$

$$= -p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \left(-\frac{2}{3}\mu\,\nabla\cdot\vec{U}\right)\left(\nabla\cdot\vec{U}\right)$$

$$+ 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \qquad (1)$$

$$\tau_{xy}\frac{\partial u}{\partial y} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial y}$$

$$= \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right]$$

$$\tau_{yx}\frac{\partial v}{\partial x} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x} = \mu\left[\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\Rightarrow \tau_{xy}\frac{\partial u}{\partial y} + \tau_{yx}\frac{\partial v}{\partial x} = \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] = \mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^2 \qquad (2)$$

Likewise

$$\tau_{xz}\frac{\partial u}{\partial z} + \tau_{zx}\frac{\partial w}{\partial x} = \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]^2 \qquad (3)$$

$$\tau_{yz}\frac{\partial v}{\partial z} + \tau_{zy}\frac{\partial w}{\partial y} = \mu\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right]^2 \qquad (4)$$

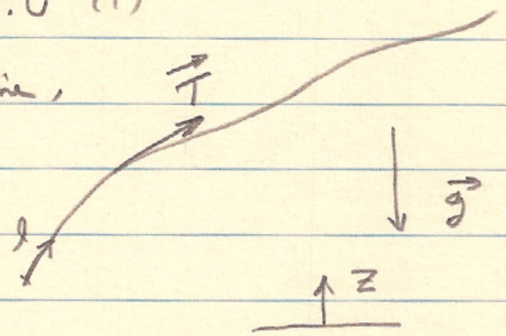
The result we are looking for can be found by summing up the right hand sides of Eqs. (1), (2), (3), and (4).

Problem 1.6

The mechanical energy equation for an inviscid fluid will be

$$\rho \vec{U} \frac{D}{Dt}\vec{U} = -U \nabla P + \rho \vec{g} \cdot \vec{U} \quad (1)$$

In view of the streamline, this equation can be rewritten as:



$$\rho u^2 \frac{du}{dl} = -u\, dP/dl - \rho g\, dz/dl\, u \quad (2)$$

where steady-state has been assumed.

Divide through by u, and manipulate the left side to get

$$\rho \frac{d}{dl}\frac{u^2}{2} = -dP/dl - \rho g \frac{dz}{dl}$$

$$\rho = \text{const.} \Rightarrow \frac{d}{dl}\left(\frac{u^2}{2} + \frac{P}{\rho} + gz\right) = 0$$

$$\Rightarrow \frac{u^2}{2} + \frac{P}{\rho} + gz = \text{const.} \quad (3)$$

Part c

The thermal energy equation, when viscosity is set equal to zero, simply gives

$$\frac{De}{Dt} = 0$$

In steady-state this would give, along a streamline.

$$u \frac{de}{dl} = 0$$

We can add this equation to Eq. (2), and follow from that point. The result, instead of Eq. (3), would then be

$$e + \frac{P}{\rho} + \frac{1}{2}u^2 + gz = \text{const.}$$

Problem 1.7

The Navier-Stokes equation for an inviscid flow is:

$$\rho \frac{D\vec{U}}{Dt} = -\nabla P + \vec{F}_{body} \qquad (1)$$

The left side of this equation can be recast as:

$$\rho \frac{D\vec{U}}{Dt} = \rho \left[\frac{\partial \vec{U}}{\partial t} + \nabla \left(\frac{1}{2} U^2 \right) - \vec{U} \times \left(\nabla \times \vec{U} \right) \right] \qquad (2)$$

Therefore:

$$\rho \left[\frac{\partial \vec{U}}{\partial t} + \nabla \left(\frac{1}{2} U^2 \right) \right] = \rho \vec{U} \times \left(\nabla \times \vec{U} \right) - \nabla P + \rho \vec{g} \qquad (3)$$

For irrotational flow, $\nabla \times \vec{U} = 0$. Furthermore, for irrotational flow a velocity potential ϕ can be defined so that,

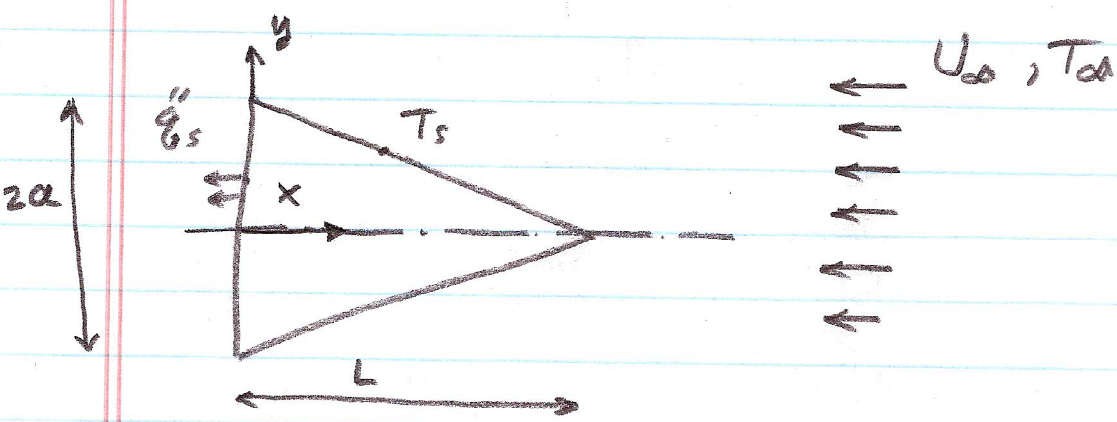
$$\vec{U} = \nabla \phi \qquad (4)$$

Combining *Eqs*. (3) and (4), and integrating between two arbitrary points *i* and *j*, gives:

$$\rho \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} U^2 \right]_i^j = P_i - P_j - \rho g \left(z_j - z_i \right) \qquad (5)$$

Problem Chp1.2D Flow Part a Rectangular Object

HW1, P.3, 2015



Assume the fluid is incompressible & constant-property. Also, assume that the process is steady-state and the flow field is 2D. Without symmetry:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g_x \qquad (2)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g_y \qquad (3)$$

$$\rho C_p\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\Phi \qquad (4)$$

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \qquad (5)$$

The equations for upper and lower surfaces are,

$$y = a(1 - x/L) \qquad (6)$$

$$y = -a(1 - x/L) \qquad (7)$$

Therefore, the boundary conditions are

$u = 0$, $v = 0$, $T = T_s$

$$\left\{\begin{array}{l} y = a(1 - x/L) \\ x \le L \end{array}\right. \qquad (8-a)$$

$$\left\{\begin{array}{l} y = -a(1 - x/L) \\ x \le L \end{array}\right. \qquad (8-b)$$

Also,

$$k\frac{\partial T}{\partial x} = q_s'' \qquad (9)$$

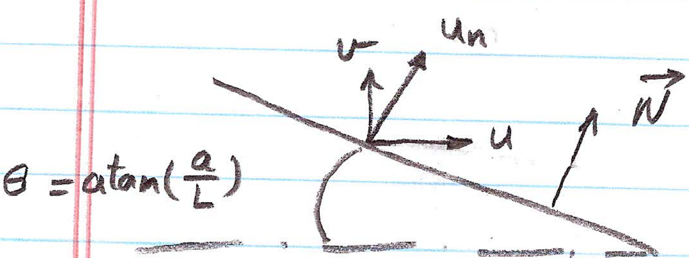
$$u = v = 0$$

at

$$x = 0$$

$$-a \le y \le a$$

Note that if the flow is inviscid, then $\mu = 0$, and the no-slip boundary condition does not apply, and the hydrodynamic b.c.'s will be



$$\vec{U} \cdot \vec{N} = 0$$

or

$$u \sin\theta + v \cos\theta = 0 \qquad (10)$$

This equation ensures that $u_n = 0$

The hydrodynamic b.c. on the back side will be

$v = 0$ at $x = 0$ and $-a < y < +a$

The energy boundary conditions remain unchanged.

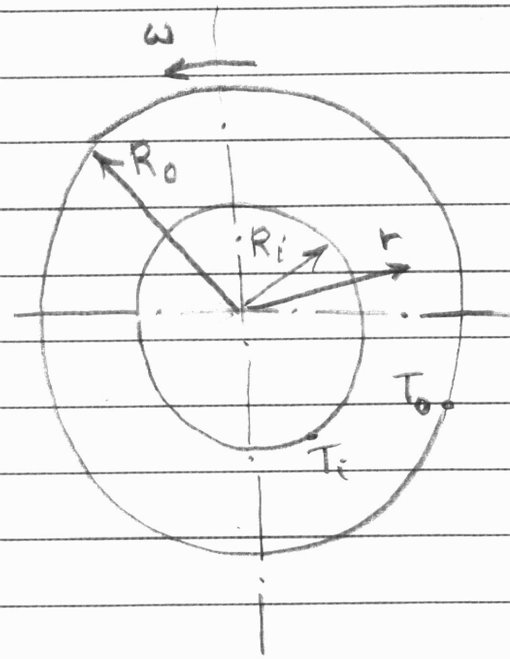
When the flow field is symmetric with respect to the $y=0$ plane, then Eqs. (1–5) all apply, except that we now must have $g_y = 0$, otherwise symmetry is mathematically not possible.

In fact, with dropping g_y term, all the above equations will apply and symmetry will automatically apply. In this case Eqs. (7) and (8-b) can be dropped, and instead we can write

$$\frac{\partial T}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0 \qquad \text{at } y=0$$

The boundary conditions on the back face will also need to be applied for $0 \le x \le a$ only.

Problem 1.8



Define
u, v, w = components of velocity along r, θ and z coordinates.

The mass and momentum conservation equations will be similar to Eqs. (B.1), and (B.2) ~ (B.4), respectively. Because there is no dependence on θ or z, we will get

$$B.1 \Rightarrow -\rho \frac{v^2}{r} = -\partial P/\partial r \quad (1)$$

$$B.2 \Rightarrow 0 = \frac{d}{dr}\left[{}^1\!/_r \frac{d}{dr}(rv)\right] \quad (2)$$

$$B.3 \Rightarrow 0 = -\frac{\partial P}{\partial z} \quad (3)$$

Note that if the cylinder was vertical and gravity was considered, then

$$B.3 \Rightarrow 0 = -\partial P/dz - \rho g \quad (4)$$

The energy equation, in its general form, is Eq. (C.1) in Appendix C. For the axi-symmetric and steady-state conditions it reduces to

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\, dT/dr\right) + \mu\left[r\frac{d}{dr}\left(\frac{v}{r}\right)\right]^2 \quad (5)$$

With no viscous dissipation, the second term drops out.

The boundary conditions for the above equations are

at $r = R_i$ we have

$v = 0$

$T = T_i$

at $r = R_o$ we have

$v = R_o \omega$

$T = T_o$

For P, we note that $P = f(r)$ only, and the solution of the momentum equations will provide the distribution of P as a function of r, provided that P at one point (e.g., at R_i) is known.

The solution of the momentum equation in r direction [Eq. (2)], will give:

$$v(r) = R_o \omega \frac{\frac{r}{R_i} - \frac{R_i}{r}}{\left(\frac{R_o}{R_i} - \frac{R_i}{R_o}\right)} \qquad (6)$$

The solution of the energy equation is [see Bird et al., 2002, p. 343]:

$$\frac{T-T_i}{T_o-T_i} = \left[1-\frac{\ln(r/R_o)}{\ln\eta}\right] + \frac{\mu\omega^2 R_o^2}{k(T_o-T_i)}\cdot\frac{\eta^4}{(1-\eta^2)^2}\cdot$$

$$\left\{\left[1-(R_o/r)^2\right] - \left(1-1/\eta^2\right)\frac{\ln(r/R_o)}{\ln\eta}\right\} \qquad (7)$$

where

$$\eta = R_i/R_o$$

Problem 2, Homework Set 1, 2016

a)

Mass:

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta} u_\theta + \frac{\partial}{\partial x} u_x = 0 \qquad (1)$$

Momentum: Eqs. (B.2), (B.3), (B.4) and (C1), with the following changes apply

- Drop time-dependent
- replace z with x (difference in notation)

Boundary conditions

For simplicity assume that the center of coordinates is at the apix of the cone. then

for $0 \le x \le L$, at $r = \frac{R}{L}x$; and for $0 < \theta \le 2\pi$:

$$T = T_s\,,\; u_r = u_\theta = u_x = 0$$

For $x = L$, at $r \le R$ and $0 < \theta \le 2\pi$

$$u_r = u_\theta = u_x = 0,\; -k\frac{\partial T}{\partial x} = q_s''$$

b) Eq. (1) applies.

Momentum

$$\rho\left[u_r \frac{\partial u_r}{\partial r} + u_\theta \frac{1}{r}\frac{\partial u_r}{\partial \theta} + u_x \frac{\partial u_r}{\partial x} - \frac{u_\theta^2}{r}\right]$$

$$= -\frac{\partial P}{\partial r} + \rho g_r$$

$$\rho\left[u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_x \frac{\partial u_\theta}{\partial x} + \frac{u_r u_\theta}{r}\right] = -\frac{1}{r}\frac{\partial P}{\partial r} + \rho$$

$$\rho\left[u_r \frac{\partial u_x}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_x}{\partial \theta} + u_x \frac{\partial u_x}{\partial x}\right] = -\frac{\partial P}{\partial x} + \rho g_x$$

Boundary conditions

Define $\alpha = \tan^{-1}\frac{R}{L}$. Then

for $0 \leq x < L$; at $r = \frac{R}{L}x$ and $0 < \theta \leq 2\pi$:

$-u_x \sin\alpha + u_r \cos\alpha = 0$

(This is the velocity component that is perpendicular to the surface)

$T = T_s$

for $x = L$; $r \leq R$, $0 \leq \theta \leq 2\pi$:

$u_x = 0$; $-k\frac{\partial T}{\partial x} = q_s''$

c) For axi-symmetric flow:

1. All θ-dependent terms disappear and although in many cases $u_\theta = 0$
2. $g_r = g_\theta = 0$ must apply.

The viscous equations will be

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial}{\partial x}(u_x) = 0$$

$$\left[u_r \frac{\partial u_r}{\partial r} + u_x \frac{\partial u_r}{\partial x}\right] = -\frac{\partial P}{\partial r} + \mu \left\{ \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] + \frac{\partial^2 u_r}{\partial x^2} \right\}$$

$$\rho\left[u_r \frac{\partial u_x}{\partial r} + u_x \frac{\partial u_x}{\partial x}\right] = -\frac{\partial P}{\partial x} + \mu \left\{ \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_x}{\partial r}\right) + \frac{\partial^2 u_x}{\partial x^2} \right\} + \rho g_x$$

Boundary Conditions:

For $0 \le x \le L$; at $r = R\frac{x}{L}$;

$u_r = u_x = 0$; $T = T_s$

For $x = L$; at $r \le R$

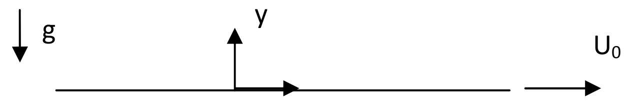
$u_r = u_x = 0$, $-k\frac{\partial T}{\partial x} = q_s''$

transient and 3D

d) The Navier-Stokes and energy conservation equations apply to laminar and turbulent flows both. For turbulent flow, however, they can be solved only by direct numerical simulation (DNS) techniques. Simpler and tractable forms can be derived by averaging to mask fluctuations.

Problem 1.9: Solution

a)



Assume incompressible, constant-property fluid, and assume the plate is infinitely large.

The x-momentum equation will them be, assuming laminar flow and negligible viscous dissipation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{dP}{dx}+\nu\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

By virtue of being infinitely large, no x-dependent terms will occur. Also, $v=0$, because finite v will not be consistent with the infinitely large plate assumption. We end up with:

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

$u=U_0$ at $y=0$

$u\to 0$ as $y\to\infty$

The solution to this p.d.e. is

$$\frac{u}{U_0}=1-\mathrm{erf}(\eta)$$

$$\eta=\frac{y}{2\sqrt{\nu t}} \qquad (3)$$

Where the error function is defined as:

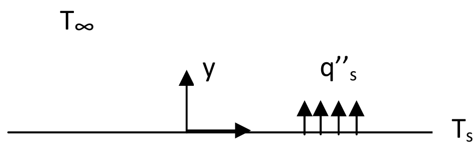
$$\mathrm{erf}(z)=\frac{2}{\pi}\int_0^z e^{-z^2}\,dz \qquad (4)$$

$$\tau_s=\mu\frac{du}{dy}|_{y=0}=\mu\frac{du}{d\eta}\frac{d\eta}{dy}|_{y=0}$$

$$=-\mu U_0\frac{2}{\sqrt{\pi}}e^{-\eta^2}|_{\eta=0}\frac{1}{2\sqrt{\nu t}}=-\mu\frac{U_0}{\sqrt{\pi\nu t}} \qquad (5)$$

The physical problem can thus be simplified as a fluid layer with thickness of $\sqrt{\pi\nu t}$, with linear velocity profile.

b)



For heat diffusion in a stagnant fluid, we have

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{6}$$

Define $\theta = T - T_\infty$, then

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial y^2} \tag{7}$$

$$\text{At } y = 0, \theta = T_s - T_\infty$$

$$\text{at } y \to \infty, \theta \to 0$$

The solution will be

$$\frac{T - T_\infty}{T_s - T_\infty} = 1 - \text{erf}(\eta')\ , \eta' = \frac{y}{2\sqrt{\alpha t}} \tag{8}$$

The wall heat flux will be found by writing,

$$q''_s = -k\frac{dT}{dy}|_{y=0} = -k\frac{d\theta}{d\eta'}\frac{d\eta'}{dy}|_{\eta'=0}$$

$$= +k(T_s - T_\infty)\frac{2}{\sqrt{\pi}}e^{-\eta'^2}|_{\eta'=0}\frac{1}{2\sqrt{\alpha t}} = k\frac{T_s - T_\infty}{\sqrt{\pi \alpha t}} \tag{9}$$

It is as if at any time t we deal with quasi-steady conduction through a slab with a thickness of $\sqrt{\pi \alpha t}$.

c) In this case, assuming that viscous dissipation is negligible, the governing equation and solutions will be exactly as before.

Problem 1.10

The momentum equations in r and z directions will be [see Eqs. (B.2) and (B.4) in Appendix B]:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right)$$

$$= -\frac{\partial P}{\partial r} + \mu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] + \frac{\partial^2 u_r}{\partial z^2}\right\} + \rho g_r \quad (1)$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right)$$

$$= -\frac{\partial P}{\partial z} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right\} + \rho g_z \quad (2)$$

The mechanical energy equation can be found by multiplying Eq. (1) by u_r, multiplying Eq. (2) by u_z, and summing up the resulting two equations

First, let us focus on the left-hand side of the equation we are seeking

$$LHS = \rho \left[u_r \frac{\partial u_r}{\partial t} + u_r^2 \frac{\partial u_r}{\partial r} + u_r u_z \frac{\partial u_r}{\partial z} + u_z \frac{\partial u_z}{\partial t} + u_r u_z \frac{\partial u_z}{\partial r} + u_z^2 \frac{\partial u_z}{\partial z} \right]$$

$$= \rho \left\{ \frac{\partial}{\partial t}\left[\frac{1}{2}(u_r^2 + u_z^2)\right] + u_r \frac{\partial}{\partial r}\left[\frac{1}{2}(u_r^2 + u_z^2)\right] + u_z \frac{\partial}{\partial z}\left[\frac{1}{2}(u_z^2 + u_r^2)\right] \right\} \qquad (3)$$

The right-hand side of the equation will be

$$RHS = -\left(\frac{\partial P}{\partial r} u_r + \frac{\partial P}{\partial z} u_z\right) + \rho (u_r g_r + u_z g_z) + \mu \phi$$

where

$$\phi = 2\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{u_r}{r}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] + \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right]^2 - \frac{2}{3}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_z) + \frac{\partial u_z}{\partial z}\right]^2$$

Problem 1.11

The momentum conservation equations in r and θ directions will be [see Eqs. (B.6) and (B.6) in Appendix B]:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}\right)$$
$$=-\frac{\partial P}{\partial r}+\mu\left\{\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2u_r)+\right.$$
$$\left.\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)\right\}+\rho g_r \qquad (1)$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\theta u_r}{r}\right)$$
$$=-\frac{1}{r}\frac{\partial P}{\partial\theta}+\mu\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)\right.$$
$$\left.+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)\right]+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right\}+\rho g_\theta \qquad (2)$$

We now must multiply Eq. (1) by u_r, multiply Eq. (2) by u_θ, and sum up the resulting two equations.

Let us first derive the left-hand side of the resulting equation.

$$LHS = \rho \left\{ u_r \frac{\partial u_r}{\partial t} + u_r^2 \frac{\partial u_r}{\partial r} + \frac{u_r u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_r u_\theta^2}{r} \right.$$

$$\left. + u_\theta \frac{\partial u_\theta}{\partial t} + u_r u_\theta \frac{\partial u_\theta}{\partial r} + \frac{u_\theta^2}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta^2}{r} \right\}$$

$$= \rho \left\{ \frac{\partial}{\partial t}\left[\frac{1}{2}(u_r^2 + u_\theta^2)\right] + u_r \frac{\partial}{\partial r}\left[\frac{1}{2}(u_r^2 + u_\theta^2)\right] \right.$$

$$\left. + \frac{u_\theta}{r} \frac{\partial}{\partial \theta}\left[\frac{1}{2}(u_r^2 + u_\theta^2)\right] \right\} \qquad (3)$$

The RHS of the resulting equation will be

$$RHS = \rho u_r g_r + \rho u_\theta g_\theta + \mu \Phi \qquad (4)$$

where it can be shown that

$$\Phi = 2\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)^2 + \left(\frac{u_r + u_\theta \cot\theta}{r}\right)^2\right]$$

$$+ \left[r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]^2 - \frac{2}{3}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r)\right.$$

$$\left. + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta)\right]^2$$

Problem 1.12

The thermal energy equation is:

$$\rho\frac{D\boldsymbol{h}}{Dt}=\nabla\cdot\left(k\nabla T\right)+\frac{DP}{Dt}+\mu\Phi \qquad (1)$$

Also, from thermodynamics, for a pure substance

$$\boldsymbol{h}=\boldsymbol{h}\left(T,P\right)$$

$$\left(\frac{\partial \boldsymbol{h}}{\partial P}\right)_{T}=\left[v-T\left(\frac{\partial v}{\partial T}\right)_{P}\right]$$

Therefore

$$d\boldsymbol{h}=\left(\frac{\partial \boldsymbol{h}}{\partial T}\right)_{T}dT+\left(\frac{\partial \boldsymbol{h}}{\partial P}\right)_{P}dP=C_P dT+\left[v-T\left(\frac{\partial v}{\partial T}\right)_{P}\right]dP$$

Equation (1) can then be cast as

$$\rho C_P\frac{DT}{Dt}=\frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{P}\frac{DP}{Dt}+\nabla\cdot\left(k\nabla T\right)+\mu\Phi$$

$$\rho v=1$$

$$\left(\frac{\partial v}{\partial T}\right)_{P}=\left(\frac{\partial\left(\frac{1}{\rho}\right)}{\partial T}\right)_{P}=-\frac{T}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_{P}$$

$$\frac{\partial \ln\rho}{\partial \ln T}=\frac{\partial \ln\rho}{\partial \rho}\frac{\partial \rho}{\partial T}\frac{\partial T}{\partial \ln T}=\frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)$$

Because $\rho v=1$, then we get

$$\rho C_P\frac{DT}{Dt}=\rho T\left(\frac{\partial v}{\partial T}\right)_{P}\frac{DP}{Dt}+\nabla\cdot\left(k\nabla T\right)+\mu\Phi \qquad (2)$$

Also, since

$$\left(\frac{\partial v}{\partial T}\right)_P = \left(\frac{\partial\left(\frac{1}{\rho}\right)}{\partial T}\right)_P = -\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P$$

Equation (2) can then be case as,

$$\rho C_P \frac{DT}{Dt} = \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \frac{DP}{Dt} + \nabla\cdot\left(k\nabla T\right) + \mu\Phi \quad (3)$$

Furthermore,

$$\frac{\partial \ln \rho}{\partial \ln T} = \frac{\partial \ln \rho}{\partial \rho}\frac{\partial \rho}{\partial T}\frac{\partial T}{\partial \ln T} = \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right) \quad (4)$$

Comparing Eq. (4) with the first term on the right hand side of Eq. (3), the problem has been solved.

Problem 1.13: Solution

a) $V_{\text{cv}}\dot{\sigma}'''_{\text{gen}} = \frac{d}{dt}\iiint_{V_{\text{cv}}} \rho s dV + \sum_{\text{Exit}} \dot{m}_{\text{i}} s_{\text{i}} - \sum_{\text{Inlet}} \dot{m}_{\text{j}} s_{\text{j}} - V_{\text{cv}} \frac{\dot{q}}{T_{\text{cv}}} - \sum_{N_A} \frac{q''_l A_l}{T_l}$

N_A= Number of boundary areas through which heat transfer takes place

b) Use the divergence theorem to write, for a control volume,

$$\iint_{A_{\text{cv}}} \rho s(\vec{U} \cdot \vec{N}) dA = \iiint_{V_{\text{cv}}} \nabla \cdot (\rho \vec{U} s) dV$$

$$\iint_{A_{\text{cv}}} \frac{\vec{q}'' \cdot \vec{N}}{T} dA = \iiint_{V_{\text{cv}}} \nabla \cdot \left(\frac{\vec{q}''}{T}\right) dV$$

Substituting in the differential equation, and taking the limit of $V_{\text{cv}} \to 0$, one gets

$$\dot{\sigma}'''_{\text{gen}} = \frac{\partial}{\partial t}(\rho s) + \nabla \cdot (\rho \vec{U} s) + \nabla \cdot \left(\frac{\vec{q}''}{T}\right) - \frac{\dot{q}}{T}$$

This equation can be rewritten as:

$$\dot{\sigma}'''_{\text{gen}} = s\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U})\right] + \rho\left[\frac{\partial s}{\partial t} + \vec{U} \cdot \nabla s\right] + \nabla \cdot \left(\frac{\vec{q}''}{T}\right) - \frac{\dot{q}}{T}$$

The first term on the right side vanishes because of the mass conservation principle. We then get:

$$\dot{\sigma}'''_{\text{gen}} = \rho \frac{Ds}{Dt} + \nabla \cdot \left(\frac{\vec{q}''}{T}\right) - \frac{\dot{q}}{T}$$

```
"Problem 1.16"
P=1.e5
T=300
M_L=18
M_CO2=44
m_CO2_L=25e-6
M_air=29
"Get mole fraction of CO2 in water"
X_CO2_L=(m_CO2_L/M_CO2)*M_L
"Air and water are at equilibrium.  Therefore Henry's laww will determine the CO2 concentration in air."
"From Appendix I:"
C_He=1710e5
"Now:"
X_CO2_L*C_He=X_CO2_G*P
m_CO2_G=(X_CO2_G*M_CO2)/((X_CO2_G*M_CO2)+(1-X_CO2_G)*M_air)
```

Problem 1.16

$P = 100000$

$T = 300$

$M_L = 18$

$M_{CO2} = 44$

$m_{CO2,L} = 0.000025$

$M_{air} = 29$

Get mole fraction of CO2 in water

$$X_{CO2,L} = \frac{m_{CO2,L}}{M_{CO2}} \cdot M_L$$

Air and water are at equilibrium. Therefore Henry's laww will determine the CO2 concentration in air.

From Appendix I:

$C_{He} = 1.71 \times 10^{8}$

Now:

$$X_{CO2,L} \cdot C_{He} = X_{CO2,G} \cdot P$$

$$m_{CO2,G} = \frac{X_{CO2,G} \cdot M_{CO2}}{X_{CO2,G} \cdot M_{CO2} + \left[1 - X_{CO2,G}\right] \cdot M_{air}}$$

SOLUTION

Unit Settings: SI K Pa J mass deg

$C_{He} = 1.710E+08$	$M_{air} = 29$	$M_{CO2} = 44$
$m_{CO2,G} = 0.0263$	$m_{CO2,L} = 0.000025$	$M_L = 18$
$P = 100000$	$T = 300$	$X_{CO2,G} = 0.01749$

$X_{CO2,L} = 0.00001023$

No unit problems were detected.

```
"Problem 1.20"
P=1.e5
T=320
M_L=18
M_Cl=35.46
m_Cl_L=50e-6
grad_m=100 "[1/m]"
"From Appendix J, at 298 K"
D_Cl_298=1.25e-9
mu_298=viscosity(water, P=P, T=298)
"To estimate D_Cl at 320 and 400 K temperatures, we cna use teh correlation of Wilke and Chang (1955), Eq. (1.5.24), whereby
D_12 is proportional to mT/mu:"
mu_320=viscosity(water, P=P, T=320)
"For a temperature of 400 K, atmospheric pressure would result in boiling.  Therefore, the pressure must be larger.  The effect of pressure on liquid  water properties will be
small, however.  Let us assume a pressure significantly larger than the saturation pressure of water at 400 K.  Assume a pressure of 3 bars."
P_400=3e5
mu_400=viscosity(water, P=P_400, T=400)
D_Cl_320/D_Cl_298=(320/mu_320)/(298/mu_298)
D_Cl_400/D_Cl_298=(400/mu_400)/(298/mu_298)
"Now calculate the diffusive mass fluxes."
m_DD_320=D_Cl_320*grad_m "[kg/m_2. s]"
m_DD_400=D_Cl_400*grad_m "[kg/m_2. s]"
```

SOLUTION

Unit Settings: SI K Pa J mass deg

$D_{Cl,298}$ = 1.250E-09	$D_{Cl,320}$ = 2.078E-09
$D_{Cl,400}$ = 6.856E-09	$grad_m$ = 100
μ_{298} = 0.0008936	μ_{320} = 0.0005773
μ_{400} = 0.0002187	M_{Cl} = 35.46
$m_{Cl,L}$ = 0.00005	$m_{DD,320}$ = 2.078E-07
$m_{DD,400}$ = 6.856E-07	M_L = 18
P = 100000	P_{400} = 300000
T = 320	

3 potential unit problems were detected.

Problem 1.21

The species conservation equation can be written by noting that the flow field is axi-symmetric (no dependence on coordinate ϕ). Therefore, from Appendix D, Eq. (D-7):

$$\rho\left(u_r \frac{\partial m_1}{\partial r} + \frac{u_\theta}{r}\frac{\partial m_1}{\partial \theta}\right) = \rho \mathcal{D}_{12}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial m_1}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial m_1}{\partial \theta}\right)\right]$$

The boundary conditions are:

at $r = R$ (for any θ):

$m_1 = m_{1,s}$

at $r \to \infty$ (for any θ):

$m_1 = m_{1,\infty}$

at $\theta = 0$ and $\theta = \pi$:

$$\frac{\partial m_1}{\partial \theta} = 0$$