Chapter 1

Note: Solutions not appearing in this complete solutions manual can be found in the student study guide.

Exercises 1.1

2.a $2i^3 - 3i^2 + 5i$

$= 2i^2 \cdot i - 3i^2 + 5i = -2i - 3(-1) + 5i$

$= -2i + 3 + 5i = 3 + 3i$

2.b $3i^5 - i^4 + 7i^3 - 10i^2 - 9$

$= 3i^4 \cdot i - i^4 + 7i^2 \cdot i - 10i^2 - 9$

$= 3i - 1 - 7i + 10 - 9$

$= -4i$

2.c $\dfrac{5}{i} + \dfrac{2}{i^3} - \dfrac{20}{i^{18}}$

$= \dfrac{5i}{i^2} + \dfrac{2}{i^2 \cdot i} - \dfrac{20}{(i^2)^9} = -5i - \dfrac{2}{i} + 20$

$= -5i - \dfrac{2i}{i^2} + 20 = -5i + 2i + 20 = 20 - 3i$

2.d $2i^6 + \left(\dfrac{2}{-i}\right)^3 + 5i^{-5} - 12i$

$= 2(i^2)^3 + \dfrac{8}{(-i)^3} + \dfrac{5}{i^5} - 12i = -2 - \dfrac{8}{i^2 \cdot i} + \dfrac{5}{i^4 \cdot i} - 12i$

$= -2 + \dfrac{8}{i} + \dfrac{5}{i} - 12i = -2 + \dfrac{8i}{i^2} + \dfrac{5i}{i^2} - 12i$

$= -2 - 8i - 5i - 12i = -2 - 25i$

3. $(5 - 9i) + (2 - 4i) = (5 + 2) + (-9 - 4)i = 7 - 13i$

4. $3(4 - i) - 3(5 + 2i)$

$= 12 - 3i - 15 - 6i = -3 - 9i$

6. $i(4 - i) + 4i(1 + 2i)$

$= 4i - i^2 + 4i + 8i^2 = ri + 1 + 4i - 8 = -7 + 8i$

7. $(2 - i)(4 + i) = 8 - 12i + 2i - 3i^2 = 8 - 10i + 3 = 11 - 10i$

8. $\left(\frac{1}{2}-\frac{1}{4}i\right)\left(\frac{2}{3}+\frac{5}{3}i\right)$

$=\frac{1}{3}+\frac{5}{6}i-\frac{1}{6}i-\frac{5}{12}i^2=\frac{1}{3}+\frac{4}{6}i+\frac{5}{12}=\frac{4+5}{12}+\frac{2}{3}i$

$=\frac{9}{12}+\frac{2}{3}i=\frac{3}{4}+\frac{2}{3}i$

10. $\frac{i}{1+i}=\frac{i}{1+i}\cdot\frac{1-i}{1-i}=\frac{i=i^2}{(1)^2-(i^2)}=\frac{i+1}{2}=\frac{1}{2}+\frac{i}{2}$

11. $\frac{2-4i}{3+5i}=\frac{2-4i}{3+5i}\frac{3-5i}{3-5i}=\frac{6-10i-12i+20i^2}{9-15i+15i-25i^2}=\frac{-14-22i}{34}=-\frac{7}{17}-\frac{11}{17}i$

12. $\frac{10-5i}{6+2i}=\frac{5(2-i)}{2(3+i)}=\frac{5(2-i)(3-i)}{2(3+i)(3-i)}=\frac{5(6-2i-3i+i^{2)}}{2(9-i^2)}$

14. $\frac{(1+i)(1-2i)}{(2+i)(4-3i)}=\frac{1-2i+i-2i^2}{8-6i+4i-3i^2}=\frac{3-i}{11-2i}=\frac{(3-i)(11+2i)}{11^2-(2i)^2}$

$\frac{33+6i-11i-2i^2}{121+4}=\frac{35-5i}{125}=\frac{35}{125}-\frac{5i}{125}=\frac{7}{25}-\frac{i}{25}$

15. $\frac{(5-4i)-(3+7i)}{(4+2i)+(2-3i)}=\frac{(5-3)+(-4-7)i}{(4+2)+(2-3)i}$

$=\frac{2-11i}{6-i}$

$=\frac{2-11i}{6-i}\frac{6+i}{6+i}$

$=\frac{12+2i-66i-11i^2}{36+6i-6i-i^2}$

$=\frac{23-64i}{37}$

$=\frac{23}{37}-\frac{64}{37}i$

16. $\frac{(4+5i)+2i^3}{(2+i)^2}=\frac{4+5i+2i\cdot i^2}{4+i^2+4i}=\frac{4+3i}{4i+3}=\frac{(4+3i)(4i-3)}{(4i)^2-9}$

$=\frac{16i-12+2i^2-9i}{-16-9}=\frac{7i-14}{-25}=\frac{14}{25}-\frac{7i}{25}$

18. $(1+i)^2(1-i)^3$

$= (1+i^2+2i)(1-i)^2(1-i)$

$= (2i)(1-2i+i^2)(1-i) = (2i)(-2i)(1-i)$

$= -4i^2(1-i) = 4(1-i) = 4-4i$

19. $(3+6i)+(4-i)(3+5i)+\dfrac{1}{2-i} = (3+6i)+(12+20i-3i-5i^2)+\dfrac{1}{2-i}\dfrac{2+i}{2+i}$

$= (3+6i)+(17+17i)+\dfrac{2+i}{4+2i-2i-i^2}$

$= (3+6i)\dfrac{5}{5}+(17+17i)\dfrac{5}{5}+\dfrac{2+i}{5}$

$= \dfrac{15+30i}{5}+\dfrac{85+85i}{5}+\dfrac{2+i}{5}$

$= \dfrac{102}{5}+\dfrac{116}{5}i$

20. $(2+3i)\left(\dfrac{2-i}{1+2i}\right)^2$

$\dfrac{2-i}{1+2i} = \dfrac{2-i}{1+2i}\cdot\dfrac{1-2i}{1-2i} = \dfrac{2-4i-i+2i^2}{1^2-(2i)^2} = \dfrac{-5i}{5} = -i$

$(2+3i)\left(\dfrac{2-i}{1+2i}\right)^2 = (2+3i)(-i)^2 = -2-3i$

22. $\left(1-\dfrac{1}{2}i\right)^3$

$= (1)^3+\dfrac{3}{1!}(1)^2\cdot\left(-\dfrac{1}{2}i\right)+\dfrac{3(2)}{2!}(1)^1\left(-\dfrac{1}{2}i\right)^2+\left(-\dfrac{1}{2}i\right)^3$

$= 1-\dfrac{3}{2}i+3\dfrac{1}{4}i^2-\dfrac{1}{8}i^3 = 1-\dfrac{3}{2}i+\dfrac{3}{4}(-1)+\dfrac{1}{8}i$

$= 1-\dfrac{3}{4}-\dfrac{3}{2}i+\dfrac{1}{8}i = \dfrac{1}{4}-\dfrac{11}{8}i$

23. $(-2+2i)^5 = (-2)^5+\dfrac{5}{1}(-2)^4(2i)^1+\dfrac{20}{2}(-2)^3(2i)^2+\dfrac{60}{6}(-2)^2(2i)^3+\dfrac{120}{24}(-2)^1(2i)^4+(2i)^5$

$= -32+(5)(16)(2i)+(10)(-8)(-4)+(10)(4)(-8i)+(5)(-2)(16)+32i$

$= -32+160i+320-320i-160+32i$

$= 128-128i$

24. $$(1+i)^8 = (1)^8 + \frac{8}{1!}(1)^7(i) + \frac{8(7)}{2!}(1)^6(i)^2 + \frac{8(7)(6)}{3!}(1)^5(i)^3$$

$$+\frac{8(7)(6)(5)}{4!}(1)^4(i)^4 + \frac{8(7)(6)(5)(4)}{5!}(1)^3(i)^5 +$$

$$\frac{8(7)(6)(5)(4)(3)}{6!}(1)^2(i)^6 + \frac{8(7)(6)(5)(4)(3)(2)}{7!}(1)(i)^7 + i^8$$

$$= 1 + 8i - 28 + 56i^3 + 70i^4 + 56i^5 + 28i^6 + 8i^7 + i^8$$

$$= 1 + 8i - 28 - 56i + 70 + 56i - 28 - 8i + 1$$

$$= 16$$

26. $$z = \frac{1}{(1+i)(1-2i)(1+3i)}$$

$$(1+i)(1-2i)(1+3i) = (1-2i+i-2i^2)(1+3i)$$

$$= (3-i)(1+3i) = 3+9i-i-3i^2 = 6+8i$$

$$\frac{1}{6+8i} = \frac{1}{6+8i}\cdot\frac{6-8i}{6-8i} = \frac{6-8i}{36-64i^2} = \frac{6-8i}{100} = \frac{6}{100} - \frac{8}{100}i$$

$$= \frac{3}{50} - \frac{4}{50}i = \frac{3}{50} - \frac{2}{25}i$$

$$\text{Re}(z) = \frac{3}{50}; \qquad \text{Im}(z) = \frac{-2}{25}$$

27. $$\text{Re}\left(\frac{1}{z}\right) = \text{Re}\left(\frac{1}{x+iy}\right)$$

$$= \text{Re}\left(\frac{1}{x+iy}\frac{x-iy}{x-iy}\right)$$

$$= \text{Re}\left(\frac{x-iy}{x^2 - ixy + ixy + i^2y^2}\right)$$

$$= \text{Re}\left(\frac{x-iy}{x^2+y^2}\right)$$

$$= \text{Re}\left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right)$$

$$= \frac{x}{x^2+y^2}$$

28. $z = x + iy$

$\text{Re}(z^2)$

$z^2 = (x+iy)^2 = x^2 + 2xyi + i^2y^2 = x^2 - y^2 + 2xyi$

$\text{Re}(z^2) = x^2 - y^2$

30. $\bar{z}^2 = (x-iy)^2 = x^2 + i^2y^2 - 2xyi = x^2 - y^2 - 2xyi$

$\bar{z}^2 + z^2 = x^2 - y^2 - 2xyi + x^2 - y^2 + 2xyi$

$= 2x^2 - 2y^2$

$\text{Im}(\bar{z}^2 + z^2) = 0$

31. $\text{Re(iz)} = \text{Re}(i(x+iy))$

$= \text{Re}(ix + i^2y)$

$= \text{Re}(ix - y)$

$= \ -y$

$= \ -\text{Im}(z)$

32. $z = x + iy$

$iz = ix + i^2y = -y + ix$

$\text{Im}(iz) = x = \text{Re}(z)$

34. $\text{Re}(z^2) = x^2 - y^2$

$= (\text{Re(z)})^2 - (\text{Im(z)})^2$

35. If $z_1 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, then

$$z_1^2 + i = \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 + i$$
$$= \frac{2}{4} - \frac{2}{4}i - \frac{2}{4}i + \frac{2}{4}i^2 + i$$
$$= \frac{1}{2} - i - \frac{1}{2} + i$$
$$= 0.$$

An additional solution is $z_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$

36. $z^4 = -4$

$z_1 = 1 + i,\ z_2 = -1 + i$

$z_3 = \dfrac{-4}{1+i} = \dfrac{-4}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{-4+4i}{2} = -2 + 2i = 2(-1+i)$

$z_4 = \dfrac{-4}{-1+i} = \dfrac{-4}{-1+i} \cdot \dfrac{-1-i}{-1-i} = \dfrac{4+4i}{(1)^2 - i^2} = \dfrac{4+4i}{2} = 2(1+i)$

38. $z - 2\overline{z} + 7 - 6i = 0$

$z - 2\overline{z} = 6i - 7$

$x + iy - 2(x - iy) = 6i - 7$

$x + iy - 2x + 2iy = 6i - 7$

$-x + 3iy = 6i - 7$

$-x = -7 \Rightarrow x = 7$

$3y = 6 \Rightarrow y = 2$

39. Let $z = a + ib$, then $z^2 = (a + ib)^2 = a^2 + iab + iab + i^2b^2 = a^2 - b^2 + 2abi$. By Definition 1.1.2, $z^2 = i$ if and only if Re (z^2) = Re(i) and Im(z^2) = Im(i). This gives the following two equations:

$$a^2 - b^2 = 0$$
$$2ab = 1.$$

From the second equation, $b = \dfrac{1}{2a}$. Substituting this into the first equation we get

$$a^2 - \frac{1}{4a^2} = 0 \quad \text{or} \quad a^4 = \frac{1}{4}.$$

Solving for a we find that $a = \pm\dfrac{\sqrt{2}}{2}$. Now since $b = \dfrac{1}{2a}$, we have two solutions, $a = \dfrac{\sqrt{2}}{2}$, $b = \dfrac{\sqrt{2}}{2}$ and $a = -\dfrac{\sqrt{2}}{2}$, $b = -\dfrac{\sqrt{2}}{2}$. Therefore, $z = \pm\left(\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i\right)$.

40. $\overline{z}^2 = 4z$

$(x-iy)^2 = 4(x+iy)$

$x^2 + i^2y^2 - 2xyi = 4x + +4yi$

$x^2 - y^2 = 4x$

$-2xy = 4y \Rightarrow -2x = 4 \Rightarrow x = -2$

$x^2 - y^2 = 4x \Rightarrow 4 - y^2 = -8 \Rightarrow 12 = y^2 \Rightarrow y = \pm 2\sqrt{3}$

42. $\dfrac{z}{1+\overline{z}} = 3+4i$

$$\frac{z}{1+\overline{z}} = \frac{z(1-\overline{z})}{1^2-\overline{z}^2} = \frac{(x+iy)(1-x+iy)}{1-(x-iy)^2}$$

$$= \frac{x-x^2+xyi+iy-xyi+i^2y^2}{1-(x^2+i^2y^2-2xyi)} = \frac{x-x^2-y^2+iy}{1-x^2+y^2+2xyi}$$

$$\frac{x-x^2-y^2+iy}{1-x^2+y^2+2xyi} = 3+4i$$

$x - x^2 - y^2 + iy = (3+4i)(1-x^2+y^2+2xyi)$

$x - x^2 - y^2 + iy = 3 - 3x^2 + 3y^2 + 6xyi + 4i - 4ix^2 + 4iy^2 + 8xyi^2$

$x - x^{2-}y^2 + iy = 3 - 3x^2 + 3y^2 + 6xyi + 4i - 4x^2i + 4y^2i - 8xy$

$x - x^2 - y^2 = 3 - 3x^2 + 3y^2 - 8xy$

$y = 6xy + 4 - 4x^2 + 4y^2$

$$\begin{cases} 2x^2 - 4y^2 + x + 8xy - 3 = 0 \\ -4x^2 + 4y^2 - y + 6xy + 4 = 0 \end{cases}$$

43. To solve we multiply the first equation by $-i$ then sum the two equations.

$$\begin{aligned} -i^2z_1 - \quad -i^2z_2 &= -i(2+1-i) \\ -z_1 + (1-i)z_2 &= 3-5i \end{aligned} \quad \Rightarrow \quad \begin{aligned} z_1 - \quad z_2 &= 10-2i \\ -z_1 + (1-i)z_2 &= 3-5i \\ \hline -iz_2 &= 13-7i \end{aligned}$$

Thus, $-iz_2 = 13 - 7i$ or $z_2 = 7 + 13i$. Now substitute this value of z_2 into the first equation and solve for z_1.

$$\begin{aligned} iz_1 - i(7+13i) &= 2+10i \\ iz_1 + 13 - 7i &= 2+10i \\ iz_1 &= -11+17i \\ z_1 &= 17+11i \end{aligned}$$

44. $iz_1 + (1+i)z_2 = 1+2i$

$(2-i)z_1 + 2iz_2 = 4i$

$$\left.\begin{aligned} = \frac{-2i}{1+i}(iz_1) - 2iz_2 &= (1+2i)\cdot\left(\frac{-2i}{1+i}\right) \\ (2-i)z_1 + 2iz_2 &= 4i \end{aligned}\right\} \oplus$$

$$\left(\frac{2}{1+i} + 2 - i\right)z_1 = \frac{-2i(1+2i)}{1+i} + 4i$$

$$(1-i+2-i)z_1 = -3i+1+4i$$

$$(3=2i)z_1 = 1+i \Rightarrow z_1 = \frac{1+i}{3-2i} \Rightarrow z_1 = \frac{1}{13} + \frac{5}{13}i$$

$$(2-i)z_1 + 2iz_2 = 4i$$

$$(2-i)\left(\frac{1}{13} + \frac{5}{13}i\right) + 2iz_2 = 4i$$

$$\frac{2}{13} + \frac{10}{13}i - \frac{1}{13}i + \frac{5}{13} + 2iz_2 = 4i$$

$$\frac{7}{13} + \frac{9}{13}i + 2iz_2 = 4i$$

$$2iz_2 = 4i - \frac{7}{13} - \frac{9}{13}i$$

$$2iz_2 = -\frac{7}{13} + \frac{43}{13}i$$

$$z_2 = \frac{-\frac{7}{13} + \frac{43}{13}i}{2i}$$

$$= \frac{-\frac{43}{13} - \frac{7}{13}i}{-2} = \frac{43}{26} + \frac{7}{26}i$$

46. $(1+i)^8 = 16$

$(1+i)^{5404} = 10808$

47. (a) Since $i^4 = 1$, it follows that $i^{4k} = (i^4)^k = 1^k = 1$. So, $n = 4k$.

48. $5+6i = (1+u)(a+ib)$

$= a + ib + ai - b$

$$5+6i = a-b+(a+b)i$$

$$\left.\begin{array}{l} a-b=5 \\ a+b=6 \end{array}\right\} \oplus \quad 2a=11 \quad a=\frac{11}{2}$$

$$a+b=6 \quad b=6-a=6-\frac{11}{2}=\frac{1}{2}$$

$$\frac{1}{3-4i}=a+ib$$

$$(a+ib)(3-4i)=1 \Rightarrow 3a-4ai+3bi+4b=1$$

$$\Rightarrow \begin{cases} 3a+4b=1 \\ 3b-4a=0 \end{cases} \quad 3b=4a \quad b=\frac{4}{3}a$$

$$3a+4b=1$$

$$3a+4\frac{4}{3}a=1 \quad 3a+\frac{16}{a}=1$$

$$\frac{25}{3}a=1 \quad a=\frac{3}{25}$$

$$b=\frac{4}{3}a=\frac{4}{3}\cdot\frac{3}{25}=\frac{4}{25}$$

50. Suppose $z_1z_2=0$ and $z_1 \neq 0$. Then divide both sides of $z_1z_2=0$ by z_1 to get $z_2=0$. Thus, either $z_1=0$ or $z_2=0$.

51. Suppose $z_1z_2 = r$ where r is a real number. Then $z_1 = r/z_2$. Now multiply the numerator and denominator of r/z_2 by $\overline{z}_2$ and explain why the result is of the form $k\overline{z}_2$ where k is a real constant.

52. $z_1\overline{z}_1+\overline{z}_1z_2=2\mathrm{Re}(z_1\overline{z}_2)$

$$z_1=x_1+iy_1 \quad \overline{z}_1=x_1-iy_1$$

$$z_2=x_2+iy_2 \quad \overline{z}_2=x_2-iy_2$$

$$z_1\overline{z}_2=(x_1+iy_1)(x_2-iy_2)=x_1x_2-x_1y_2i+x_2y_1i+y_1y_2$$

$$\overline{z}_1z_2=(x_1-iy_1)(x_1+iy_2)=x_1x_2+x_1y_2i-x_2y_1i+y_1y_2$$

$$z_1\overline{z}_2+\overline{z}_1z_2=2x_1x_2+2y_1y_2$$

$$\mathrm{Re}(z_1\overline{z}_2)=x_1x_2+y_1y_2$$

$$2\mathrm{Re}(z_1\overline{z}_2)=2x_1x_2+2y_1y_2$$

55. Assume that such a subset P exists and that i is in P. By the second property (applied two times) i^3 is in P. Explain why this leads to a contradiction.

Exercises 1.2

2.
$$\begin{aligned} z_1 &= 1-i \\ z_2 &= 1+i \\ z_1+z_2 &= 1-i+1+i \\ &= 2 \\ z_1-z_2 &= 1-i-1-i \\ &= -2i \end{aligned}$$

4.
$$\begin{aligned} z_1 &= 4-3i \\ z_2 &= -2+3i \\ 2z_1+4z_2 &= 2(4-3i)+4(-2+3i) \\ &= 8-68-8+12i \\ &= 6i \\ z_1-z_2 &= 4-3i+2-3i \\ &= 6-6i \end{aligned}$$

6.
$$\begin{aligned} z_1 &= -2-8i \\ z_2 &= 3i \\ z_3 &= -6-5i \\ z_2z_3 &= z_2-z_3 \\ z_1z_2 &= z_1-z_2 \\ z_1z_3 &= z_1-z_3 \end{aligned}$$

7. Two adjacent sides of the triangle are

$$\begin{aligned} z_1-z_3 &= (-2-8i-(-6-5i)=4-3i, \text{ and} \\ z_2-z_3 &= \quad (3i)-(-6-5i)=6+8i. \end{aligned}$$

These sides are represented by the two vectors $4i-3j$ and $6i+8j$. the angle θ between these vectors can be determined using the dot product (see Review Topic: Vectors):

$$(4\mathbf{i}-3\mathbf{j})\cdot(6\mathbf{i}+8\mathbf{j}) = 24-24=0.$$

This implies that $\cos\theta = 0$, and so $\theta = \pi/2$. Therefore, z_1, z_2, and z_3 are the vertices of a right triangle.

8.

$$\begin{aligned}
z_1 &= 1+5i \\
z_2 &= -4-i \\
z_3 &= 3+i \\
z_3 - z_2 &= 3+i+4+i \\
&= 7+2i \\
d &= \sqrt{(7-1)^2+(2-5)^2} \\
&= \sqrt{6^2+(-3)^2} \\
&= \sqrt{36+9} = \sqrt{45} \\
d &= 3\sqrt{5}
\end{aligned}$$

10. $i(2-i)-4(1+\frac{1}{4}i) \qquad |z| = \sqrt{x^2+y^2}$

$$\begin{aligned}
&2i - i^2 - 4 - i \\
&= 2i + 1 - 4 - i \\
&= i - 3 \\
&|z| = \sqrt{(-3)^2+(1)^2} = \sqrt{9+1} = \sqrt{10}
\end{aligned}$$

11.

$$\begin{aligned}
\left|\frac{2i}{3-4i}\right| &= \frac{|2i|}{|3-4i|} \\
&= \frac{\sqrt{(0)^2+(2)^2}}{\sqrt{(3)^2+(-4)^2}} \\
&= \frac{\sqrt{4}}{\sqrt{25}} \\
&= \frac{2}{5}
\end{aligned}$$

12.

$$\frac{1-2i}{1+i} + \frac{2-i}{1-i}$$

$$= \frac{(1-2i)(1-i)}{(1)^2-(i)^2} + \frac{(2-i)(1+i)}{(1)^2-(i)^2} = \frac{1-i-2i+2i^2}{1+1} + \frac{2+2i-i-i^2}{1+1}$$

$$= \frac{-1-3i}{2} + \frac{3+i}{2} = \frac{-1-3i+3+i}{2} = \frac{-2i+2}{2} = -i+1$$

$$|z| = \sqrt{(-1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}$$

14. $|z+\overline{5}|$

$$\begin{aligned} z+\overline{5} &= x+iy+5(x-iy) \\ &= x+iy+5x-5yi \\ &= 6x-4yi \end{aligned}$$

$$\begin{aligned} |z+5\overline{z}| &= \sqrt{(6x)^2+(-4y)^2} = \sqrt{36x^2+16y^2} \\ &= \sqrt{4(9x^2=4y^2)} = 2\sqrt{9x^2+4y^2} \end{aligned}$$

15. In order to determine which complex number is closest to the origin we compute the distance from z_1 to 0 and the distance from z_2 to 0.

$$|z_1-0| = |10+8i| = \sqrt{(10)^2+(8)^2} = \sqrt{164}$$
$$|z_2-0| = |11-6i| = \sqrt{(11)^2+(-6)^2} = \sqrt{157}$$

Therefore, z_2 is closest to the origin. On the other hand, the distance from z_1 to $1+i$ and the distance from z_2 to $1+i$ is:

$$|z_1-(1+i)| = |9+7i| = \sqrt{(9)^2+(7)^2} = \sqrt{130}$$
$$|z_2-(1+i)| = |10-7i| = \sqrt{(10)^2+(-7)^2} = \sqrt{149}$$

Therefore, z_1 is closest to $1+i$.

16. $\frac{1}{2}-\frac{1}{4}i \qquad \frac{2}{3}+\frac{1}{6}i$

$$d_1 = \sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{4}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{16}} = \sqrt{\frac{4+1}{16}} = \sqrt{\frac{5}{16}} = \sqrt{\frac{5}{4}}$$

$$d_2 = \sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{1}{6}\right)^2} = \sqrt{\frac{4}{9}+\frac{1}{36}} = \sqrt{\frac{16+1}{36}} = \sqrt{\frac{17}{36}} = \sqrt{\frac{17}{6}}$$

$\frac{1}{2}-\frac{1}{4}i$ is closest to origin.

For $1+i$

$$d_1 = \sqrt{\left(1-\frac{1}{2}\right)^2+\left(1+\frac{1}{4}\right)^2} = \sqrt{\frac{1}{4}+\frac{25}{16}} = \sqrt{\frac{4+25}{16}} = \sqrt{\frac{29}{16}} = \sqrt{\frac{29}{4}}$$

$$d_2 = \sqrt{\left(1-\frac{2}{3}\right)^2+\left(1-\frac{1}{6}\right)^2} = \sqrt{\frac{1}{9}+\frac{25}{36}} = \sqrt{\frac{29}{36}} = \sqrt{\frac{29}{6}}$$

18. $[\text{Im}(i\bar{z})]^2 = 2$

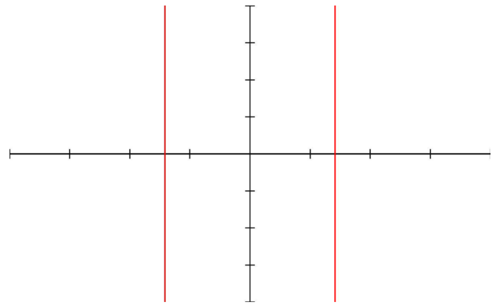
$i\bar{z} = i(x-iy) = y+ix$

$\text{Im}(i\bar{z}) = x$

$[\text{Im}(i\bar{z})]^2 = x^2$

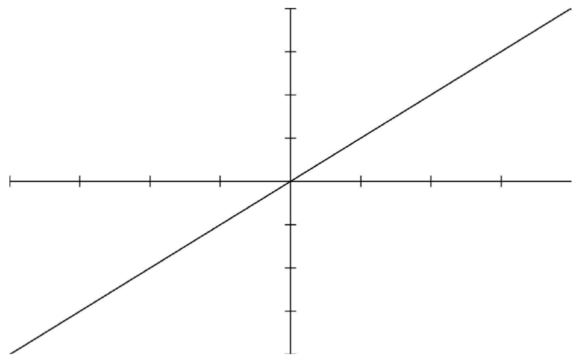
$\Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$



19. If $|z-i| = |z-1|$, then $|z-i|^2 = |z-1|^2$. Let $z = x+iy$, then:

$$|(x+iy)-i|^2 = |(x+iy)-1|^2$$
$$|x+i(y-1)|^2 = |(x-1)+iy|^2$$
$$x^2+(y-1)^2 = (x-1)^2+y^2$$
$$\cancel{x^2}+\cancel{y^2}-2y+\cancel{1} = \cancel{x^2}-2x+\cancel{1}+\cancel{y^2}$$
$$-2y = -2x$$
$$y = x.$$

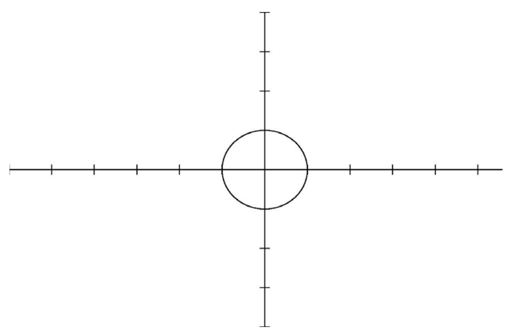


The set of points satisfying the equation $|z-i| = |z-1|$ lie on the line $y = x$ in the plane.

20. $\bar{z} = z^{-1}$

$\bar{z} = \frac{1}{z} \Rightarrow z\bar{z} = 1 \Rightarrow x^2+y^2 = 1$

circle centered at origin with radius 1



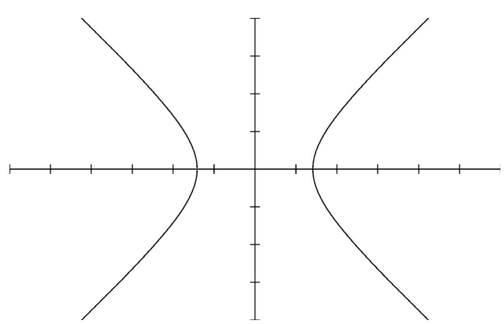
22. $\text{Re}(z^2) = |\sqrt{3}-i|$

$z^2 = (x+iy)^2 = x^2-y^2+2xyi$

$\text{Re}(z^2) = x^2-y^2$

$|\sqrt{3}-i| = \sqrt{(\sqrt{3})^2+(-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

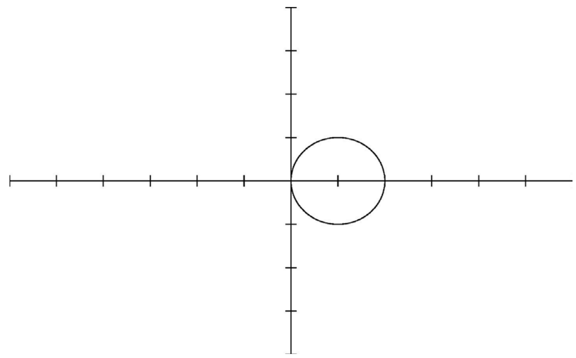
$x^2-y^2 = 2$ Hyperbola.



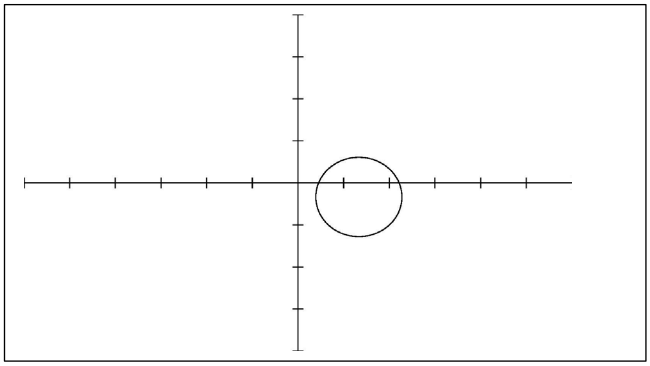
23. If $|z-1|=1$, then $|z-1|^2=1$.

Let $z=x+iy$, then:

$$|(x+iy)-1|^2=1$$
$$|(x-1)+iy|^2=1$$
$$(x-1)^2+y^2=1.$$



The set of points satisfying the equation $|z-1|=1$ lie on the circle $(x-1)^2+y^2=1$ with center (1, 0) and radius 1.



24. $|z-i|=2|z-1|$

$$z-i=x+iy-i=x+(y-1)i$$
$$|z-i|=\sqrt{x^2+(y-1)^2}$$
$$z-1=x+iy-1=(x-1)+yi$$
$$|z-1|=\sqrt{(x-1)^2+y^2}\Rightarrow 2|z-1|=2\sqrt{(x-1)^2+y^2}$$
$$\sqrt{x^2+(y-1)^2}=2\sqrt{(x-1)^2+y^2}$$
$$\Rightarrow x^2+(y-1)^2=4\left((x-1)^2+y^2\right)$$
$$\Rightarrow x^2+y^2-2y+1=4(x^2-2x+1+y^2)$$
$$\Rightarrow x^2+y^2-2y+1=4x^2-8x+4+4y^2$$
$$\Rightarrow 3y^2+3x^2-8x+2y+3=0$$

because the coefficients of x^2 and y^2 are equal $\Rightarrow$ circle.

26. $|z|=\text{Re}(z)$

$$\sqrt{x^2+y^2}=x$$
$$x^2+y^2=x^2\Rightarrow y^2=0\Rightarrow y=0\Rightarrow x\text{-axis}$$

27. We use the triangle inequalities:

$$\big||z|-|6+8i|\big| \le |z+6+8i| \le |z|+|6+8i|.$$

Making the substitutions $|z|=2$ and $|6+8i|=\sqrt{6^2+8^2}=10$ we obtain

$$|2-10| \le |z+6+8i| \le 2+10.$$

Thus, $8 \le |z+6+8i| \le 12.$

28. $|z|=1$

$|\le|z^2-3| \le 4$

$|z^2-3| \le |z^2|+|-3|$

$|z^2-3| \le 1+3$

$|z^2-3| \le 4$

30. $z^4-5z^2+6 \qquad |z|=2$

$z^4-5z^2+6=(z^2-3)(z^2-2)$

$\left|\dfrac{1}{z^4-5z^2+6}\right| \le M$

$|z^4-5z^2+6| = |(z^2-3)(z^2-2| = |z^2-3||z^2-2|$

$\le (|z^2|+|-3|)(|z^2|+|-2|)$

$\le (4+3)(4+2) \le (7)(6) \le 42$

$\Rightarrow \left|\dfrac{1}{z^4-z^2+6}\right| \le \dfrac{1}{42}$

31. Let $z=x+iy$. From Definition 1.1.2, $|z|-z=2+i$ if and only if $\text{Re}(|z|-z)=\text{Re}(2+i)=2$ and $\text{Im}(|z|+z)=\text{Re}(2+i)=1$. This gives the following two equations:

$$\text{Re}\left(\sqrt{x^2+y^2}-(x+iy)\right)=\sqrt{x^2+y^2}-x=2$$
$$\text{Im}\left(\sqrt{x^2+y^2}-(x+iy)\right)= \quad -y \quad =1.$$

Since $y=-1$ from the second equation, the first equation becomes

$$\sqrt{x^2+1}-x=2$$
$$\sqrt{x^2+1}=2+x$$
$$x^2+1=(2+x)^2$$
$$\cancel{x^2}+1=4+4x+\cancel{x^2}$$
$$-4x=3$$
$$x=-\frac{3}{4}$$

The complex number $z=-\frac{3}{4}-i$ satisfies the given equation.

32. $|z|^2+1+12i=6z$

$$x^2+y^2+1+12i=6(x+iy)$$
$$x^2+y^2+1+12i=6x+6yi$$
$$\begin{cases} x^2+y^2+1=6x \\ 12=6y \Rightarrow y=\frac{12}{6}=2 \end{cases}$$
$$x^2+y^2+1=6x$$
$$\Rightarrow x^2+4+1=6x \Rightarrow x^2-6x+5=0$$
$$\Rightarrow (x-1)(x-5)=0$$
$$\Rightarrow x=1 \text{ or } x=5$$
$$\begin{cases} z=1+2i \\ \text{or} \\ z=5+2i \end{cases}$$

34.a $z=a+ib$

$-z=-a-ib$

symmetric with respect to origin

34.b $z=a+ib$

$$z^{-1}=\frac{a-ib}{a^2+b^2}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$$

symmetric with respect to x-axis

35. In each case consider the angle between z and iz.

36. Suppose $|z| = 0$ and $z = x + iy$.

$\sqrt{x^2 + y^2} = 0$

$x^2 + y^2 = 0$

$x = 0$ and $y = 0$

Thus $z = 0 + i0 = 0$.

38. $z = x + iy$ radius = 5

center $(3, -6)$

$|z^2| = x^2 + y^2$

$(x-3)^2 + (y+6)^2 = 25$

$x^2 - 6x + 9 + y^2 + 12y + 36 = 25$

$x^2 + y^2 - 6x + 12y + 45 = 25$

$x^2 + y^2 - 6x + 12y = -20$

$|z|^2 - 6x + 12y = 20i^2$

39. Consider $|z|$.

42. $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

equation of line $y - y_1 = m(x - x_1)$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

43. Rewrite the equation as $(z_3 - z_2) = -k(z_1 - z_2)$. Now both $z_3 - z_2$ and $z_1 - z_2$ can be viewed as vectors with the same initial point z_2. Since k is a real number, $-k(z_1 - z_2)$ is just a scalar multiple of $z_1 - z_2$. Use the equality above to make a statement regarding the directions of these two vectors.

46. $z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

$\text{Re}(z) = x \Rightarrow |\text{Re}(z)| = \sqrt{x^2} = x$

$x \le \sqrt{x^2 + y^2}$ it could be equal if $y = 0$

$\text{Im}(z) = y \Rightarrow |\text{Im}(z)| = \sqrt{y^2} = y$

$y \le \sqrt{x^2 + y^2}$ it could be equal if $x = 0$

47. For both parts, set $z = x + iy$ and use the definition of modulus.

48. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$

$|z_1 + z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2$

$z_1 - z_2 = x_1 + iy_1 - x_2 - iy_2 = (x_1 - x_2) + i(y_1 - y_2)$

$|z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$z_1 = x_1 + iy_1 \Rightarrow |z_1|^2 = x_1^2 + y_1^2$

$z_2 = x_2 + iy_2 \Rightarrow |z_2|^2 = x_2^2 + y_2^2$

$2(|z_1|^2 + |z_2|^2) = 2(z_1^2 + y_1^2 + x_2^2 + y_2^2)$

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2x_1^2 + 2y_1^2 + 2x_2^2 + 2y_2^2$

Exercises 1.3

2. -10

$x = -10 \qquad y = 0$

$$r = |z| = \sqrt{(-10)^2 + 0^2} = \sqrt{100} = 10$$

$$\frac{y}{x} = \frac{0}{-10} = 0 \qquad \theta = \tan^{-1}(0) = \pi$$

$$z = r(\cos\theta + i\sin\theta) = 10(\cos\pi + i\sin\pi)$$

3. Since $-3i = 0 + (-3)i$, we identify $x = 0$ and $y = -3$. Then $\sqrt{0^2 + (-3)^2} = 3$. Since $x = 0$ and $y = -3 < 0$, $\text{Arg}(-3i) = -\frac{\pi}{2}$ A different argument for $-3i$ is given by $\theta = \text{Arg}(-3i) + 2\pi = \frac{3\pi}{2}$. Therefore,

$$-3i = 3\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right), \quad \text{using } \theta = \text{Arg}(-3i)$$

$$-3i = 3\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right), \quad \text{using } \theta \neq \text{Arg}(-3i).$$

4. $6i$

$x = 0 \qquad y = 6$

$$r = |z| = \sqrt{0^2 + (6)^2} = 6$$

$$\frac{y}{x} = \frac{6}{0} = \infty \qquad \theta = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$z = r(\cos\theta + i\sin\theta) = 6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

6. $5 - 5i$

$x = 5 \qquad y = -5$

$$r = |z| = \sqrt{(5)^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\frac{y}{x} = \frac{-5}{5} = -1 \qquad \theta = \tan^{-1}(-1) = \frac{-\pi}{4}$$

$$z = r(\cos\theta + i\sin\theta) = 5\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right)$$

7. For $-\sqrt{3}+i$, we identify $x=-\sqrt{3}$ and $y=1$. Then $r=\sqrt{(\sqrt{3})^2+1^2}=2$. Since $z=-\sqrt{3}+i$ is in the second quadrant,

$$\text{Arg}(-\sqrt{3}+i)=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\pi=-\frac{\pi}{6}+\pi=\frac{5\pi}{6}.$$

A different argument for $-\sqrt{3}+i$ is given by $\theta=\text{Arg}(-\sqrt{3}+i)+2\pi=\frac{17\pi}{6}$. Therefore,

$$-\sqrt{3}+i = 2\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right),\quad \text{using } \theta=\text{Arg}(-\sqrt{3}+i)$$

$$-\sqrt{3}+i = 2\left(\cos\left(\frac{17\pi}{6}\right)+i\sin\left(\frac{17\pi}{6}\right)\right),\quad \text{using } \theta\neq\text{Arg}(-\sqrt{3}+i).$$

8. $-2-2\sqrt{3}i$

$x=-2 \qquad y=-2\sqrt{3}$

$r=|z|=\sqrt{(-2)^2+\left(-2\sqrt{3}\right)^2}=\sqrt{4+12}=\sqrt{16}=4$

$\dfrac{y}{4}=\dfrac{-2\sqrt{3}}{-2}=\sqrt{3} \qquad \theta=\tan^{-1}\left(\sqrt{3}\right)=\dfrac{4\pi}{3}$

$z=r(\cos\theta+i\sin\theta)=4\left(\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}\right)$

10. $\dfrac{12}{\sqrt{3}+i}=\dfrac{12}{\sqrt{3}+i}\cdot\dfrac{\sqrt{3}-i}{\sqrt{3}-i}=\dfrac{12\sqrt{3}-12i}{3-i^2}=\dfrac{12\sqrt{3}-12i}{4}=3\sqrt{3}-3i$

$x=3\sqrt{3} \qquad y=-3$

$r=|z|=\sqrt{\left(3\sqrt{3}\right)^2}=\sqrt{27+9}=\sqrt{36}=6$

$\dfrac{y}{x}=\dfrac{-3}{3\sqrt{3}}=\dfrac{-1}{\sqrt{3}}=-\dfrac{\sqrt{3}}{3} \qquad \theta=\tan^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)=-\dfrac{\pi}{6}$

$z=r(\cos\theta+i\sin\theta)=6\left(\cos\left(\dfrac{-\pi}{6}\right)+i\sin\left(\dfrac{-\pi}{6}\right)\right)$